Algebra, Geometry and Analysis of Commuting Ordinary Differential Operators

Alexander Zheglov*

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For any questions please write to: azheglov@math.msu.su or to abzv24@mail.ru

1 Introduction

There are two classical problems related to integrable systems, appeared and studied already in the works of I. Schur, J. Burchnall, T. Chaundy in the beginning of 20th century: how to construct explicitly a pair of commuting differential operators and how to classify all commutative subalgebras of differential operators. Both problems have broad connections with many branches of modern mathematics, first of all with integrable systems, since explicit examples of commuting operators provide explicit solutions of many non-linear partial differential equations.

The theory of commuting differential operators is far to be complete, but it is well developed for commuting *ordinary differential operators*. In particular, the classification of rings of commuting ordinary differential operators in terms of spectral data (Krichever's theorem), as well as its various generalizations, is known. However, for high rank rings or for rings with special spectral curves, this theory is not complete enough and continues to evolve. Recently, this theory has been associated with such well-known open conjectures as the Dixmier conjecture and the Jacobian conjecture.

This course involves an explanation of basic ideas and constructions from the theory of commuting ordinary differential operators as well as an overview of related open problems from algebra, algebraic geometry and complex analysis. One of the objectives of the course is to propose new tasks for research.

We meet ordinary differential operators every time when we want to solve a *linear* differential equation:

$$(a_n\partial^n + \ldots + a_0)\psi = 0,$$

where a_i, ψ are (usually) smooth functions, and even *non-linear* equations.

Consider a ring $R = C^\infty(\mathbb{R})$ of smooth (or analytic) functions on the line (or on a open neighbourhood of zero), denote $\partial := \partial/\partial x$. For any function $f \in R$ denote by $\hat{f}$ the *operator* of multiplication on f in R: $\hat{f}(g) := f \cdot g$. Then the Leibniz rule

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

is equivalent to the equality of *operators*: $\partial\hat{f} = \widehat{f'} + \hat{f}\partial$. Later we will omit the sign ˆ to simplify the notation.

Example 1.1. Consider two operators:

$$L = \partial^2 + \hat{u}, \quad P = 4\partial^3 + 6\hat{u}\partial + 3\widehat{u'}.$$

Then $[P, L] = 6\widehat{uu'} + \widehat{u'''}$ (check it! Hint: in order to check it, apply $P \circ L$ and $L \circ P$ to a test function $\varphi \in R$. Then the following equation must hold: $(P \circ L - L \circ P)\varphi = (u''' + 6uu')\varphi$). Now if we take $u = u(x,t)$, where t is a new variable, and set $\frac{\partial}{\partial t}(\partial) = 0$, we obtain a famous non-linear equation of mathematical physics, the Korteweg de Vries equation:

$$u_t = 6uu_x + u_{xxx} \tag{KdV}$$

Namely, the equation

$$\frac{\partial L}{\partial t} = [P, L]$$

is equivalent to it.

First explicit examples appeared already in 1903 in the work of Wallenberg [115]:

Example 1.2. Let $\Lambda \simeq \mathbb{Z}^2 \subset \mathbb{C}$ be a lattice and

$$\wp(x) = \sum_{\lambda \in \Lambda \backslash \{0,0\}} \left(\frac{1}{(x+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

be the corresponding Weierstrass function (a meromorphic function on a torus $\mathbb{C}/\mathbb{Z}^2$, we'll return to special functions later). Wallenberg observed that the ordinary differential operators L, P from previous example with $u(x) = -2\wp(x+\alpha)$, $\alpha \in \mathbb{C}$, or with $u(x) = -2/(x+\alpha)^2$ or with $u(x) = -2/\sin^2(x+\alpha)$ (degenerations of $\wp(x)$), commute.

I. Schur in 1905 and Burchnall, Chaundy in 1920-th got more examples of operators of relatively prime orders (Burchnall and Chaundy even classified such pairs).

In 1968 Dixmier discovered another interesting example [21]: for any $\lambda \in \mathbb{C}$ put $Q = \partial^2 + x^3 + \lambda$ and consider operators

$$L = Q^2 + 2x, \quad P = 2Q^3 + 3(Qx + xQ).$$

Then L and P commute and satisfy the relation $Q^2 = P^3 - \lambda$.

Two problems mentioned in the beginning appear to be connected with many problems from different branches of mathematics (search for them in internet), e.g.:

- Complex analysis (the Schottky problem, solved)
- Non-linear partial differential equations (find new exact solutions)
- Deformation quantisation
- Algebra (the Dixmier or Jacobian or Poisson conjectures, highly non-trivial and still open)

Preparation knowledges.

It is highly recommended to be familiar with the basic topics from Algebra and Commutative algebra (though all auxiliary results from these topics will be reminded as necessary) such as:

1. Resultants, transcendence basis, factorial rings
2. Rings and modules, their tensor product
3. Localisation of rings and modules
4. Noetherian rings, Hilbert's basis theorem
5. Integral elements, Noether's normalisation lemma, Hilbert's Nullstellensatz
6. Krull dimension
7. Completions of noetherian rings

The main reference for these topics are the books of Lang [47] and of Atiyah, Macdonald [4].

It will also be useful to know basic definitions from algebraic geometry (e.g. as in Ch.1, Sec. 1-5 of the book [32] or as in Ch. 1-3 of the book [104]; see also short useful preliminary course [106]). For the reader convenience, I included all necessary results from this list (sometimes even with proofs, for undergraduate students) into Appendix.

The main emphasis in these lectures is on a detailed analysis of the *algebraic* theory of commuting operators, which is important for understanding generalizations of this theory in higher dimensions, cf. problem 13.5.

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2 Lectures guide and List of notations

The lectures consist of Theorems, Propositions, Lemmas, Remarks, Exercises, Problems and Comments. The comments are not necessary for the first reading, but they contains useful information for curious or advanced readers. Problems are yet open questions or tasks for further investigation. Sometimes they are difficult, sometimes not, and sometimes they are already known (folklore) facts, which are, however, have never been written anywhere. We tried to keep the exposition of our lectures as self-contained as it is possible. The material is based upon various texts from references.

Calligraphic letters denote generic algebras and rings. Usual capital letters denote commutative algebras and rings. Almost always the letter D (combined together with various indices) is reserved to denote rings of ordinary differential operators.

Throughout these lectures K will denote a field of characteristic zero.

Recall the following commonly used definitions.

Definition 2.1. Let K be a field. An *algebra* over K or K-algebra is a vector space over K equipped with a bilinear product $\cdot$. A K-linear map $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is a *homomorphism* of algebras if $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \quad \forall x, y \in \mathcal{A}_1$.

A *Lie algebra* over K is an algebra with a product usually denoted by $[,]$ that satisfies the axioms of alternativity, i.e. $[y, y] = 0 \quad \forall y$, and the Jacoby identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

A *ring* over K is an associative K-algebra with the multiplicative unity 1. In particular, the ring over K contains a copy of K. A K-linear map $\varphi : \mathcal{R}_1 \to \mathcal{R}_2$ is a *homomorphism* of rings if it is a homomorphism of algebras and $\varphi(1) = 1$.

A left (right) $\mathcal{R}$-module M, where $\mathcal{R}$ is a ring, is an additive group with the left (right) action of $\mathcal{R}$: $\mathcal{R} \times M \to M$ ($M \times \mathcal{R} \to M$) with usual axioms. Homomorphisms of $\mathcal{R}$-modules are defined as linear maps compatible with the action of the ring.

A left (right) ideal I in $\mathcal{R}$ is an abelian subgroup such that $ry \in I$ ($yr \in I$) $\forall r \in \mathcal{R}$, $\forall y \in I$. A (two-sided) ideal is a left and right ideal.

A ring $\mathcal{R}$ is an *integral domain* is it contains no non-zero zero divisors, i.e. $xy \neq 0$ for all non-zero $x, y \in \mathcal{R}$.

If M is a $\mathcal{R}$-module (left or right), then $1 \in \mathcal{R}$ acts trivially on M. An R-module M is of finite type if it is generated by finitely many elements, that is, if there exist $a_1, \ldots, a_n \in M$ such that any $y \in M$ can be written as $y = r_1 a_1 + \ldots + r_n a_n$ for some $r_i \in \mathcal{R}$. If a module A over a ring R also has a ring structure (compatible with that of R in the sense that the map $R \to A$ given by $r \mapsto r \cdot 1_A$ is a ring homomorphism), then A is called an R-algebra. An R-algebra A is of finite type (or finitely generated) if there exist $a_1, \ldots, a_n \in A$ such that any $y \in A$ can be written as a polynomial in $a_1, \ldots, a_n$ with coefficients in R.

An ideal I is called *principal* if it is generated by one element: $I = (y)$.

If $M \neq 0$ and M has exactly two submodules, namely M and 0, then M is a *simple*, or *irreducible*, module. A module which is a direct sum of simple modules is called *semisimple*; and if the simple modules are pairwise isomorphic, it is called *isotypic*.

If a module M has the property that each descending chain

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots$$

must terminate after a finite number of steps, then M satisfies the *descending chain condition*, or d.c.c, and then M is called an *Artinian* module.

The dual concept, using ascending chains of submodules of M, is the *ascending chain condition*, or a.c.c. Modules satisfying the a.c.c. are called *Noetherian*.

If $\mathcal{R}$ is Noetherian as a right $\mathcal{R}$-module, then $\mathcal{R}$ is a *right Noetherian ring*. Similarly, one defines left Noetherian, right Artinian and left Artinian rings. Then a Noetherian, or Artinian, ring is one which has both the right- and left-hand properties.

A ring $\mathcal{R}$ is a right (left) principal ideal domain (PID for short) if all right (left) ideals are principal.

A ring $\mathcal{R}$ is *simple* if it has precisely two ideals, 0 and R.

List of notations.

- $\mathrm{Der}_K(\mathcal{A})$ denotes the space of K-derivation of the algebra $\mathcal{A}$

- $C_n^i = \frac{n(n-1)\dots(n-i+1)}{i!}$
- For any ring $\mathcal{R}$ we denote by $\mathcal{R}[[z]] = \{\sum_{i=0}^{\infty} u_i z^i, u_i \in \mathcal{R}\}$ the *ring of formal power series* with usual multiplication (i.e. z is a formal variable which commutes with all elements from $\mathcal{R}$)
- $\mathcal{R}((z)) = \{\sum_{i=N\in\mathbb{Z}}^{\infty} u_i z^i, u_i \in \mathcal{R}\}$ denotes the *ring of formal Laurent series*
- $\mathrm{End}(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathcal{A})$ denotes the space of all endomorphisms of an algebra (ring) $\mathcal{A}$
- $\mathrm{Aut}(\mathcal{A})$ denotes the group of automorphisms, i.e. invertible endomorphisms
- $\mathcal{R}^*$ denotes the group of units
- $D(\mathcal{R}) = \mathcal{R}[\partial]$ denotes the ring of ordinary differential operators with coefficients in $\mathcal{R}$
- $E(\mathcal{R}) = \mathcal{R}((\partial^{-1}))$ denotes the ring of pseudo-differential operators with coefficients in $\mathcal{R}$
- Affine spectral curve C_0 : see definition 8.12
- Spectral module F : see definition 6.4
- Rank of a ring, $\mathrm{rk}\, B$: see definition 6.1
- Rank of the spectral module (sheaf): see definition 6.5
- (Projective) spectral sheaf $\mathcal{F}$: see definition 8.13
- Projective spectral curve C : see section 10.2
- Projective spectral data: see section 10.2
- True (fake) rank of a ring: see exercise 11.6
- Analytic spectral data: see section 12.1
- Vector BA-function: see definition 12.1

3 Basic constructions

First recall the most important for us basic definitions.

Definition 3.1. Let $\mathcal{A}$ be an algebra over K . A K-*derivation* of $\mathcal{A}$ is a K-linear map $\partial : \mathcal{A} \to \mathcal{A}$ such that the Leibniz rule hold:

$$\partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b), \text{ for any } a, b \in \mathcal{A},$$

where $\cdot$ means the multiplication in the algebra $\mathcal{A}$. For shortness we'll write $a^{(k)}$ instead of $\partial^k(a)$, and we'll omit $\cdot$ in formulas with multiplication of elements. The collection of all K-derivations of $\mathcal{A}$ is denoted by $\mathrm{Der}_K(\mathcal{A})$.

Remark 3.1. If $\mathcal{A}$ has a unit 1, then $\partial(1) = \partial(1^2) = 2\partial(1)$, so that $\partial(1) = 0$. Thus by K-linearity, $\partial(k) = 0$ for all $k \in K$.

$\mathrm{Der}_K(\mathcal{A})$ is a Lie algebra with Lie bracket defined by the commutator:

$$[\partial_1, \partial_2] = \partial_1 \circ \partial_2 - \partial_2 \circ \partial_1$$

(here $\circ$ means the composition; check it!).

Now we want to give a formal definition of the ring of ordinary differential operators. In particular, we'd like to understand the role of this ring in abstract Algebra. we can define this ring in various ways.

The first way is to define it as a skew polynomial ring. Let $\mathcal{R}$ be a ring over K; let $\sigma \in \mathrm{End}(\mathcal{R})$ be a ring endomorphism.

Definition 3.2. A linear map $\delta : \mathcal{R} \to \mathcal{R}$ is called σ-derivation, if

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$$

for any $a, b \in \mathcal{R}$. In particular, $\sigma(1) = 1$ and $\delta(1) = 0$.

Idea: We want to consider polynomials over a ring $\mathcal{R}$ in a variable x which is not assumed to commute with the elements of $\mathcal{R}$. It is desired, however, that each polynomial should be expressible uniquely in the form $\sum x^i a_i$ for some $a_i \in \mathcal{R}$. This applies, of course, to the elements ax. Another desired property is

$$\deg(f(x)g(x)) \leq \deg f(x) + \deg g(x), \tag{1}$$

where the degree function is defined in obvious way (we'll call it also the order function, see definition below). This property implies that

$$ax = x\sigma(a) + \delta(a) \tag{2}$$

for some endomorphism σ and a σ-derivation δ.

Exercise 3.1. Show that the property (1) implies equation (2).

Construction. Given $\mathcal{R}, \sigma, \delta$, consider the ring $E = \mathrm{End}(\mathcal{R}^{\mathbb{N}})$ of abelian group endomorphisms. Note that $\mathcal{R} \hookrightarrow E$, acting by right multiplication. Also there is an element $x \in E$ defined by

$$x(r_i) := (r_i)x := \sigma(r_{i-1}) + \delta(r_i),$$

where $(r_i) = (r_0, r_1, \dots)$ and $r_{-1} = 0$. Consider the subring $S \subset E$ generated by $\mathcal{R}$ and x. One can check that for any $a \in \mathcal{R}$ holds equation (2). Therefore, every element of S can be written in the form $\sum x^i a_i$. Since

$$(1, 0, 0, \dots)(\sum x^i a_i) = (a_i),$$

this expression is unique.

The ring S thus constructed is called a *skew polynomial ring*, and is denoted by $\mathcal{R}[x; \sigma, \delta]$. If $\delta = 0$ this is written as $\mathcal{R}[x; \sigma]$; and, if $\sigma = 1$, as $\mathcal{R}[x; \delta]$.

Remark 3.2. The ring $\mathcal{R}[x; \sigma, \delta]$ can also be described as being the ring T generated freely over $\mathcal{R}$ by an element x subject only to the relation $ax = x\sigma(a) + \delta(a)$ for each $a \in \mathcal{R}$. To see this, note that each element of T can be written in the form $\sum x^i a_i$ and that there is an obvious surjection $T \to \mathcal{R}[x; \sigma, \delta]$ (using the freeness of T). Since the x^i are $\mathcal{R}$-independent in $\mathcal{R}[x, \sigma, \delta]$, they are also $\mathcal{R}$-independent in T. Hence $T \simeq \mathcal{R}[x; \sigma, \delta]$.

Clearly $\mathcal{R}[x; \sigma, \delta]$ has the universal property that if $\psi : R \to S$ is a ring homomorphism, and $y \in S$ has the property that

$$\psi(a)y = y\psi(\sigma(a)) + \psi(\delta(a)) \quad \text{for all } a \in \mathcal{R}$$

then there exists a unique ring homomorphism $\chi : \mathcal{R}[x, \sigma, \delta] \to S$ such that $\chi(x) = y$ and the diagram

$$\begin{array}{ccc} \mathcal{R} & \to & \mathcal{R}[x; \sigma, \delta] \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

is commutative.

Remark 3.3. One could follow an alternative convention, forming a skew polynomial ring using the relation $xa = \sigma(a)x + \delta(a)$ and writing elements in the form $\sum a_i x^i$. In the case when σ is an *automorphism* (and this will be exactly the case in our lectures) the ring obtained is simply $\mathcal{R}[x; \sigma^{-1}, -\delta\sigma^{-1}]$. This alternative convention will be used when convenient.

On each skew polynomial ring there is a natural order (or degree) function:

Definition 3.3. For any non-zero operator $P = \sum_{i=0}^{n} a_i x^i$ of the ring $\mathcal{R}[x; \sigma, \delta]$ we define its *order* to be

$$\mathrm{ord}(P) = n = \max\{k| \quad a_k \neq 0\}.$$

The non-zero coefficient $a_n \in \mathcal{R}$ is called *the highest term* $HT(P)$ of the operator P. Conventionally, $\mathrm{ord}(0) := -\infty$, $HT(0) = 0$.

Definition 3.4. Let $\mathcal{R}$ be a ring over K and let ∂ be a K-derivation. We define the ring of *ordinary differential operators with coefficients in* $\mathcal{R}$ as the skew polynomial ring

$$D(\mathcal{R}) := \mathcal{R}[x; \partial]$$

(it will be denoted also by $\mathcal{R}[\partial]$).

The second way. An alternative way to define the ring of ODOs (which will be useful for our further constructions) is to use the following idea.

If we have a K-algebra $\mathcal{A}$ with a K-derivation ∂, we can consider formal symbols of the form $\sum_{i=0}^{n} u_i \partial^i$ and think of these symbols as acting on elements of $\mathcal{A}$ by multiplication and differentiation: $(u\partial)(f) = u \cdot \partial(f)$. Thus we obtain a big space of K-linear operators acting on $\mathcal{A}$. The Leibniz rule can be considered as an equality of operators:

$$\partial f = f' + f\partial,$$

what motivates the following definition

Definition 3.5. Let $\mathcal{R}$ be a ring over K and let ∂ be a K-derivation. We define the ring of *ordinary differential operators with coefficients in* $\mathcal{R}$ as the set

$$D(\mathcal{R}) := \mathcal{R}[\partial] = \{\sum_{i=0}^{n} u_i \partial^i, \; u_i \in \mathcal{R}\}$$

(which is obviously a linear space over K) with the composition rule

$$\partial^n u = \sum_{i=0}^{n} C_n^i u^{(i)} \partial^{n-i},$$

where $C_n^i = \frac{n(n-1)\dots(n-i+1)}{i!}$, and $u^{(0)} = u$.

Exercise 3.2. Extending the composition rule by linearity we can write down its general form: if $P = \sum_{k=0}^{n} a_k \partial^k$, $Q = \sum_{l=0}^{m} b_l \partial^l$, then

$$PQ = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{0 \leq i \leq k} C_k^i a_k b_l^{(i)} \partial^{k+l-i}. \tag{3}$$

Proposition 3.1. *The space* $\mathcal{R}[\partial]$ *with the composition rule* (3) *is a ring over* K.

Proof. The distributivity of the multiplication can be easily checked directly. Obviously, the multiplicative identity of $\mathcal{R}$ is the multiplicative identity of $\mathcal{R}[\partial]$. So, we need to check only the associativity of the multiplication. We use the following trick (cf. [73, Ch.III,§11]). Let's extend the derivation ∂ on $\mathcal{R}[\partial]$ by setting $\partial(\partial)=0$. Introduce a new derivation δ on $\mathcal{R}[\partial]$ by setting $\delta(a\partial^n)=na\partial^{n-1}$ (check that ∂,δ are derivations). Then for any $P,Q\in\mathcal{R}[\partial]$ we have

$$PQ=\sum_{k=0}^{\infty}\frac{1}{k!}\delta^k(P)*\partial^k(Q), \tag{4}$$

where $*$ means the multiplication of series from the *ring of formal power series* $\mathcal{R}[[z]]$, where z is just replaced by ∂, and $*$ has the effect of bringing all elements from $\mathcal{R}$ to the left and powers of ∂ to the right, e.g. $(a\partial^n)*(b\partial^m)=(ab)\partial^{n+m}$ (in fact, the sum is finite, since P,Q are polynomials in ∂). Indeed, note that it is enough to check this equality for monomials of the form $f\partial^n$, $g\partial^m$. We have

$$f\partial^n g\partial^m=\sum_{k=0}^{n}C_n^k f g^{(k)}\partial^{n-k+m}=\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{n!}{(n-k)!}f\partial^{n-k}\right)*g^{(k)}\partial^m=\sum_{k=0}^{\infty}\frac{1}{k!}\delta^k(f\partial^n)*\partial^k(g\partial^m).$$

Then,

$$\delta(P*Q)=\delta(P)*Q+P*\delta(Q),\quad \partial(P*Q)=\partial(P)*Q+P*\partial(Q)$$

and $\delta\circ\partial=\partial\circ\delta$. Therefore,

$$\delta^k(P*Q)=\sum_{i=0}^{k}C_k^i\delta^i(P)*\delta^{k-i}(Q),\quad \partial^l(P*Q)=\sum_{i=0}^{l}C_l^i\partial^i(P)*\partial^{l-i}(Q)$$

and

$$(PQ)T=\sum_{k,l\geq 0}\frac{1}{l!k!}\delta^l(\delta^k(P)*\partial^k(Q))*\partial^l(T)=\sum_{k,l,p\geq 0}\frac{1}{l!k!}C_l^p\delta^{p+k}(P)*\delta^{l-p}\partial^k(Q)*\partial^l(T)$$

and

$$P(QT)=\sum_{l',k'\geq 0}\frac{1}{k'!l'!}\delta^{k'}(P)*\partial^{k'}(\delta^{l'}(Q)*\partial^{l'}(T))=\sum_{l',k',p'\geq 0}\frac{1}{k'!l'!}C_{k'}^{p'}\delta^{k'}(P)*\partial^{k'-p'}\delta^{l'}(Q)*\partial^{p'+l'}(T).$$

Replacing $p'+l'$ by l, $k'-p'$ by k and p' by p in the second formula, we obtain the first one, since

$$\frac{1}{k'!l'!}C_{k'}^{p'}=\frac{1}{l'!p'!(k'-p')!}=\frac{1}{(l-p)!p!k!}=\frac{1}{l!k!}C_l^p$$

for $p'\leq k'$ and $p\leq l$. Thus, the multiplication is associative and $\mathcal{R}[\partial]$ is a ring. □

Definition 3.6. Let $\mathcal{R}$ be a ring. By a *discrete valuation* on $\mathcal{R}$ we will understand a function v on $\mathcal{R}$ with values in $\mathbb{Z}\cup\infty$ ($\mathbb{Z}\cup\infty$ form a monoid with the operation $y+\infty=\infty+y=\infty$ for all $y\in\mathbb{Z}\cup\infty$) subject to the conditions:

1. $v(y)\in\mathbb{Z}\cup\infty$ and v assumes at least two values,
2. $v(xy)=v(x)+v(y)$,
3. $v(x+y)\geq\min\{v(x),v(y)\}$

The set

$$\ker v = \{y \in \mathcal{R} | v(y) = \infty\}$$

is easily verified to be an ideal of $\mathcal{R}$, which is proper by (3). If $\ker v = 0$, v is said to be *proper*; e.g. on a field every valuation is proper, because 0 is the only proper ideal. If $\mathcal{R}$ is a ring over K we will consider discrete K-valuations, i.e. discrete valuations trivial on K: $v(K) = 0$. In our lectures we'll meet only proper discrete K-valuations. In general it follows easily from the conditions above that $v(1) = v(-1) = 0$ and that $v(-y) = v(y)$ $\forall y \in \mathcal{R}$.

If we have a discrete valuation v, we can define a metric on $\mathcal{R}/\ker v$ by choosing a real constant c between 0 and 1 and defining

$$d(x, y) = c^{v(x-y)}.$$

It is easily verified, using the conditions above, that the usual axioms of a metric hold, and moreover $d(x + a, y + a) = d(x, y)$ (check it). Thus, if v is proper, $\mathcal{R}$ becomes a topological ring with a Hausdorff topology. As with every metric space, one can form the completion of R, which plays an important role in commutative ring theory.

Exercise 3.3. 1) Prove that in fact $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, i.e. every triangle is isosceles. In terms of the original valuation this states that if $v(x + y) > \min\{v(x), v(y)\}$, then $v(x) = v(y)$.

2) An easiest example of a complete discrete valuated ring is the ring $K[[z]]$ of formal power series with a proper valuation defined as $v(u) = n$, if $u = \sum_{i=n}^{\infty} c_i z^i$. Recall the multiplication of two series:

$$(\sum_{i=0}^{\infty} a_i z^i)(\sum_{j=0}^{\infty} b_j z^j) = \sum_{k=0}^{\infty}(\sum_{i+j=k} a_i b_j) z^k$$

Show that $K[[z]]$ is complete. In particular, if $u \in K[[z]]$ is such that $u = c - \tilde{u}$ with $0 \neq c \in K$ and $\tilde{u}(0) = 0$, then the inverse element $u^{-1} = c^{-1}(1 + \sum_{i=1}^{\infty} c^{-i}\tilde{u}^i)$ is well defined.

Comment 3.1. For further reading about the theory of valuations for non-commutative rings see e.g. books [98], [15].

On each ring of ordinary differential operators the order function defines a discrete valuation and the corresponding metric topology on this ring. Below we'll need several additional notions:

Definition 3.7. For any non-zero operator $P = \sum_{i=0}^{n} u_i \partial^i$, $u_n \neq 0$, of the ring $\mathcal{R}[\partial]$ the term $u_n \partial^n$ is called *the highest symbol* $\sigma(P)$ of the operator P.

The operator P is called *monic* if $HT(P) = 1$. It is called *normalized* if it has the form

$$P = \partial^n + u_{n-2}\partial^{n-2} + \ldots + u_0.$$

From the composition rule (3) immediately follows

Lemma 3.1. *Let $\mathcal{R}[\partial]$ be a ring of ODOs. For any non-zero elements $P, Q \in \mathcal{R}[\partial]$ we have*

- $\mathrm{ord}(PQ) \leq \mathrm{ord}(P) + \mathrm{ord}(Q)$, *and the equality holds iff $HT(P)HT(Q) \neq 0$;*
- $HT(PQ) = HT(P)HT(Q)$, *provided $HT(P)HT(Q) \neq 0$.*

Exercise 3.4. Let $\mathcal{R}$ be an integral domain. Show that $\mathcal{R}[\partial]$ is an integral domain and that $-\mathrm{ord}$ is a proper discrete valuation.

4 Basic algebraic properties of the ring $\mathcal{R}[\partial]$

In this lecture we present first some basic properties of the ring $\mathcal{R}[\partial]$ and then we'll proceed to more special properties.

The properties of $\mathcal{R}$ are, of course, reflected in those of $\mathcal{R}[x;\partial]$ as the next result shows. Before this result let's introduce additional definitions (recall that basic definitions are formulated at the beginning of lectures).

Definition 4.1. Let δ be a derivation of $\mathcal{R}$. An ideal $I \subset \mathcal{R}$ is called δ-stable if $\delta(I) \subset I$.

The derivation δ is called *inner* if $\delta = ad(a)$ for some $a \in \mathcal{R}$, where $ad(a)$ is a derivation defined as

$$ad(a)(b) := [b, a] = ba - ab.$$

Theorem 4.1. *Let $S = \mathcal{R}[x;\sigma,\delta]$.*

1. *If σ is injective and $\mathcal{R}$ is an integral domain, then S is an integral domain.*

2. *If σ is injective and $\mathcal{R}$ is a division ring, then S is a principal right ideal domain.*

3. *If σ is an automorphism and $\mathcal{R}$ is right (or left) Noetherian, then S is right (respectively left) Noetherian.*

4. *Let $S = \mathcal{R}[x;\delta]$. Then S is a simple ring if and only if $\mathcal{R}$ has no proper non-zero δ-stable ideals and δ is not an inner derivation of $\mathcal{R}$.*

Proof. 1) If $f = \sum_{i=0}^{n} x^i a_i$ and $g = \sum_{j=0}^{m} x^j b_j$ with a_n, b_m nonzero, then fg has order $n+m$ and leading coefficient $\sigma^m(a_n)b_m$, which is nonzero.

2) If I is a nonzero right ideal containing a nonzero element of order n, then I contains a monic polynomial of order n, i.e. $\sum_{i=0}^{n} x^i b_i$, $b_n = 1$. A division (Euclidean) algorithm shows that I is generated by the monic element of least order belonging to I. Recall the Euclidean algorithm: given elements M and L with (say) $\operatorname{ord} M \geq \operatorname{ord} L$, there are elements Q_i, R_i such that $M = Q_1 L + R_1$, $\operatorname{ord} R_1 < \operatorname{ord} L$, $L = Q_2 R_1 + R_2$, $\operatorname{ord} R_2 < \operatorname{ord} R_1$, and so on. If $R_i \neq 0$, $R_{i+1} = 0$, then R_i is called the *right GCD* of L and M (the left GCD is analogously defined).[1]

3) (This is a variation on a standard proof of the Hilbert basis theorem.) Suppose $\mathcal{R}$ is right Noetherian. Since σ is an automorphism, each element of S can be written in the alternative form $\sum b_i x^i$. If I is a right ideal of S, let I_n be the set of leading coefficients, when written in this form, of elements in I of order $\leq n$. It is clear that I_n is a right ideal of $\mathcal{R}$ (called the n-*th leading right ideal* of I) and that $I_n \subset I_{n+1}$. Furthermore, if I' is a right ideal of S with $I \subseteq I'$ and with $I_n = I'_n$ for each $n \geq 0$, then $I = I'$. (To prove this, suppose otherwise. Choose an element in $I' \backslash I$ of least possible order, m say. Then $I_m \neq I'_m$.)

Now, suppose that $L_0 \subset L_1 \subset \dots$ is an ascending chain of right ideals of S, and denote by L_{in} the n-th leading right ideal of L_i. Consider the array $\{L_{in} | i, n \geq 0\}$. Note first that $L_{ij} \subseteq L_{km}$ whenever $i \leq k$ and $j \leq m$. The ascending chain $\{L_{ii} | i \geq 0\}$ of right ideals of $\mathcal{R}$ stabilizes, say at L_{jj}. For each n with $0 \leq n \leq j-1$ the chain $\{L_{in} | i \geq 0\}$ stabilizes, say at k_n. Choose

$$m = \max\{j, k_0, \dots, k_{j-1}\}.$$

Then, for all $i \geq m$ and all $n \geq 0$, $L_{in} = L_{mn}$. Thus $L_i = L_m$, and so S is right Noetherian. The left Noetherian case is similar.

4) First note that both conditions are necessary: if $\mathcal{R}$ has a proper δ-stable ideal I then $I[x]$ is a proper ideal of S (check it!); if $\delta = ad(a)$, then $ad(x-a)(b) = 0$ for all $b \in \mathcal{R}$, wherefrom $S = \mathcal{R}[x-a;0]$, but this ring is not simple (try to find an ideal).

[1] Thus, $M = M'R_i$, $L = L'R_i$ for some operators M', L'. Note that the GCD is well defined up to left (right) multiplication by a unit in the ring $\mathcal{R}$.

Now assume $0 \neq I$ is an ideal of S. It is easy to check that the ideals I_n are δ-stable. Choose the least n with $I_n \neq 0$. Then $I_n = \mathcal{R}$, since $\mathcal{R}$ has no proper non-zero δ-stable ideals. Note that if $n = 0$, then $I = S$.

If $n > 0$ then there exists a monic element

$$f = x^n + x^{n-1} r_{n-1} + \ldots + r_0 \in I.$$

As it follows from the commutation relations in S,

$$fr - rf = x^{n-1}(r_{n-1}r - rr_{n-1} - n\delta(r)) + \ldots.$$

Since n is minimal, we have $r_{n-1}r - rr_{n-1} - n\delta(r) = 0$, whence δ is an inner derivation, a contradiction. □

Now let's consider more special properties of the ring of ODOs. Of course, they appear if we consider more special types of coefficient rings. Recall that ordinary differential operators appears naturally every time when we study *linear differential equations* $P(\psi) = 0$. In this case coefficients of P and the function ψ are usually assumed to be smooth or analytic in some open neighborhood of 0. Since analytic functions admit the Taylor series expansion in appropriate neighborhoods, it is reasonable to study rings of ODO's with coefficients in the *commutative* ring $R = K[[x]]$ or in its field of fractions $\mathrm{Quot}(R) = K((x))$, with the derivation $\partial = \partial/\partial x$. Let's collect basic algebraic facts about these rings.

Theorem 4.2. *Let $D = K[[x]][\partial]$ be the ring of ODOs with coefficients in the commutative ring $R = K[[x]]$; let $\tilde{D} = K((x))[\partial]$ be the ring of ODOs with coefficients in the field of fractions $\mathrm{Quot}(R)$. Then the following additional properties hold:*

1. *The rings $D, \tilde{D}$ are simple.*
2. *The units of rings D, $\tilde{D}$ (i.e. invertible elements) are the units of R or $\mathrm{Quot}(R)$ correspondingly (the units R^* are just elements $w \in R$ with $w(0) \neq 0$).*
3. *(baby version of the Dixmier conjecture)*

   Let φ be a non-zero ring endomorphism of D. Then this is an automorphism, i.e. $\mathrm{End}(D)\backslash\{0\} = \mathrm{Aut}(D)$. More precisely, there exist $u \in K[[x]]$ satisfying $u(0) = 0$ and $u'(0) \neq 0$, and $v \in K[[x]]$ such that

$$\begin{cases} x & \overset{\varphi}{\mapsto} & u \\ \partial & \overset{\varphi}{\mapsto} & \dfrac{1}{u'}\partial + v. \end{cases} \tag{5}$$

4. *(normalisation)*

   Let $P = u_n\partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0 \in D$, where $u_n(0) \neq 0$. Assume that K contains a root ξ of $u_n(0)$, i.e. that $\xi^n = u_n(0)$.

   Then there exists $\varphi \in \mathrm{Aut}(D)$ such that $\varphi(P)$ is normalized. Moreover, if $Q \in D$ is a normalized differential operator of positive order and ψ is an inner automorphism of D (i.e. is of the form $\psi : y \mapsto w^{-1}yw$, $w \in R^*$) such that $\psi(Q) = Q$, then $\psi = \mathrm{id}$.

Remark 4.1. The condition about the existence of a root is purely formal and not so restrictive for our aims. Usually we'll have a possibility to multiply P by a non-zero constant, so that we can assume that $u_n(0) = 1$.

Remark 4.2. Assume that K is a complete field (e.g. $K=\mathbb{R},\mathbb{C},\mathbb{Q}_p$). Let $w\in K[[x]]$ be a unit. Then for the *inner automorphism* $\mathsf{Ad}_w : D\to D,\ P\mapsto w^{-1}Pw$, we have:

$$\begin{cases} x & \mapsto & x \\ \partial & \mapsto & \partial+\dfrac{w'}{w}. \end{cases}$$

Note that for any $K[[x]]\ni v=\sum\limits_{i=0}^{\infty}\beta_i x^i=\beta_0+\tilde{v}$, the formal power series $w:=\exp(v)=e^{\beta_0}\exp(\tilde{v})$ is well defined and is a unit in $K[[x]]$.

Therefore, any automorphism $\varphi\in\operatorname{Aut}(D)$ satisfying $\varphi(x)=x$ is inner, see (5). Indeed, in this case $\varphi(\partial)=\partial+v$ for some v, and we can always solve the equation $dlog(w)=v$: $w=\exp(\int v)$. Note that we can choose the integral in such a way that $\tilde{v}=\int v$ has zero free term. In this case the exponent $\exp(\tilde{v})$ is defined over any field K of characteristic zero. So, in fact we don't need the assumption about the completeness of the field K for this statement.

Proof. 1) This item follows directly from theorem 4.1, item 4. Namely, since the coefficient rings are commutative in our case, the derivation can not be inner. Moreover, no non-zero proper ideal can be δ-stable. Indeed, if there is such an ideal $I\subset K[[x]]$, take $0\neq f\in I$ and let $f=a_rx^r+h.o.t$. Then $\partial^r(f)=a_rx^r+\dots$ is a unit in $K[[x]]$, i.e. it can not belong to any proper ideal.

2) The description of units follows from lemma 3.1. Indeed, the order of a unit must be zero, i.e. any unit is an element of R or $\operatorname{Quot}(R)$. The description of units in R follows from the observation that elements $u\in R$ with $u(0)=0$ form the (unique) maximal ideal in R, i.e. such elements are not invertible. Other elements are invertible by exercise 3.3.

3) Let $u:=\varphi(x)\in D$. First note that $\operatorname{ord}(u)=0$. For, if $\operatorname{ord}(u)>0$, the image of any infinite series from $K[[x]]$ will not belong to D. By the same reason $u(0)=0$. Let $P:=\varphi(\partial)=a_n\partial^n+a_{n-1}\partial^{n-1}+\dots+a_0\in D$ for some $n\in\mathbb{N}$, where $a_n\neq 0$. Clearly, $[P,u]=nu'a_n\partial^{n-1}+\text{l.o.t}$, hence $[\partial,x]=1=[P,u]$ if and only if $n=1$ and $a_1=\dfrac{1}{u'}$.

Exercise 4.1. Show that (5) is indeed an automorphism.

4) By assumption, a_n is a unit in $K[[x]]$. Therefore, there exists $a\in K[[x]]$ such that $a^n=u_n$. It implies that $P=\big(a\partial\big)^n+\text{l.o.t}$. Hence, there exists a change of variables as in (5) transforming P into an operator of the form $\widetilde{P}:=\partial^n+c_{n-1}\partial^{n-1}+\dots+c_0$. Applying now to $\widetilde{P}$ an automorphism (5) with $u=x$ and $v=-\dfrac{c_{n-1}}{n}$, we get a normalized operator Q. This proves the first statement. The proof of the second statement is straightforward. □

Remark 4.3. The ring D contains another well known ring, called the *first Weyl algebra*: $A_1=K[x][\partial]$. Amazingly the fourth property from theorem 4.2 is still unknown for it. This problem is called the *Dixmier conjecture*: is it true that $\operatorname{End}(A_1)\backslash\{0\}=\operatorname{Aut}(A_1)$? It was the first problem (among six) posed by J. Dixmier in [21].

The same conjecture exists for many variables: the n-th Weyl algebra is defined as $A_n=K[x_1,\dots,x_n][\partial_1,\dots\partial_n]$, where $\partial_i=\partial/\partial x_i$. The conjecture says that every non-zero endomorphism of A_n is an automorphism. Let's denote by DC_n the Dixmier conjecture for the algebra A_n.

The Dixmier conjecture is equivalent to other famous open conjectures: the Jacobian conjecture and the Poisson conjecture. This relation can serve as an illustration of unity in mathematics. Recall the

Jacobian conjectures J_n, $n>1$*:* Any $\phi\in\operatorname{End}_k(K[x_1,\dots,x_n])$ such that $J\phi\in K^*$ is an automorphism, where $J\phi=\det(\partial(\phi(x_i))/\partial x_j)$.

Tsushimoto in [109], [110] and independently Belov-Kanel and Kontsevich in [6] proved the following implications: $DC_n\Rightarrow J_n$, $J_{2n}\Rightarrow DC_n$.

The simple and attractive problem we want to study in our lectures dates back to works of Wallenberg ([115]), Schur ([99]) and Burchnall-Chaundy ([12], [13], [14], [5]). The problem asks to find and classify all non-trivial commutative subrings of $D = K[[x]][\partial]$ in the sense that we are looking for subrings not isomorphic to $K[P]$. Originally this problem was considered for ODOs with analytic coefficients. In this case for each operator P there is a shift of variables $x \mapsto x + \varepsilon$, $\partial \mapsto \partial$ making the highest coefficient of P not vanishing at zero (note that such a shift is not an endomorphism of D, but is an endomorphism of some smaller rings, e.g. of the first Weyl algebra). Therefore, due to theorem 4.2, item 4) the problem reduces to the classification of commutative subrings containing monic operators.

Definition 4.2. A differential operator $P = u_n\partial^n + u_{n-1}\partial^{n-1} + \dots + u_0 \in D$ of positive order n is called *formally elliptic* if $u_n \in K^*$.

Exercise 4.2. Let B be a commutative subring of D containing a formally elliptic element P. Show that *all* elements of B are formally elliptic.

Remark 4.4. According to theorem 4.2, item (4), we can transform P into a normalized formally elliptic differential operator. Therefore, to eliminate redundant degrees of freedom in the problem of classification of commutative subalgebras of differential operators, we can consider commutative subrings which of D are assumed

- contain an elliptic operator of positive order (i.e. being *elliptic*)
- are *normalized*, meaning e.g. that all elements of B of *minimal* positive order are normalized.

Exercise 4.3. If $B \subset D$ is a normalized elliptic subring as in previous remark and φ is an inner automorphism of D such that $\varphi(B) = B$, then $\varphi = \mathrm{id}$.

Problem 4.1. Investigate commutative subrings $B \subset D$ (even in the simplest case $K = \mathbb{C}$) containing an operator $P = u_n\partial^n + \ldots + u_0$ such that $u_n(0) = 0$ and there are no shift $x \mapsto x + \varepsilon$ making u_n not vanishing at zero (i.e. u_n is not analytic in any neighbourhood of zero).

To study basic algebraic properties of *commutative subrings* of differential operators, we need to explain the Schur theory of pseudo-differential operators.

5 Pseudo-differential operators and the Schur theory.

First we define a *complete* ring, called the ring of pseudo-differential operators, which contains the ring of differential operators. The most easy way to do it is to use the trick from the second way of definition of the ring of ODOs.

5.1 Pseudo-differential operators

Definition 5.1. Let $\mathcal{R}$ be a ring over K and let ∂ be a K-derivation. We define the ring of *pseudo-differential operators with coefficients in* $\mathcal{R}$ as the set

$$E(\mathcal{R}) := \mathcal{R}((\partial^{-1})) = \{ \sum_{i=-\infty}^{N} u_i\partial^i, \, u_i \in \mathcal{R}, \; N \in \mathbb{Z} \}$$

(which is obviously a linear space over K) with the composition rule

$$\partial^n u = \sum_{i=0}^{\infty} C_n^i u^{(i)} \partial^{n-i}$$

for all $n \in \mathbb{Z}$.

Exercise 5.1. Extending the composition rule by linearity we can write down its general form: if $P = \sum_{k=-\infty}^{n} a_k \partial^k$, $Q = \sum_{l=-\infty}^{m} b_l \partial^l$, then

$$PQ = \sum_{k=-\infty}^{n} \sum_{l=-\infty}^{m} \sum_{i=0}^{\infty} C_k^i a_k b_l^{(i)} \partial^{k+l-i}. \tag{6}$$

Note that for each $n \in \mathbb{Z}$ the number of terms with $k + l - i = n$ is finite, so the sum is well defined.

Proposition 5.1. *The space* $\mathcal{R}((\partial^{-1}))$ *with the composition rule* (6) *is a ring over* K *.*

Exercise 5.2. Check that the proof of proposition 3.1 works also for this proposition.

Obviously, $D(\mathcal{R}) \subset E(\mathcal{R})$ and the order function can be extended to the ring $E(\mathcal{R})$ just in the same way (see definition 3.7). I particular, the function $(-\operatorname{ord})$ is a proper discrete valuation on $E(\mathcal{R})$ if $\mathcal{R}$ is an integral domain. The notions of *monic* and *normalized* pseudo-differential operators are defined in the same way.

Lemma 5.1. *Let* $\mathcal{R}$ *be an integral domain. Then* $E(\mathcal{R})$ *is a complete ring with respect to the valuation topology defined by the valuation* $v = -\operatorname{ord}$ *.*

Proof. Let $\{P_n \in E(\mathcal{R})\}$, $n \in \mathbb{N}$ be a Cauchy sequence. By definition of the Cauchy sequence for each N there exists $n(N) \in \mathbb{N}$ such that for all $m, k > n(N)$ $v(P_m - P_k) > N$. Consider the sequence $\{-\operatorname{ord}(P_n)\}$. Then we have two possibilities: either it stabilizes (i.e. there exists $n_0 \in \mathbb{N}$ such that $-\operatorname{ord}(P_n) = const$ for all $n \geq n_0$), or not.

In the second case we claim that the limit of the sequence $\{P_n\}$ is zero. Indeed, if 0 is not the limit, there exists N such that for all $n \in \mathbb{N}$ there is $m(n) > n$ such that $v(P_{m(n)}) < N$. But then for all $k > n(N)$ we must have $v(P_{m(n(N))}) = v(P_k)$, i.e. the sequence $\{-\operatorname{ord}(P_n)\}$ stabilizes, a contradiction.

If the sequence $\{-\operatorname{ord}(P_n)\}$ stabilizes, then we can built the limit recursively, by finding the sequence of its coefficients. The order of the limit must be, obviously, equal to $-n_0$. For $N = n_0$ we must have $v(P_m - P_k) > n_0$ for all $k, m > n(N)$ and therefore the operators P_m and P_k must have equal highest coefficients. Thus we take this highest coefficient as the first coefficient of the limit. Taking $N = n_0 + 1$ we obtain by analogous arguments that the operators P_m and P_k have equal coefficients at ∂^{-n_0} and at ∂^{-n_0-1} for all $m, k > n(N)$. The coefficient at ∂^{-n_0-1} is the second coefficient of the limit. Continuing this line of reasoning we'll find all coefficients of the limit operator (we leave to the reader to check that it is indeed the limit). $\square$

Theorem 5.1. *The following results are true.*

1. *The spaces* $E(\mathcal{R})^{\leq i} = \{P \in E(\mathcal{R}) | \operatorname{ord}(P) \leq i\}$ *define a structure of filtered ring on* $E(\mathcal{R})$ *:* $E(\mathcal{R})^{\leq i} E(\mathcal{R})^{\leq j} \subset E(\mathcal{R})^{\leq i+j}$ *.*
2. $E(\mathcal{R})$ *is a graded Lie algebra with respect to the commutator bracket, besides* $[E(\mathcal{R})^{\leq i}, E(\mathcal{R})^{\leq j}] \subset E(\mathcal{R})^{\leq i+j-1}$ *.*
3. *There is a decomposition of the vector space* $E(\mathcal{R})$ *into direct sum of subalgebras* $E(\mathcal{R}) = E(\mathcal{R})^{\leq -1} \oplus D(\mathcal{R})$ *. The projections of an operator* P *onto these subrings are denoted by* P_- *and* P_+ *correspondingly.*
4. *For any monic operator* $P = \partial^d + a_{d-1}\partial^{d-1} + \ldots$ *there exists the inverse operator* $P^{-1} = \partial^{-d} + b_{-d-1}\partial^{-d-1} + \ldots$ *.*
5. *For any monic operator* $P = \partial^d + a_{d-1}\partial^{d-1} + \ldots$ *there exists a unique monic* d *-th root, i.e. a monic operator* $P^{1/d} = \partial + u_0 + u_{-1}\partial^{-1} + \ldots$ *such that* $(P^{1/d})^d = P$ *.*

6. *Assume that the derivation $\partial : \mathcal{R} \to \mathcal{R}$ is surjective. Assume also that the equation $dLog(y) := y^{-1}y' = c$ has a solution in $\mathcal{R}$ for any $c \in \mathcal{R}$ (these properties hold e.g. for $\mathcal{R} = K[[x]]$).*

   Then for every first order operator $L = \partial + u_0 + u_{-1}\partial^{-1} + \ldots$ there exists an invertible zero-th order operator
$$S = s_0 + s_1\partial^{-1} + s_2\partial^{-2} + \ldots$$
   (the Schur operator) such that $S^{-1}LS = \partial$. If $\bar{S}$ is another operator such that $\bar{S}^{-1}L\bar{S} = \partial$, and $\ker(\partial) = K$, then there is an invertible zero-th order operator S_c with constant coefficients such that $\bar{S} = S \cdot S_c$.

Proof. 1) follows from the multiplication law 6. 2) follows from 1). 3) is obvious. 4) If we rewrite the operator P as $P = (1 + a_{d-1}\partial^{-1} + \ldots)\partial^d = (1 - P_0)\partial^d$ (note that $\mathrm{ord}(P_0) < 0$), then $P^{-1} = \partial^{-d}(1 + P_0 + P_0^2 + \ldots)$.

5) We will find the operator $Y = P^{1/d}$ as the limit of a Cauchy sequence. Set $Y_1 = \partial$. Then $Y_1^d = P + O(d-1)$ (here we denote by $O(k)$ elements from $E(\mathcal{R})^{\le k}$). Now let's construct the sequence by induction. Let Y_k be such that $Y_k^d = P + O(d-k)$ and $Y_k - Y_{k-1} = O(-k+2)$. Consider the operator $Y_{k+1} = Y_k + b\partial^{-k+1}$ (here b is unknown coefficient). Then
$$Y_{k+1}^d = Y_k^d + dbY_k^{d-1}\partial^{-k+1} + O(d-k-1) = Y_k^d + db\partial^{d-k} + O(d-k-1).$$
On the other hand, $Y_k^d = P + a\partial^{d-k} + O(d-k-1)$ for some $a \in \mathcal{R}$. Thus, setting $b = -a/d$, we define Y_{k+1}, after that we proceed by induction. Clearly, $\{Y_k\}$ is a Cauchy sequence. So, there is the limit Y by lemma 5.1, and $Y^d = P$.

6) As in 5) we will find the operator S as the limit of a Cauchy sequence, which can be found by induction. Set $S_0 = w$, where w is a solution of the equation $dLog(w) = u_0$ (cf. remark 4.2). Then $S_0^{-1}LS_0$ is a normalized operator. Assume we have found the operator S_k, $k > 0$ such that $S_k^{-1}LS_k = \partial + a\partial^{-k-1} + O(-k-2)$. It is enough to find $\bar{S}_{k+1} = 1 + s_{k+1}\partial^{-k-1}$ such that
$$\bar{S}_{k+1}^{-1}(\partial + a\partial^{-k-1} + O(-k-2))\bar{S}_{k+1} = \partial + O(-k-2).$$
Then the $(k+1)$-th operator from the Cauchy sequence is $S_{k+1} = S_k\bar{S}_{k+1}$. It is easy to check that this sequence is indeed a Cauchy sequence.

Now direct calculations show that
$$\bar{S}_{k+1}^{-1}(\partial + a\partial^{-k-1} + O(-k-2))\bar{S}_{k+1} = \partial + s'_{k+1}\partial^{-k-1} + a\partial^{-k-1} + O(-k-2).$$
Since ∂ gives a surjective map, the equation $s'_{k+1} = -a$ has a solution, and we are done.

If $\bar{S}$ is another operator with these properties, then $S^{-1}\bar{S}\partial = \partial S^{-1}\bar{S} = S^{-1}\bar{S}\partial + \partial(S^{-1}\bar{S})$. Hence, $\partial(S^{-1}\bar{S}) = 0$, i.e. the operator $S_c = S^{-1}\bar{S}$ is a zero-th order operator with constant coefficients. □

Corollary 5.1. *Let $P \in K[[x]][\partial]$ be a monic operator. Denote by B_P the set of operators commuting with P. Then B_P is a commutative ring over K.*

Moreover, there is an embedding $B_P \hookrightarrow K[[z]]$ of the ring B_P into the ring of formal power series $K((z))$.

Proof. Let S be a Schur operator for the operator $L = P^{1/d}$, $d = \mathrm{ord}(P)$ from theorem 5.1, item (6). Then $S^{-1}B_PS$ is a set of pseudo-differential operators commuting with $\partial^{\mathrm{ord}(P)}$.

Exercise 5.3. Show that operators from $S^{-1}B_PS$ commute with ∂.

As we have seen above, each such operator has constant coefficients. But all operators with constant coefficients commute. Thus they form a commutative subring. At last, $S^{-1}B_PS \subset K((\partial^{-1})) \simeq K((z))$. □

Here are some applications of the Schur theory.

Definition 5.2. An operator $S = s_0 + s_1\partial^{-1} + \dots$ is called *admissible* if $S^{-1}\partial S \in K((\partial^{-1}))$.

Exercise 5.4. Show that $S = TS_0$, where $T = ce^{\alpha x}$, where $c, \alpha \in K$ and $S_0 = 1 + s_1\partial^{-1} + \dots$ with $s_i \in K[x]$, $\deg s_i \leq i$.

Problem 5.1. Give a description of monic admissible operators in terms of their coefficients.

Exercise 5.5. Show that if $P \in A_1$, where P is monic, then $B_P \subset A_1$.

Example 5.1. If P is not monic, this is not true. To construct a counterexample let's consider the example of Wallenberg from the beginning of these lectures, example 1.2. Consider the following change of variables (automorphism of D): $z := u(x)$, where $u(x)$ is one of functions from example 1.2. Then $\partial_x = u'(x)\partial_z$. It is well known (and it is easy to check, say for rational or trigonometric function) that there is an equality

$$u'(x)^2 = 2u(x)^3 - g_2u(x) - g_3$$

for some $g_2, g_3 \in \mathbb{C}$ (say, for rational u $g_2 = g_3 = 0$). Then we have

$$\frac{\partial(u'(x)^2)}{\partial z} = 6z^2 - g_2$$

and

$$\partial_x^2 + u(x) = (u'(x)^2)\partial_z^2 + \frac{\partial(u'(x)^2)}{2\partial z}\partial_z + z = (2z^3 - g_2z - g_3)\partial_z^2 + (3z^2 - g_2/2)\partial_z + z \in \mathbb{C}[z][\partial_z]$$

However, $HT(P) = 4u'(x)^3 \notin \mathbb{C}[z]$.

Exercise 5.6. Check whether the m.c. system $\{\partial^k, k \geq 0\}$ satisfies the right Ore condition (see Appendix).

Exercise 5.7. Let $F \in K[X,Y]$, $F(X,Y) = Y^2 - X^{2g+1} - c_{2g}X^{2g} - \dots - c_0$. Assume that $F(P,Q) = 0$, where $P, Q \in A_1$. Prove that $[P,Q] = 0$.

Exercise 5.8 (*)**.** Let $F \in K[X,Y]$, $F(X,Y) = \sum_{i+j \leq N} c_{ij}X^iY^j$ be a polynomial. Assume that $F(P,Q) = \sum c_{ij}P^iQ^j = 0$, where $P, Q \in A_1$. Is it true that $[P,Q] = 0$?

Comment 5.1. This exercise is connected with the following interesting conjecture of Y. Berest, cf. [59].

The group of automorphisms of the first Weyl algebra A_1 acts on the set of solutions of the equation $F(X,Y) = 0$, i.e. if $X, Y \in A_1$ satisfy the equation and $\varphi \in Aut(A_1)$, then $\varphi(X), \varphi(Y)$ also satisfy the equation. The group $Aut(A_1)$ is generated by the following automorphisms

$$\varphi_1(x) = \alpha x + \beta\partial_x, \quad \varphi_1(\partial_x) = \gamma x + \delta\partial_x, \quad \alpha, \beta, \gamma, \delta \in K, \quad \alpha\delta - \beta\gamma = 1,$$
$$\varphi_2(x) = x + P_1(\partial_x), \quad \varphi_2(\partial_x) = \partial_x,$$
$$\varphi_3(x) = x, \quad \varphi_2(\partial_x) = \partial_x + P_2(x),$$

where P_1, P_2 are arbitrary polynomials (see [21]). So, $Aut(A_1)$ consists of tame automorphisms. A natural and important problem is to describe the orbit space of the group action of $Aut(A_1)$ in the set of solutions. If one describes the orbit space it gives a chance to compare $End(A_1)$ and $Aut(A_1)$ ($End(A_1)$ consists of endomorphisms $\varphi : A_1 \to A_1$, i.e. $[\varphi(\partial_x), \varphi(x)] = 1$). Berest has proposed the following interesting conjecture:

If the Riemann surface corresponding to the equation $F = 0$ with generic $c_{ij} \in \mathbb{C}$ has genus $g = 1$ then the orbit space is infinite, and if $g > 1$ then there are only finite number of orbits.

One can prove that if there are finite number of orbits for some equation F then $End(A_1) \backslash \{0\} = Aut(A_1)$.

5.2 Pseudodifferential operators

There are other completions of the ring D. Let's define one such remarkable ring which can be thought of as a simple purely algebraic analogue of the algebra of (analytic) pseudodifferential operators on a manifold (see e.g. [100] for basics of pseudodifferential operators).

Denote $\hat{R} := K[[x]]$. Consider the K-vector space

$$\mathcal{M} := \hat{R}[[\partial]] = \left\{ \sum_{k\geq 0} a_k \partial^k \;\middle|\; a_k \in \hat{R} \text{ for all } k \in \mathbb{N}_0 \right\},$$

Let $\upsilon_x : \hat{R} \to \mathbb{N}_0 \cup \infty$ be the discrete valuation defined by the unique maximal ideal $\mathfrak{m} = (x)$ of $\hat{R}$.

Definition 5.3. For any element $0 \neq P := \sum\limits_{k\geq 0} a_k \partial^k \in \mathcal{M}$ we define its *order* to be

$$\mathbf{ord}(P) := \sup\{k - \upsilon_x(a_k) \mid k \in \mathbb{N}_0\} \in \mathbb{Z} \cup \{\infty\}, \tag{7}$$

and define $\mathbf{ord}(0) := -\infty$. Define

$$\hat{D} := \{Q \in \mathcal{M} \mid \mathbf{ord}(Q) < \infty\}.$$

Let $P \in \hat{D}$. Then we have uniquely determined $\alpha_{k,i} \in K$ such that

$$P = \sum_{k,i\geq 0} \alpha_{k,i}\, x^i \partial^k. \tag{8}$$

For any $m \geq -d = -\mathbf{ord}(P)$ we put:

$$P_m := \sum_{i-k=m} \alpha_{k,i}\, x^i \partial^k$$

to be the m-th *homogeneous component* of P. Note that $\mathbf{ord}(P_m) = -m$ and we have a decomposition $P = \sum\limits_{m=-d}^{\infty} P_m$.

Remark 5.1. Note that for a *differential* operator P with *constant* highest term the order $\mathbf{ord}(P)$ and the usual order coincide.

Definition 5.4. Define the *highest symbol* of $P \in \hat{D}$ as $\sigma(P) := P_{\mathbf{ord}(P)} = P_{-d}$. We say that $P \in \hat{D}$ is *homogeneous* if $P = \sigma(P)$.

Theorem 5.2. *There are the following properties of* $\hat{D}$ *:*

1. $\hat{D}$ *is a ring (with natural operations* $\cdot$ *,* $+$ *coming from* D *);* $\hat{D} \supset D$ *.*

2. $\hat{R}$ *has a natural structure of a left* $\hat{D}$ *-module, which extends its natural structure of a left* D *-module.*

3. *We have a natural isomorphism of* K *-vector spaces*

$$F := \hat{D}/\mathfrak{m}\hat{D} \to K[\partial].$$

4. *Operators from* $\hat{D}$ *can realise arbitrary endomorphisms of the* K *-algebra* $\hat{R}$ *which are continuous in the* $\mathfrak{m}$ *-adic topology.*

5. *There are Dirac delta functions, operators of integration, difference operators.*

Proof. (1) The main point is to show that the natural product $\cdot$ is well–defined for any pair of elements $P, Q \in \hat{D}$. Let $d = \mathbf{ord}(P)$ and $e = \mathbf{ord}(Q)$. Assume first that P and Q are homogeneous. Then we have presentations $P = \sum\limits_{k\geq 0} a_k \partial^k$ and $Q = \sum\limits_{l\geq 0} b_l \partial^l$, where $a_k \in B_{k-d}$ and $b_l \in B_{l-e}$ for any $k, l \geq 0$.
Having the Leibniz formula in mind, we *define*:

$$P \cdot Q := \sum_{k\geq 0} \sum_{l\geq 0} \sum_{0\,\leq\, i\,\leq\, k} C_k^i a_k \frac{\partial^i\, b_l}{\partial x^i} \partial^{k+l-i}. \tag{9}$$

Since for any $j \geq 0$, there exist only finitely many $k, l, i \geq 0$ such that

$$j = k + l - i \quad \text{and} \quad k \geq i, \quad l \geq i$$

the right–hand side of (9) is a well–defined *homogeneous* element of $\hat{D}$. Moreover, $\mathbf{ord}(P \cdot Q) = \mathbf{ord}(P) + \mathbf{ord}(Q)$ provided $P \cdot Q \neq 0$.

Now, let $P, Q \in \hat{D}$ be arbitrary elements and $P = \sum\limits_{m=-d}^{\infty} P_m$ respectively $Q = \sum\limits_{l=-e}^{\infty} Q_l$ be the corresponding homogeneous decompositions. Then we put:

$$P \cdot Q := \sum_{p=-(d+e)}^{\infty} \left(\sum_{\substack{m+l=p \\ m\geq -d \\ l\geq -e}} P_m \cdot Q_l \right). \tag{10}$$

It is a tedious but straightforward computation to verify that $\hat{D}$ is indeed a K–algebra with respect to the introduced operations $\cdot$ and $+$. Note that $\sigma(P \cdot Q) = \sigma(P) \cdot \sigma(Q)$, provided $\sigma(P) \cdot \sigma(Q) \neq 0$.

Exercise 5.9. Verify that $\hat{D}$ is indeed a K–algebra.

(2) In order to define the natural left action of the K–algebra $\hat{D}$ on $\widehat{R}$, take first $P \in \hat{D}$ homogeneous of order $d \in \mathbb{Z}$ and $f \in R_e$ for some $e \in \mathbb{N}_0$. Then we have an expansion $P = \sum\limits_{k\geq 0} a_k \partial^k$ with $a_k \in R_{k-d}$ for any $k \geq 0$. Since $\partial^k \circ f = 0$ for any $k \geq 0$ such that $k \geq e + 1$, we have a well–defined element $P \circ f \in R_{e-d}$.

Now, let $P \in \hat{D}$ and $f \in \widehat{R}$ be arbitrary elements and $d = \mathbf{ord}(P)$. Since we have homogeneous decompositions $P = \sum\limits_{m=-d}^{\infty} P_m$ and $f = \sum\limits_{e=0}^{\infty} f_e$, we can define:

$$P \circ f := \sum_{k=0}^{\infty} \Big(\sum_{\substack{m\geq -d,\, e\geq 0 \\ m+e=k}} P_m \circ f_e \Big).$$

It follows from the definition that $P \circ f \in \mathfrak{m}^k$ provided $f \in \mathfrak{m}^{k+d}$. This shows that the action of $\hat{D}$ on $\widehat{R}$ is indeed continuous in the $\mathfrak{m}$–adic topology. So, we have a natural algebra homomorphism $\hat{D} \to \mathrm{End}_K^{\mathrm{c}}(\widehat{R})$, where $\mathrm{End}_K^{\mathrm{c}}(\widehat{R})$ denotes the algebra of K–linear operators of $\widehat{R}$, which are *continuous* in the $\mathfrak{m}$–adic topology.
It remains to prove that the algebra homomorphism $\hat{D} \to \mathrm{End}_K^{\mathrm{c}}(\widehat{R})$ is injective. For this, it is sufficient to show that for any homogeneous operator

$$P = \sum_{\substack{k,i\,\geq\, 0 \\ k-i=d}} \alpha_{k,i}\, x^i \partial^k \in \hat{D}$$

of order d, there exists $f \in \widehat{R}$ such that $P \circ f \neq 0$. Let l be an element of the set

$$\{k \geq 0 \big| \text{ there exists } i \geq 0 \text{ such that } \alpha_{k,i} \neq 0\}$$

with l smallest possible. Then $P \circ x^l = l!\, \alpha_{l,i}\, x^i \neq 0$, implying the statement.
(3) Note that we have a well–defined injective K–linear map

$$\hat{D}/\mathfrak{m}\hat{D} \to K[[\partial]], \quad P = \sum_{k \geq 0} a_k \partial^k \mapsto P\big|_0 := \sum_{k \geq 0} a_k(0)\partial^k,$$

whose image contains the subspace $K[\partial]$. Let $d = \mathbf{ord}(P)$, then by the definition we have: $v(a_k) \geq k - d$ for any $k \geq 0$. In particular, $a_k(0) = 0$ for any $k \geq 0$ such that $k \geq d+1$, hence $P\big|_0 \in K[\partial]$ as claimed.
(4) Let $\alpha \in \mathrm{End}^c_{K-alg}\, \hat{R}$ be a continuous algebra endomorphism. Then it is defined by the image $\alpha(x) \in \mathfrak{m}$. Put $u = \alpha(x) - x$, and define $\hat{D} \ni P_\alpha = \sum_{i \geq 0} (i!)^{-1} u^i \partial^i$. Then for any $f \in \hat{R}$ we have

$$P_\alpha \circ f(x) = f(x+u);$$

in particular, P_α realize the endomorphism α.

(5) Note that $\delta := \exp((-x) * \partial)$ realize the delta-function:

$$\delta \circ f(x) = f(0);$$

the operator

$$\int := (1 - \exp((-x) * \partial)) \cdot \partial^{-1} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}(-\partial)^k$$

realise the operators of integration

$$\int \circ x^m = \frac{x^{m+1}}{m+1};$$

the ordinary difference operators $\sum_{i=0}^M f_i(n) T^i$ with $T \circ f(n) = f(n+1)$ can be embedded e.g. as follows

$$\sum_{i=0}^M f_i(n) T^i \hookrightarrow \hat{D} \quad \text{via} \quad T \mapsto x, n \mapsto -x\partial.$$

□

Remark 5.2. Unlike the usual ring of PDOs the ring $\hat{D}$ contains zero divisors (e.g. the delta-function δ). There are the following properties of the order function (contained in the proof of theorem):

1. $\mathbf{ord}(P \cdot Q) \leq \mathbf{ord}(P) + \mathbf{ord}(Q)$, and the equality holds if $\sigma(P) \cdot \sigma(Q) \neq 0$,
2. $\sigma(P \cdot Q) = \sigma(P) \cdot \sigma(Q)$, provided $\sigma(P) \cdot \sigma(Q) \neq 0$,
3. $\mathbf{ord}(P + Q) \leq \max\{\mathbf{ord}(P), \mathbf{ord}(Q)\}$.

In particular, the function $-\mathbf{ord}$ determines a discrete pseudo-valuation on the ring $\hat{D}$.

Exercise 5.10. Show that $\hat{D}$ is complete with respect to the topology defined by the pseudo-valuation $-\mathbf{ord}$.

Problem 5.2. Describe all zero divisors in $\hat{D}$.

Comment 5.2. There are other possible ways to define a "symmetric" completion of the ring D (see [119, §2.1.5]). E.g. we can define for each sequence in $\mathfrak{m}D$, $\{(P_n)_{n\in\mathbb{N}}\}$, such that $P_n(R)$ converges uniformly in $\hat{R}$ (i.e. for any $k>0$ there is $N>0$ such that $P_n(\hat{R})\subseteq \mathfrak{m}^k$ for $n\geq N$) a k-linear operator $P:\hat{R}\to\hat{R}$ by

$$P(f)=\varinjlim_{n\to\infty}\sum_{v=0}^{n}P_v(f), \quad P:=\sum_n P_n,$$

and define a completion to be the ring consisting of such operators. This completion is bigger, but $\hat{D}$ has finer properties sufficient for many aims. More details and generalisations see in [11].

Problem 5.3. Classify commutative rings of difference operators. This problem was partially solved in [71] and [41], but it is still open in general.

As we will see, commutative subrings of ODOs can be classified in terms of algebro-geometric *spectral data*, which in particular consist of an algebraic curve and a spectral sheaf (or spectral bundle). These objects are very well known in *algebraic geometry*.

6 Basic algebraic properties of commutative subrings of ODOs and elements of the differential Galois theory

6.1 Basic algebraic properties of commutative subrings

Notation 6.1. Let's denote the filtration on D induced by the filtration $E(K[[x]])^{\leq n}$ by

$$D^{(n)}:=D\cap E(K[[x]])^{\leq n}$$

and for any subring $B\subset D$

$$B^{(n)}:=B\cap D^{(n)}.$$

Proposition 6.1. *Let $B\subset D$ be a commutative subring over K (not necessarily elliptic or normal), containing an operator of positive order. Then B is finitely generated over K.*

Proof. We'll need the following claim.
Claim. Let $N_B=\{\mathrm{ord}\,P| \quad P\in B\}\subset\mathbb{N}\cup\{0\}$. Then there exists a finite subset $F_B\subset\mathbb{N}$ such that $N_B=r(\mathbb{N}\backslash F_B)\cup\{0\}$, where $r=GCD\{\mathrm{ord}\,P| \quad P\in B\}$.

Proof. Since r is a GCD there exist operators $P,Q\in B$ such that $\mathrm{ord}(P)i+\mathrm{ord}(Q)j=r$ for some $i,j\in\mathbb{Z}$.

Exercise 6.1. Prove this statement.

Since $r>0$, i or $j>0$. Without loss of generality let $i>0$. Since $r|\,\mathrm{ord}(P)$ and $r|\,\mathrm{ord}(Q)$ we must have $j\leq 0$. Note that if $j=0$ then $\mathrm{ord}(P)=r$, so $B=K[P]$ and we are done.

So let $j<0$, $\alpha=\mathrm{ord}\,P$, $\beta=\mathrm{ord}\,Q$, $\alpha=\alpha' r$, $\beta=\beta' r$. Obviously, $N_B\subset r\mathbb{N}\cup\{0\}$. Now it suffices to show that $N_B\supset rn$ for any $n\gg 0$.

We claim that $rn\in N_B$ for any $n\geq -j\alpha'\beta'$. Indeed, let $n>-j\alpha'\beta'$. Applying the Euclidean algorithm to $n+j\alpha'\beta'$, we find unique numbers $m\geq 0$, $0\leq l<\alpha'$ such that $n=-j\alpha'\beta'+m\alpha'+l$. Therefore,

$$rn=-rj\alpha'\beta'+m\alpha+l(i\alpha+j\beta)=(m+il)\alpha-(\alpha'-l)j\beta=\mathrm{ord}(P^{m+il}Q^{-(\alpha'-l)j})\in N_B$$

Now $F_B=\{n\in\mathbb{N}| \quad rn\notin N_B\}$, therefore F_B is finite. □

Let's prove that B is finitely generated. Let's denote by $\tilde{F}_B = \{\mathrm{ord}(P)| \quad P \in B^{(-rj\alpha'\beta'-r)}\}$. Then $\tilde{F}_B \cup \{nr| \quad n \ge -j\alpha'\beta'\} = N_B$. Let $s = \sharp\tilde{F}_B$. Choose operators $T_1, \ldots, T_s \in B$ such that $\{\mathrm{ord}(T_1), \ldots \mathrm{ord}(T_s)\} = \tilde{F}_B$. Then we claim that $B = K[P, Q, T_1, \ldots, T_s]$.

Indeed, let $L \in B$, $\mathrm{ord}(L) = t \in N_B$. Then there is an operator $L' \in K[P, Q, T_1, \ldots, T_s]$ such that $\mathrm{ord}\, L' = t$. Let $L = a\partial^t + l.o.t.$, $L' = b\partial^t + l.o.t.$. Then

$$0 = [L, L'] = tab'\partial^{2s-1} - tba'\partial^{2s-1} + l.o.t.$$

Hence $ab' = ba'$, where from $b = ac$, $c \in K$. Then $\mathrm{ord}(L - c^{-1}L') < t$ and $(L - c^{-1}L') \in B$. Repeating the same arguments with this new operator and so on, we will come to an operator of order < 0, i.e. to the zero operator. Hence $L \in K[P, Q, T_1, \ldots, T_s]$, and we are done. □

Now let's prove the second property of a commutative ring of ODOs.

Proposition 6.2. *Let $B \subset D$ be a commutative subring as in proposition 6.1. Then any two non-algebraic over K elements are algebraically dependent.*

Proof. We use notations from the proof of proposition 6.1.

Take any $P \in B$ and choose $Q \in B$ such that $GCD(\mathrm{ord}\, P, \mathrm{ord}\, Q) = r$ (note: such Q exists, because for any $n \gg 0$ $nr \in N_B$, see proposition 6.1). Consider now the ring $\tilde{B} = K[P, Q] \subset B$ and repeat arguments from proposition 6.1. Then $N_{\tilde{B}} \supset rn$, $n \ge -j\alpha'\beta'$. Again note that $j < 0$, for if $j = 0$ then $Q \in K[P]$ (or $P \in K[Q]$), therefore P, Q are algebraically dependent.

Let $\{u_1, \ldots, u_q\}$ be a K-linear basis for $\tilde{B}^{(-rj\alpha'\beta'-r)}$ and let $\varphi_n = P^{m+il}Q^{-(\alpha'-l)j}$ for $n \ge -j\alpha'\beta'$. Then $\{u_1, \ldots, u_q\} \cup \{\varphi_n\}_{n \ge -j\alpha'\beta'}$ form a K-basis for $\tilde{B}$. Since any φ_n is not the power of P, for $N \gg 0$ P^N is a linear combination of $u_1, \ldots, u_q, \varphi_{n_1}, \ldots, \varphi_{n_k}$ (and u_i include restricted powers of P, so that N can be chosen bigger than they together). Thus we get a non-trivial polynomial relation (with the highest coefficient one in variable P):

$$P^N = a_{k_1}P^{k_1} + \ldots + a_0, \quad k_i < N,\ a_{k_i} \in K[Q],$$

and P, Q are algebraically dependent. Note that for any $P_1, P_2 \in B$ there exists $Q \in B$ such that $GCD(\mathrm{ord}\, P_1, \mathrm{ord}\, Q) = r$ and $GCD(\mathrm{ord}\, P_2, \mathrm{ord}\, Q) = r$. Then P_1, Q and P_2, Q are algebraically dependent and by Lemma 14.3 P_1, P_2 are algebraically dependent. □

Corollary 6.1. *Let $B \subset D$ be a commutative subring as in proposition 6.1. Then* $\mathrm{trdeg}(\mathrm{Quot}(B)/K) = 1$.

Proof. Indeed, by corollaries 14.2 and 14.3 any two elements in $\mathrm{Quot}(B)$ are algebraically dependent over K. Obviously, an operator of positive order is transcendental over K. □

Definition 6.1. Let B be a commutative subring of D. We call the natural number

$$r = \mathrm{rk}(B) = \gcd\left\{\mathsf{ord}(P)\middle|P \in B\right\}$$

the (algebraic) *rank* of B.

Remark 6.1. It is interesting to note that $\mathrm{rk}(K[P, Q]) \ne \gcd(\mathrm{ord}(P), \mathrm{ord}(Q))$ in general. See [31] for counterexamples.

Problem 6.1. (S.P. Tsarev) Let D_S be the right quotient ring with respect to $S = C_D(0)$. Let $P, Q \in D_S$ be two commuting elements. Is it true that they are algebraically dependent? Conjecturally, it is not true.

On the other hand, it is true if we replace D by A_1 (Goodearl).

Problem 6.2. (folklore, L. Makar-Limanov) Let $P, Q \in A_1$ be two commuting operators. Assume P is monic (so, Q is formally elliptic) and $K[P,Q] \nsubseteq K[\tilde{P}]$ for any $\tilde{P} \in A_1$. Show that $\gcd(\mathrm{ord}(P), \mathrm{ord}(Q)) > 1$. So, in particular, $\mathrm{rk}(K[P,Q]) > 1$.

Is it true for P, Q not monic?

Remark 6.2. First proof of the fact that two commuting operators are algebraically dependent belongs to Burchnall and Chaundy [12]. Their famous lemma was proved by different method, which gives in particular an explicit form of the equation of the algebraical dependence. Below we present an updated version of their lemma with a proof, cf. [83, Lemma 1.11]. To understand the proof in full generality we need to introduce several basic results from *differential algebra*.

6.2 Elements of the differential Galois theory

In this section we present several very basic notions of the differential Galois theory, following in part the exposition of [54]; see the books [91], [87] for further reading.

Definition 6.2. A commutative ring R with a derivation ∂ is called a *differential ring* (or integral domain, field, etc., respectively). In such a ring R elements a such that $a' = \partial(a) = 0$ are known as *constants* and the set Z of constants comprises a subring of R. If R is a field, Z is a subfield of R.

An ideal I of such a ring R is known as a *differential ideal* if $a \in I$ implies $a' \in I$.

Let $\tilde{K} \subset \tilde{L}$ be fields, with ∂ a derivation on $\tilde{K}$ which extends to $\tilde{L}$. Then $\tilde{L}$ is a *differential extension* of $\tilde{K}$. If $\eta \in \tilde{L}$, then the smallest differential field containing $\tilde{K}, \eta, \eta', \eta'', \ldots$ is denoted by $\tilde{K}\langle\eta\rangle$.

Example 6.1. The rings $R = K[[x]]$ or the ring of smooth functions on an interval with usual derivations are, clearly, differential rings. Their fields of constants are K and $\mathbb{R}$ correspondingly. The field $K((x))$ is a differential field.

Example 6.2. Let (R, ∂) be a differential ring. Let $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$ be distinct indeterminates over R. Put $\partial(x^{(i)}) = x^{(i+1)}$ for all $i \geq 0$. Then ∂ can be extended to a derivation on the polynomial ring $R\{x\} := R[x^{(0)}, x^{(1)}, \ldots]$ in a natural way, and we denote this extension also by ∂.

If $(\tilde{K}, \partial)$ is a differential field then $K\{x\}$ is a differential integral domain, and its derivation extends uniquely to the quotient field $\tilde{K}\langle x\rangle$; its elements are *differential rational functions* of x over $\tilde{K}$.

In particular, we can construct differential extensions by adding formal integrals, i.e. solutions of the equation $\partial(x) = y$, $y \in \tilde{K}$, or formal logarithms, i.e. solutions of the equation $\partial(x) = \partial(y)/y$, $y \in \tilde{K}$. For this we can just take the field $\tilde{K}\langle x\rangle$ and set $x^{(1)} = y$ (or $\partial(y)/y$ correspondingly).

Theorem 6.1. *(Ritt-Kolchin) Assume that the differential field $\tilde{K}$ has characteristic zero and that its field Z of constants is algebraically closed. Then, for any linear differential operator $P = a_n\partial^n + \ldots + a_0$, $n > 0$, there exists n roots $\eta_1, \ldots, \eta_n$ (i.e. $P(\eta_i) = 0$) in a suitable extension of $\tilde{K}$, such that the η_i are linearly independent over Z. Moreover, the field $\tilde{K}\langle\eta_1, \ldots, \eta_n\rangle$ contains no constants not in Z.*

This result is stated and proved in [36] using results from [91] and [35]. The field $\tilde{K}\langle\eta_1, \ldots, \eta_n\rangle$ is known as a *Picard-Vessiot extension* of $\tilde{K}$ (for P).

It follows from theorem 6.1 that if the operators $A, B \in \tilde{K}[\partial]$ have a common factor F of positive order on the right, i.e. $A = \bar{A}F$ and $B = \bar{B}F$, then they have a non-trivial common root in a suitable extension of K. For, by theorem 6.1 F has a root $\eta \neq 0$ in an extension of $\tilde{K}$. We have $A(\eta) = \bar{A}(F(\eta)) = \bar{A}(0) = 0$ and similarly $B(\eta) = 0$.

On the other hand, if A and B have a non-trivial common root η in a suitable extension of $\tilde{K}$, let's show that they have a common right factor of positive order in $\tilde{K}[\partial]$. Let F be a nonzero differential operator of lowest order s.t. $F(\eta)=0$. Then F has a positive order. Since $\tilde{K}[\partial]$ is left-Euclidean, F is unique up to multiplication of non-zero elements of $\tilde{K}$. This F is a right divisor of both A and B. To see this, apply the Euclidean algorithm:

$$A = QF + R,$$

with the order of R less than the order of F, or $R=0$. Apply both sides of this equation to η:

$$A(\eta) = (QF)(\eta) + R(\eta).$$

Since $A(\eta)=0$ and $F(\eta)=0$, $R(\eta)=0$. Therefore, by minimality of F, $R=0$. Hence F is a right divisor of A. We see that F is a right divisor of B similarly. We summarize our result in the following theorem:

Theorem 6.2. *Assume that $\tilde{K}$ has characteristic zero and that its filed of constants is algebraically closed. Let A, B be differential operators of positive orders in $\tilde{K}[\partial]$. Then the following are equivalent:*

(i) A and B have a common non-trivial root in an extension of K,

(ii) A and B have a common factor of positive order on the right in $K[\partial]$.

Remark 6.3. Note that a given operator $P = a_n\partial^n + \ldots + a_0$, $n > 0$ can not have more than n linearly independent over Z roots in any differential extension $\tilde{L}$ of $\tilde{K}$ with the same field of constants. Indeed, assume a converse, and let $\eta_1, \ldots, \eta_{n+1} \in \tilde{L}$ are linearly independent roots over Z. Note that η_1 is also a root of the operator $\partial - \eta_1'/\eta_1$. Then by theorem 6.2 $P = \bar{P}(\partial - \eta_1'/\eta_1)$ with $\operatorname{ord}(\bar{P}) = n-1$.

Note that $\ker(\partial - \eta_1'/\eta_1) = \langle \eta_1 \rangle$. Indeed, if η is a root of this operator, then

$$\frac{(\eta_1\eta^{-1})'}{(\eta_1\eta^{-1})} = \frac{\eta_1'}{\eta_1} - \frac{\eta'}{\eta} = 0,$$

whence $\eta_1\eta^{-1} \in Z$ and $\eta \in \langle \eta_1 \rangle$. Thus, the space $(\partial - \eta_1'/\eta_1)(\langle \eta_1, \ldots, \eta_{n+1} \rangle)$ has dimension n and consists of roots of the operator $\bar{P}$. Continuing this line of reasoning, we will obtain a *two-dimensional space* of roots of a *first order* operator $(\partial - \eta'/\eta)$ for some $\eta \in \tilde{L}$, a contradiction.

Exercise 6.2. Show that the existence of a non-trivial factor of operators A and B is equivalent to the existence of a non-trivial order-bounded linear combination

$$\tilde{C}A + \tilde{G}B = 0, \tag{11}$$

with $\operatorname{ord}(\tilde{C}) < \operatorname{ord}(B)$, $\operatorname{ord}(\tilde{G}) < \operatorname{ord}(A)$ and $(\tilde{C}, \tilde{G}) \neq (0,0)$.

For given $A, B \in \tilde{K}[\partial]$ with $m = \operatorname{ord}(A)$, $n = \operatorname{ord}(B)$, consider the linear map

$$S : \tilde{K}^{m+n} \to \tilde{K}^{m+n}, \quad (c_{n-1}, \ldots, c_0, d_{m-1}, \ldots, d_0) \mapsto \text{coefficients of } \tilde{C}A + \tilde{G}B$$

Obviously the existence of a non-trivial linear combination (11) is equivalent to S having a non-trivial kernel, and therefore to S having determinant 0.

Theorem 6.3. $\det(S) = 0$ *if and only if A and B have a common right factor in $\tilde{K}[\partial]$ of positive order.*

Proof. Suppose $\det(S) = 0$. Then S is not surjective. Since the right GCD of A and B can be written as an order-bounded linear combination of A and B, so it is in the image of S. Therefore, it is not a constant (since otherwise S would be surjective).

On the other hand, if $\det(S) \neq 0$, then S is invertible and surjective. In particular, there exist operators $\tilde{C}, \tilde{G}$ such that $1 = \tilde{C}A + \tilde{G}B$. Therefore, every common right divisor of A and B is a right divisor of 1, i.e. it can not have a positive order. □

Theorem 6.4. *The linear map* S *is given by the matrix whose rows are coefficients of* $\partial^{n-1}A, \dots, \partial A, A, \partial^{m-1}B, \dots, B$ *.*

Proof. Let $v = (c_{n-1}, \dots, c_0, d_{m-1}, \dots, d_0)$. Consider an index i between 1 and n . If $c_{n-i} = 1$, and all the other components of v are 0 , then v is mapped by S to $\partial^{n-i} \cdot A + 0 \cdot B = \partial^{n-i}A$. So, the i -th row of S has to consist of the coefficients of $\partial^{n-i}A$.

Consider an index j between 1 and m . If $d_{m-j} = 1$, and all the other components of v are 0 , then v is mapped by S to $0 \cdot A + \partial^{m-j} \cdot B = \partial^{m-j}B$. So, the $(n+j)$ -th row of S has to consist of the coefficients of $\partial^{m-j}B$. □

Definition 6.3. Let $A, B \in R[\partial]$ be operators of orders $\mathrm{ord}(A) = m$, $\mathrm{ord}(B) = n$, with $m, n > 0$.

By $\partial Syl(A, B)$ we denote the *differential Sylvester matrix*, i.e. the $(m+n) \times (m+n)$ matrix whose rows contain the coefficients of

$$A, \partial A, \dots, \partial^{n-1}A, B, \partial B, \dots, \partial^{m-1}B,$$

i.e.

$$\partial Syl(A,B) = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n} & 0 & 0 & \dots & 0 \\ a_{1,0} & a_{1,1} & \dots & a_{1,n} & a_{1,n+1} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \ddots & & 0 \\ a_{m-1,0} & a_{m-1,1} & \dots & \dots & \dots & \dots & \dots & a_{m-1,n+m-1} \\ b_{0,0} & b_{0,1} & \dots & b_{0,m} & 0 & 0 & \dots & 0 \\ b_{1,0} & b_{1,1} & \dots & b_{1,n} & b_{1,m+1} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \dots & \ddots & & 0 \\ b_{n-1,0} & b_{n-1,1} & \dots & \dots & \dots & \dots & \dots & b_{n-1,n+m-1} \end{pmatrix}$$

The *differential Sylvester resultant* of A and B , $\partial res(A, B)$, is the determinant of $\partial syl(A, B)$.

From theorems 6.2 and 6.3 the following result is immediate.

Theorem 6.5. *Assume that* $\tilde{K}$ *is a differential field of characteristic zero and that its filed of constants is algebraically closed. Let* A, B *be linear differential operators over* $\tilde{K}$ *of positive orders. Then the condition* $\partial res(A, B) = 0$ *is both necessary and sufficient for there to exist a common non-trivial root of* A *and* B *in an extension of* $\tilde{K}$ *.*

Given a differential operator $A = a_n\partial^n + \ldots + a_0$ and its linearly independent roots $\eta_1, \dots, \eta_n \in \tilde{K}$, it is sometimes more convenient to work with the Wronskian matrix: namely, as it is easy to see, the equations $A(\eta_i) = 0$ are equivalent to the matrix equation

$$\begin{pmatrix} \eta_1 & \dots & \eta_n \\ \eta_1' & \dots & \eta_n' \\ \vdots & \dots & \vdots \\ \eta_1^{(n-1)} & \dots & \eta_n^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & & \dots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} \eta_1 & \dots & \eta_n \\ \eta_1' & \dots & \eta_n' \\ \vdots & \dots & \vdots \\ \eta_1^{(n-1)} & \dots & \eta_n^{(n-1)} \end{pmatrix}$$

Note that the matrix

$$\Psi(\eta_1, \dots, \eta_n) := \begin{pmatrix} \eta_1 & \dots & \eta_n \\ \eta_1' & \dots & \eta_n' \\ \vdots & \dots & \vdots \\ \eta_1^{(n-1)} & \dots & \eta_n^{(n-1)} \end{pmatrix}$$

is invertible. Indeed, if we assume the converse, this would mean that the rows are linearly dependent over $\tilde{K}$, i.e. for some $\xi_1, \ldots, \xi_n \in \tilde{K}$ not all vanishing we have $\xi_1 \eta_i^{(n-1)} + \ldots + \xi_n \eta_i = 0$ for $i = 1, \ldots, n$, i. e. $\eta_1, \ldots, \eta_n$ are linearly independent roots of a differential operator of order less than n, a contradiction with remark 6.3. In particular, we see that $\Psi(\eta_1, \ldots, \eta_n)$ is invertible if and only if $\eta_1, \ldots, \eta_n$ are linearly independent over the field of constants Z.

Remark 6.4. Recall that the Wronskian $W(\eta_1, \ldots, \eta_n) = \det \Psi(\eta_1, \ldots, \eta_n)$ can be defined also for any $(n-1)$ times differentiable functions on an interval. Clearly, if $\eta_1, \ldots, \eta_n$ are linearly dependent over the field of constants, then $W(\eta_1, \ldots, \eta_n) = 0$. In general, the converse is not true: as Peano shows already in the 19 century, $W(x^2, |x|x) = 0$, although the functions are not linearly dependent in any neighbourhood of zero. So, the elements from a differential field can be thought of as "smooth" (i.e. infinitely times differentiable) functions.

Let us note that there is a criterion due to Bocher which guarantees the equivalence of linear dependence and vanishing of the Wronskian: if the Wronskian of n functions is equal to zero and n Wronskians of $(n-1)$ of them do not all vanish at any point, then these functions are linearly independent.

6.3 The Burchnall-Chaundy lemma

Now we are ready to prove the Burchnall-Chaundy lemma.

Lemma 6.1. *(Burchnall, Chaundy) Let $P, Q \in D$ be two commuting operators of orders $m = \mathrm{ord}(P)$, $n = \mathrm{ord}(Q)$. Then P, Q are algebraically dependent.*

Moreover, the algebraic relation is given by the equation $\partial res(P - \lambda, Q - \mu) a_n^{-m} = 0$, where a_n is the highest coefficient of $P = a_n \partial^n + \ldots + a_0$, which is a polynomial in λ, μ with coefficients in K of order m w.r.t λ and of order n w.r.t. μ: $\partial res(P - \lambda, Q - \mu) a_n^{-m} = c(\alpha \lambda^m + \ldots \pm \mu^n)$, $c, \alpha \in K^*$.

Proof. We'll consider the operators P, Q as operators from the bigger ring $\tilde{D}$ with coefficients in the differential field $\tilde{K} = \bar{K}((x))$, where $\bar{K}$ is the algebraic closure of K (cf. theorem 4.2). Consider the space $V_\lambda = \{f \in \tilde{L} | P(f) = \lambda f\}$, $\lambda \in \bar{K}$ of roots of the operator $(P - \lambda)$ in some differential extension $\tilde{L}$ of $\tilde{K}$.

Exercise 6.3. Prove that $\dim_{\bar{K}} V_\lambda = m$ (hint: use the Ritt-Kolchin theorem 6.2). If P is monic, prove that $V_\lambda \subset \bar{K}[[x]]$ and $\dim_{\bar{K}} V_\lambda = m$ (even more precisely, $V_\lambda \subset K(\lambda)[[x]]$) without the Ritt-Kolchin theorem.

The space V_λ is invariant with respect to Q (because $[P, Q] = 0$). Therefore, the restriction of Q on V_λ can be represented by a $m \times m$ matrix with entries from $\bar{K}$. For the proof of lemma we'll need to consider the family of such spaces. To play with families it is more convenient to consider λ as a formal variable, add it to the field of constants of $\tilde{K}$, and then consider the space V_λ of roots in some differential extension of this new differential field. Namely, consider the differential field $\tilde{K}(\lambda) := \overline{K(\lambda)}((x))$ with the derivation $\partial = \partial / \partial x$ and the coefficient field $Z = \overline{K(\lambda)} \supset \bar{K}$. Consider a differential extension $\tilde{L}(\lambda)$ of $\tilde{K}(\lambda)$ which contains n linearly independent roots of the operator $(P - \lambda)$. Denote again by $V_\lambda = \{f \in \tilde{L}(\lambda) | P(f) = \lambda f\}$ and notice that, by specifying $\lambda \in \bar{K}$, we'll obtain old spaces V_λ. Again the operator $Q|_{V_\lambda}$ can be represented by a $m \times m$ matrix with entries from $\overline{K(\lambda)}$.

Remark 6.5. One can consider instead an extension "made by hands" as follows. Instead of the field $\tilde{K}(\lambda)$ consider the differential ring $\tilde{K}[[\lambda]] := \bar{K}((x))[[\lambda]]$ with the ring of constants $\bar{K}[[\lambda]]$. Now consider a differential extension field $\tilde{L}$ of $\tilde{K}$ which contains all solutions of the equations $P\eta = 0$, $P\psi_{i,j} = \psi_{i,j-1}$, where $j \geq 1$, $\psi_{i,0} = \eta_i$, the i th basis vector of the space of solutions of $P\eta = 0$.

Then the differential ring $\tilde{L}[[\lambda]]$ contains all solutions of the equation $P\psi = \lambda\psi$. Indeed, we can look for solutions in the form $\psi_i = \psi_{i,0} + \psi_{i,1}\lambda + \dots$, then our equation is equivalent to the system of equations $P\psi_{i,j} = \psi_{i,j-1}$, which is solvable over $\tilde{L}$. Obviously, the ring of constants remains the same.

Note that the matrix $\Psi(\psi_1, \dots, \psi_n)$ is invertible in the ring $\tilde{L}[[\lambda]]$, because the matrix of its zero terms, $\Psi(\eta_1, \dots, \eta_n)$, is invertible (so that the determinant is invertible in $\tilde{L}[[\lambda]]$). Similarly, the matrix $Q|_{V_\lambda}$ has entries from this ring.

Consider the characteristic polynomial $\chi(\lambda, \mu) = \det(Q|_{V_\lambda} - \mu E)$. Recall that by the Gamilton-Cayley theorem, this polynomial vanishes by substituting the operator $Q|_{V_\lambda}$. Note also that $\deg_\mu \chi = n$. Let's prove first that $\chi(\lambda, \mu)$ is a polynomial in λ, μ with coefficients from $\bar{K}$.

Let $V_\lambda = \langle \eta_1, \dots, \eta_n \rangle$, where $\eta_i \in \tilde{L}(\lambda)$. Let $J = (J_i^k)$, $J_i^k \in \overline{K(\lambda)}$ denotes the matrix of $Q|_{V_\lambda}$ in this basis, i.e. $Q(\eta_i) = \sum_{k=1}^n \eta_k J_i^k$. Note that, since $(Q(\eta_i))^{(l)} = \sum_{k=1}^n \eta_k^{(l)} J_i^k$ for any $i = 1, \dots, n$, $l \ge 0$, we have the following matrix equality:

$$Q(\Psi(\eta_1, \dots, \eta_n)) := \begin{pmatrix} Q(\eta_1) & \dots & Q(\eta_n) \\ (Q(\eta_1))' & \dots & (Q(\eta_n))' \\ \vdots & \dots & \vdots \\ (Q(\eta_1))^{(n-1)} & \dots & (Q(\eta_n))^{(n-1)} \end{pmatrix} = \Psi(\eta_1, \dots, \eta_n) \cdot J.$$

Now let's subdivide the differential Sylvester matrix $\partial Syl(P - \lambda, Q)$ into four matrices:

$$\partial Syl(P - \lambda, Q) = \begin{array}{ccl} G_1 & G_2 & \}m \\ G_3 & G_4 & \}n \\ \underbrace{n} & \underbrace{m} & \end{array}$$

and consider the vectors $\eta_{i,n} := (\eta_i, \eta_i', \dots, \eta_i^{(n-1)})^T$, $\eta_{i,m} := (\eta_i^{(n)}, \eta_i^{(n+1)}, \dots, \eta_i^{(n+m-1)})^T$. Note that

$$G_1\eta_{i,n} + G_2\eta_{i,m} = 0 \quad \text{and } G_3\eta_{i,n} + G_4\eta_{i,m} = Q(\Psi(\eta_1, \dots, \eta_n))_i,$$

where $Q(\Psi(\eta_1, \dots, \eta_n))_i$ denotes the i-th column of the matrix $Q(\Psi(\eta_1, \dots, \eta_n))$. In particular,

$$Q(\Psi(\eta_1, \dots, \eta_n)) = (G_3 - G_4 G_2^{-1} G_1)\Psi(\eta_1, \dots, \eta_n).$$

Since $\Psi(\eta_1, \dots, \eta_n)$ is invertible, we have therefore

$$\Psi(\eta_1, \dots, \eta_n)^{-1}(G_3 - G_4 G_2^{-1} G_1)\Psi(\eta_1, \dots, \eta_n) = J, \tag{12}$$

whence $\mathrm{Tr}\, J^i = \mathrm{Tr}(G_3 - G_4 G_2^{-1} G_1)^i$ for any $i \ge 0$. Now note that G_2 is a low-triangular matrix with the element a_n on the diagonal, where a_n is the highest coefficient of $P = a_n\partial^n + \dots + a_0$. So, $\det G_2 = a_n^m$, and G_2^{-1} is a triangular matrix with entries from $\tilde{K}[\lambda]$! Therefore, the matrices $(G_3 - G_4 G_2^{-1} G_1)^i$ have entries also from $\tilde{K}[\lambda]$, in particular, the elements $\mathrm{Tr}(G_3 - G_4 G_2^{-1} G_1)^i$ are also from $\tilde{K}[\lambda]$. Since the entries of the matrix J are from $\overline{K(\lambda)}$, it follows that the coefficients of the polynomials $\mathrm{Tr}(G_3 - G_4 G_2^{-1} G_1)^i$ belong to the field of constants of $\tilde{K}$, i.e. to the field $\bar{K}$. But then the coefficients of the characteristic polynomial $\det(J - \mu E) = \det(Q|_{V_\lambda} - \mu E) = \chi(\lambda, \mu)$ are from $\bar{K}[\lambda]$, i.e. $\chi(\lambda, \mu)$ is a polynomial in λ, μ with coefficients from $\bar{K}$.

Now notice the following matrix identity:

$$\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} 0 & E \\ G_2^{-1} & -G_2^{-1} G_1 \end{pmatrix} = \begin{pmatrix} E & 0 \\ G_4 G_2^{-1} & G_3 - G_4 G_2^{-1} G_1 \end{pmatrix}$$

From this identity it follows that

$$\pm\partial res(P-\lambda,Q)a_n^{-m} = \det(G_3 - G_4G_2^{-1}G_1) = \det J$$

and by analogous arguments

$$\pm\partial res(P-\lambda,Q-\mu)a_n^{-m} = \det(J-\mu E) = \chi(\lambda,\mu),$$

so that $\deg_\lambda \chi = m$ and $\chi(\lambda,\mu) = \alpha\lambda^m + \ldots \pm \mu^n$. At last, note that the coefficients of the polynomial $\partial res(P-\lambda,Q-\mu)a_n^{-m}$ belong to K, because they belong, on the one hand side, to $K((x))$, and on another hand side, to $\bar{K}$. So, $\chi(\lambda,\mu)$ is a polynomial with coefficients from K!

Now consider the equation $\chi(P,Q)(\varphi) = 0$. Note that for any $\lambda \in \bar{K}$ it has a solution $\varphi \in V_\lambda$. Obviously, these solutions are linearly independent for different λ. On the other hand, the differential operator $\chi(P,Q)$ has a finite-dimensional kernel (see the exercise above). Therefore, $\chi(P,Q) = 0$.

Exercise 6.4. Prove that for any irreducible factor $\chi_1(\lambda,\mu)$ of χ we have $\chi_1(P,Q) = 0$.

□

Remark 6.6. We can say more about the polynomial $\chi(\lambda,\mu) = c\cdot\partial res(P-\lambda,Q-\mu)a_n^{-m}$.

First let's summarize all facts from the proof together: first, the pair $\lambda,\mu \in \bar{K}$ satisfies the equation $\chi(\lambda,\mu) = 0$, if and only if λ and μ is a pair of common eigenvalues of the operators P,Q:

$$P(\varphi) = \lambda\varphi, \quad Q(\varphi) = \mu\varphi, \quad \varphi \neq 0.$$

Second, λ and μ is a pair of common eigenvalues of the operators P,Q iff $P-\lambda, Q-\mu$ have a non-trivial common root, which, by theorem 6.5 holds if and only if $\partial res(P-\lambda,Q-\mu) = 0$.

Note that $\bar{K}[P,Q] \subset \bar{K}[[x]][\partial]$ is an integral domain, therefore, $\bar{K}[P,Q] \simeq \bar{K}[T_1,T_2]/\wp$, where $\wp$ is a prime ideal of height 1, see section 14.8, theorem 14.8. Then by theorem 14.14 $\wp$ is principal, say $\wp = (f)$, where $f = f(T_1,T_2)$ is an irreducible polynomial. By exercise 6.4 it follows that any $\chi_1(T_1,T_2) = c_1\cdot f(T_1,T_2)^{q_1}$ for some $c_1 \in \bar{K}$, $q_1 \in \mathbb{N}$, and therefore $\chi(T_1,T_2) = c\cdot f(T_1,T_2)^q$ for some $c \in \bar{K}$, $q \in \mathbb{N}$.

In particular, it follows that $\chi(T_1,T_2)$ is irreducible if $(n,m) = 1$. Note also that the same arguments work over the field K.

Remark 6.7. Another interesting issue is to understand whether there is a basis in the space V_λ where $Q|_{V_\lambda}$ is represented by a matrix with polynomial in λ entries. We'll show it by using a helpful method from basic algebraic geometry (cf. also the next section).

Consider the extension from remark 6.5 and consider the matrix equation (12). It can be rewritten as the equation

$$G = \Psi_0 J \Psi_0^{-1},$$

where G denote a matrix with entries from $\tilde{L}[\lambda]$, and Ψ_0 is a matrix with entries from $\tilde{L}[[\lambda]]$ normalised as $\Psi_0|_{\lambda=0} = E$ (so, $\Psi_0 = \Psi(\eta_1,\ldots,\eta_n)^{-1}\Psi(\psi_1,\ ldots,\psi_n)$ in notations of remark 6.5). Note that Ψ_0^{-1} is also a matrix with entries from $\tilde{L}[[\lambda]]$ normalised in the same way. Since all entries of all matrices on the right hand side are taylor series in λ, the entries of their product are taylor series in λ with coefficients being *polynomials* (with coefficients from $\bar{K}$) in coefficients of the entries of the matrices Ψ_0, Ψ_0^{-1}.

Now denote by k_{ij} the degree $\deg_\lambda(g_{ij})$. Collect all coefficients at λ^k, $k > k_{ij}$ of each i,j th entry of the product $\Psi_0 J \Psi_0^{-1}$. This is a set of polynomials from the polynomial ring in (infinite) number of variables x_i, where the variables are in one to one correspondence with the coefficients at λ^k, $k \geq 0$ of the entries of the matrices Ψ_0, Ψ_0^{-1}.

Note that our matrix equation implies that any ideal I generated by any *finite number* of these polynomials in a polynomial ring over finite number of variables does not contain a unit, because there is a solution of these polynomial equations in the field $\tilde{L}$. Since $\tilde{L} \supset \bar{K}$, there is a solution of these equations also in $\bar{K}$ (just consider any maximal ideal in $\bar{K}[x_1, \ldots, x_n]$ containing $\bar{K}[x_1, \ldots, x_n] \cap I$ and apply theorem 14.10). So, we can find such values of the variables x_i from $\bar{K}$ that the matrix $\Psi_0(x_i) J \Psi_0^{-1}(x_j)$ will have entries from $\bar{K}[\lambda]$ (note that $\Psi_0(x_i)$ is still invertible). Now by changing the basis $\psi_1, \ldots, \psi_n$ with the help of the matrix $\Psi_0(x_i)^{-1}$ we are done.

Remark 6.8. The proof of this lemma can be easily carried over to the case of difference operators (see [41]). In fact, in this case it is even much easier. At the same time, the purely algebraic idea of comparing the dimension growth of subspaces from filtrations from proposition 6.2 can be also carried over to this case, though in a more non-trivial manner (see [71]).

Corollary 6.2. *Let $\tilde{K}$ be a differential field with the algebraically closed field of constants Z of characteristic zero. Let $P, Q \in \tilde{K}[\partial]$ be two commuting operators of positive orders m, n. Then $\partial res(P-\lambda, Q-\mu) a_n^{-m}$, where a_n is the highest coefficient of P, is a polynomial in λ, μ with coefficients from Z.*

Proof. We can just repeat the arguments from the proof of lemma by replacing the fields $\tilde{K}(\lambda), \tilde{L}(\lambda)$, by $\tilde{K} \otimes_Z \overline{Z(\lambda)}$ (see theorem 14.12) and its suitable extension. □

6.4 Spectral module

Definition 6.4. Let $B \subset D$ be a commutative subring. Consider the right D–module $F := D/xD \simeq K[\partial]$, $\overline{a(x)\partial^n} \mapsto a(0)\partial^n$. The *right* action of D on $K[\partial]$ is given by the following rules:

$$\left\{ \begin{array}{lcl} p(\partial) \circ \partial & = & \partial \cdot p(\partial) \\ p(\partial) \circ x & = & p'(\partial). \end{array} \right. \tag{13}$$

Restricting the action (13) on the subalgebra B, we endow F with the structure of a B–module. Since the algebra B is commutative, we shall view F as a *left* B–module (although having the natural right action in mind). The module F is called the *spectral module.*

Note that if B is elliptic, then F is torsion free, i.e. for any non-zero $f \in F$ and for any non-zero $b \in B$ $fb \neq 0$.

Definition 6.5. The rank of the spectral module is the dimension of its localisation at the zero ideal (0) of B:

$$\mathrm{rk}(F) = \dim_{\mathrm{Quot}(B)} F \otimes_B \mathrm{Quot}(B).$$

This number is also called the *analytic rank* of B (the sense of this notion will be clear later).

Theorem 6.6. *Let $B \subset D$ be a commutative subring of rank r. Then the spectral module F is finitely generated and torsion free B-module of rank r.*

Proof. Since r divides $\mathrm{ord}(P)$ for any $P \in B$, it is easy to see that the elements $1, \partial, \ldots, \partial^{r-1}$ of F are linearly independent over B. Let $F^\circ := \langle 1, \partial, \ldots, \partial^{r-1} \rangle_B \subset F$. It is sufficient to prove that the quotient F/F° is finite dimensional over K. Let $\Sigma := \{ d \in \mathbb{Z}_+ \mid \text{there exists } P \in B \text{ with } \mathrm{ord}(P) = d \}$. Obviously, Σ is a sub–semi–group of $r\mathbb{Z}_+$ (as in proposition 6.1). Moreover, one can find $l \in \mathbb{N}$ such that for all $m \geq l$ there exists some element $P_m \in B$ such that $\mathrm{ord}(P_m) = mr$. One can easily prove that F/F° is spanned over K by the classes of $1, \partial, \ldots, \partial^{lr}$, hence F is finitely generated.

Now note: $F \cdot \mathrm{Quot}(B)$ is torsion free over $\mathrm{Quot}(B)$ and there is an obvious embedding

$$(\mathrm{Quot}(B))^{\oplus r} \hookrightarrow F \cdot \mathrm{Quot}(B), \quad (w_1, \ldots, w_r) \mapsto w_1 \cdot 1 + \ldots + w_r \cdot \partial^{r-1}.$$

Exercise 6.5. Show that $F \cdot \mathrm{Quot}(B) (= F \otimes_B \mathrm{Quot}(B)) \simeq (\mathrm{Quot}(B))^{\oplus r}$.

□

7 Reminder of necessary facts and constructions from affine algebraic geometry

Affine algebraic geometry studies the solutions of systems of polynomial equations with coefficients in k (k is any field). Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring in n variables. We can consider elements of A as functions on the *affine space* k^n. Let

$$Z(T) = \{Q \in k^n | \quad f(Q) = 0 \text{ for all } f \in T\}$$

be the set of zeros of a subset $T \subset A$. Instead of a set of polynomials it is better to consider the ideal of the polynomial ring A generated by them. The subsets of k^n consisting of common zeros of the subset of polynomials are called *closed algebraic sets.* They define the *Zariski topology* on k^n.

7.1 Zariski topology

Let us prove some easy facts about closed algebraic sets. If $X \subset k^n$ we denote by $I(X) \subset k[X_1, \ldots, X_n]$ the ideal consisting of polynomials vanishing at all the points of X. It is a tautology that $X \subset Z(I(X))$ and $J \subset I(Z(J))$. If X is a closed algebraic set, then $X = Z(I(X))$ (if $X = Z(J)$, then $I(Z(J)) \supset J$, hence $Z(I(Z(J))) \subset Z(J))$.

It is clear that the function $J \to Z(J)$ reverses inclusions; associates the empty set to the whole ring, and the whole affine space k^n to the zero ideal; sends the sum of (any number of) ideals to the intersection of corresponding closed sets; and sends the intersection $I_1 \cap I_2$ to $Z(I_1) \cup Z(I_2)$.

Because of these properties we can think of closed algebraic sets as the closed sets for some topology on k^n (any intersections and finite unions are again closed, as are the empty set and the whole space). This topology is called *Zariski topology.* In the case when $k = \mathbb{C}$ or $k = \mathbb{R}$ we can compare it with the usual topology on $\mathbb{C}^n$ where closed sets are the zeros of continuous functions. Any Zariski closed set is also closed for the usual topology but not vice versa. Hence the Zariski topology is weaker. Another feature is that any open subset of k^n is dense (its closure is the whole k^n).

Definition 7.1. A closed algebraic subset $X \subset k^n$ is irreducible if there is no decomposition $X = X_1 \cup X_2$, where $X_1 \neq X$ and $X_2 \neq X$ are closed algebraic sets.

Proposition 7.1. *A closed algebraic subset $X \subset k^n$ is irreducible iff $I(X)$ is a prime ideal.*

Any closed set has a unique decomposition into a finite union of irreducible subsets $X = \cup_i X_i$ such that $X_i \nsubseteq X_j$ for $i \neq j$ (these X_i's are called the irreducible components of X).

Proof. Suppose X is irreducible. If $fg \in I(X)$, then $X \subset Z(fg) = Z(f) \cup Z(g)$. Therefore, $X = (X \cap Z(f)) \cup (X \cap Z(g))$. Since X is irreducible, we have either $X = X \cap Z(f)$ and $X \subset Z(f)$ or $X = X \cap Z(g)$ and $X \subset Z(g)$. Therefore, either $f \in I(X)$ or $g \in I(X)$, i.e. the ideal $I(X)$ is prime.

Conversely, let $\wp$ be a prime ideal. Suppose that $Z(\wp) = X_1 \cup X_2$. Then $\wp = I(X_1) \cap I(X_2)$, i.e. either $\wp = I(X_1)$ or $\wp = I(X_2)$. Therefore, $Z(\wp) = X_1$ or $Z(\wp) = X_2$, i.e. $Z(\wp)$ is irreducible.

Now let's prove the existence of the finite decomposition of X. Let Σ be the set of non-empty closed subsets of X which *can not be represented* as a finite union of irreducible closed subsets.

Suppose that Σ is not empty. Since the ring $k[X_1,\ldots,X_n]/I(X)$ is Noetherian, any chain of closed subsets $X \supset Y_1 \supset Y_2 \supset \ldots$ stabilizes, i.e. there exists $r > 0$ such that $Y_r = Y_{r+1} = \ldots$. Therefore, Σ has a minimal element, say Y. The subset Y can not be irreducible by definition of Σ, hence $Y = Y' \cup Y''$, where Y' and Y'' are proper closed subsets in Y. Since Y is minimal, the sets Y', Y'' can be represented as finite union of closed irreducible subsets. Therefore, Y also can be represented in such a way, a contradiction. Thus, $X = X_1 \cup \ldots \cup X_r$, and we can assume w.l.o.g. that $X_i \not\subseteq X_j$ for $i \neq j$ (by deleting proper subsets from the union).

Assume that there exists another representation $X = X'_1 \cup \ldots X'_s$. Then $X'_1 = \cup(X'_1 \cap X_i)$. But since X'_1 is irreducible, we have $X'_1 \subset X_i$ for some i, say $i = 1$. Analogously, $X_1 \subset X'_j$ for some j. Then $X'_1 \subset X'_j$ whence $j = 1$ and $X_1 = X'_1$. Now set $Z = \overline{X - X_1}$. Then $Z = X_2 \cup \ldots X_r = X'_2 \cup \ldots X'_s$. By induction on r we obtain the uniqueness of the decomposition. □

Let k be an algebraically closed field. Let us call an ideal $I \subset k[X_1,\ldots,X_n]$ *radical* if $\sqrt{I} = I$. A corollary of Hilbert's Nullstellensatz is that radical ideals bijectively (via operations I and Z) correspond to closed algebraic sets. The most important class of radical ideals are prime ideals. Again, by Hilbert's Nullstellensatz, these bijectively correspond to irreducible closed algebraic sets. A particular case of prime ideals are maximal ideals, they correspond to points of k^n.

Definition 7.2. An *affine variety* is a closed irreducible algebraic subset of k^n for some n. The variety k^n will be also denoted $\mathbb{A}^n_k$, and called the affine space of dimension n.

Let $X \subset \mathbb{A}^n_k$ be an affine variety. Let $J = I(X)$ be the corresponding prime ideal. Let us denote $k[X] := k[X_1, ..., X_n]/J$. Then $k[X]$ is an integral k-algebra of finite type. $k[X]$ is called the coordinate ring of X. The fraction field of $k[X]$ is denoted by $k(X)$, and is called the function field of X. Its elements are called *rational functions* as opposed to the elements of $k[X]$ which are called *regular functions*.

The function field $k(X)$ is an important object defined by X. Two affine varieties X and Y are called *birationally equivalent* if $k(X) = k(Y)$. A variety X is called rational if $k(X)$ is a purely transcendental extension of k, that is, $k(X) = k(T_1,\ldots,T_l)$.

Zariski topology on $\mathbb{A}^n_k$ induces a topology on a variety $X \subset \mathbb{A}^n_k$. An open subset $U \subset X$ is an intersection of X with an open set of $\mathbb{A}^n_k$. Such sets are called *quasi-affine* varieties.

Definition 7.3. The *dimension* of the topological space X can be defined as the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of distinct closed irreducible subsets of X.

From the Nullstellensatz it follows immediately that $\dim(X) = \dim(k[X])$. By proposition 14.8 the dimension is equal to the transcendence degree of the field $k(X)$.

7.2 Regular functions and morphisms of affine varieties

A morphism of affine varieties $X \to Y$, $X \subset \mathbb{A}^n_k$, $Y \subset \mathbb{A}^m_k$, is given by a function representable by m polynomials in n variables (thus affine varieties form a *category*). The varieties X and Y are called isomorphic if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that fg and gf are identities. The following proposition tells that the category of affine varieties is equivalent to the category of finitely generated integral domains over k.

Proposition 7.2. *Let $X \subset \mathbb{A}^n_k$ and $Y \subset \mathbb{A}^m_k$ be affine algebraic varieties.*

1. *A morphism $f : X \to Y$ defines a homomorphism of k-algebras $f^* : k[Y] \to k[X]$ via the composition of polynomials.*

2. *Any homomorphism of k-algebras $\varphi : k[Y] \to k[X]$ is of the form $\varphi = f^*$ for a unique morphism $f : X \to Y$.*

3. $f : X \to Y$ *is an isomorphism of affine varieties if and only if* $f^* : k[Y] \to k[X]$ *is an isomorphism of* k*-algebras.*

Proof. 1) follows from the fact that the composition of polynomials is a polynomial.

2) Let $x_1, \dots, x_n$ be the coordinates on X, and $t_1, \dots, t_m$ be the coordinates on Y. Let Φ be the composition of the following homomorphisms of rings:

$$k[t_1, \dots, t_m] \to k[Y] = k[t_1, \dots, t_m]/I(Y) \to k[X] = k[x_1, \dots, x_n]/I(X)$$

Let $f_i = \Phi(t_i), i = 1, \dots, m$. The polynomial map $f = (f_1, \dots, f_m)$ maps X to $\mathbb{A}_k^m$. Let $F(t_1, \dots, t_m)$ be a polynomial. Since we consider homomorphisms of rings we have

$$F(f_1, \dots, f_m) = F(\Phi(t_1), \dots, \Phi(t_m)) = \Phi(F(t_1, \dots, t_m)).$$

If $F \in I(Y)$, then $\Phi(F) = 0$. Hence all the polynomials from $I(Y)$ vanish on $f(X)$, that is, $f(X) \subset Z(I(Y)) = Y$.

Finally, $f^* = \varphi$, since these homomorphisms take the same values on the generators t_i of the ring $k[Y]$.

3) follows from (1) and (2).

□

Definition 7.4. A rational function $f \in k(X)$ is called *regular* at a point P of X if $f = g/h$, where $g, h \in k[X]$ and $h(P) \neq 0$. A function is regular on an open set $U \subset X$ if it is regular at every point of U.

The ring of regular functions on an open subset $U \subset X$ is denoted by $k[U]$. Since $k[X] \subset k[U] \subset k(X)$, the fraction field of $k[U]$ is $k(X)$. To a rational function $f \in k(X)$ one associates "the ideal of denominators" $D_f \subset k[X]$ consisting of regular functions h such that $hf \in k[X]$ (check this is an ideal!). The set of all points P where f is regular is $X \backslash Z(D_f)$. Indeed, we can write $f = g/h$, $g, h \in k[X]$, $h(P) \neq 0$, if and only if $P \notin Z(D_f)$. An immediate corollary of the Nullstellensatz says that if $I \subset k[X]$ is an ideal, and $f \in k[X]$ vanishes at all the common zeros of I in X, then $f^s \in I$ for some $s > 0$. (Apply the Nullstellensatz to the pre-image of I in $k[x_1, \dots, x_n]$ under the natural surjective map.) This is a little more general form of the Nullstellensatz.

It can be shown that the open subsets of an affine variety X of the form $h \neq 0$, $h \in k[X]$, form a base of Zariski topology on X. The following lemma gives a connection of these open subsets with the localised ring $k[X]_h$.

Lemma 7.1. *Let* X *be an affine variety. The subset of* $k(X)$ *consisting of functions regular at all the points of* X *is* $k[X]$*. A function is regular on the open subset given by* $h \neq 0$*, for* $h \in k[X]$*, if and only if* $f \in k[X]_h$*, in other words, if* $f = g/h^s$ *for some* $g \in k[X]$ *and* $s > 0$*.*

Proof. Let f be such a function. Then $Z(D_f) = \emptyset$. By corollary 14.4 D_f must be the whole ring, hence contains 1, hence $f \in k[X]$. This proves the fist statement. To prove the second statement we note that $Z(D_f)$ is contained in the closed set given by $h = 0$. By Nullstellensatz if h vanishes on $Z(D_f)$, then a power of h is in D_f. □

At last, for each point $P \in X$ there is a notion of a *stalk*:

Definition 7.5. The stalk $\mathcal{O}_{X,P}$ of the ring of regular functions on the variety X at point P is the set of pairs (U, f), where U is a (Zariski) open subset of X, containing P and f is a regular function on U. Two pairs are said to be equivalent if $f = g$ on $U \cap V$.

Exercise 7.1. Prove that $\mathcal{O}_{X,P}$ is a local ring isomorphic to the ring $k[X]_\wp$, where $\wp$ is the maximal ideal corresponding to P by the corollary from Nullstellensatz.

We finish this section with a very convenient theorem about intersections of affine varieties. To formulate this theorem we need two preliminary statements which we leave as standard exercises (see [32, Ch. 1, §7]).

First let's define the *product* of affine varieties:

Exercise 7.2. Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be affine varieties.

(i) Show that the direct product $X \times Y \subset \mathbb{A}^{n+m}$ with the induced Zariski topology is irreducible. (Hint: Suppose that $X \times Y$ is a union of two closed subsets Z_1, Z_2. Set $X_i = \{x \in X | x \times Y \subset Z_i\}$, $i = 1, 2$. Show that X_i are closed and $X = X_1 \cup X_2$. Then $X = X_1$ or $X = X_2$, whence $X \times Y = Z_1$ or $X \times Y = Z_2$.)

Note that the topology of $X \times Y$ does not coincide with the direct product topology.

(ii) Show that $k[X \times Y] \simeq k[X] \otimes_k k[Y]$

(iii) Show that $X \times Y$ is a *product in the category of affine varieties*, i.e. the projections $X \times Y \to X$, $X \times Y \to Y$ are morphisms and for any given variety Z with morphisms $Z\ rightarrowX$, $Z \to Y$ there exists a unique morphism $Z \to X \times Y$ such that the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ X & \leftarrow & X \times Y & \rightarrow & Y \end{array}$$

is commutative.

(iv) Show that $\dim X \times Y = \dim X + \dim Y$.

Theorem 7.1. *([32, Prop. 7.1]) Let X, Y be affine varieties of dimensions r, s in $\mathbb{A}^n$. Then every irreducible component W of the intersection $X \cap Y$ has dimension $\geq r + s - n$.*

7.3 From algebraic geometry to complex geometry: smooth and singular points of algebraic varieties

If $K = \mathbb{C}$ a natural question arises: when an algebraic variety form a complex manifold? The answer is simple: when algebraic variety is smooth, i.e. all its points are smooth points:

Definition 7.6. Let $X \subset \mathbb{A}^n_K$ is an affine variety and suppose that the ideal of X is generated by polynomials $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$. Let $P = (a_1, \ldots, a_n)$ be a point of X. The variety X is called *non-singular* or *smooth* at $P \in X$ if the rank of the matrix $(\partial f_i / \partial X_j)(P)$ is equal to $n - r$, where r is the dimension of X. X is non-singular or smooth if it is smooth at all its points.

Definition 7.7. A Noetherian local ring R with maximal ideal $\mathcal{M}$ and residue field k is regular if $\dim R = \dim_k(\mathcal{M}/\mathcal{M}^2)$.

Theorem 7.2 (Zariski)**.** *Let $X \subset \mathbb{A}^n_K$ be an affine variety. It is smooth at $P \in X$ if and only if the local ring $\mathcal{O}_{X,P}$ ($\simeq K[X]_\wp$) is regular.*

Proof. Let $P = (a_1, \ldots, a_n)$ be coordinates of P in $\mathbb{A}^n_K$ and let $\wp = (X_1 - a_1, \ldots, X_n - a_n)$ be the corresponding maximal ideal in $A = K[X_1, \ldots, X_n]$. Define a linear map $\theta : A \to K^n$ by setting

$$\theta(f) = \left\langle \frac{\partial f}{\partial X_1}(P), \ldots, \frac{\partial f}{\partial X_n}(P) \right\rangle, \quad f \in A.$$

Clearly, the elements $\theta(x_i - a_i)$, $i = 1, \ldots, n$ form a basis in K^n and $\theta(\wp^2) = 0$. Therefore, θ induces an isomorphism $\theta' : \wp/\wp^2 \simeq K^n$.

Let $I = I(X)$ and $f_1, \ldots, f_t$ be its generators. Then the rank of the jacobian matrix $J = (\partial f_i / \partial X_i)(P))$ is equal to the dimension of the subspace $\theta(I)$ and (since θ' is an isomorphism) to the dimension of the subspace $(I+\wp^2)/\wp^2$ in $\wp/\wp^2$. On the other hand, the local ring $\mathcal{O}_{X,P}$ is isomorphic to $(A/I)_\wp$. Therefore, if $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,P}$, then we have an isomorphism

$$\mathcal{M}/\mathcal{M}^2 \simeq \wp/(I+\wp^2).$$

Calculating the dimensions of vector spaces we get the equality

$$\dim \mathcal{M}/\mathcal{M}^2 + \operatorname{rk} J = n$$

Now assume $\dim X = r$. Then the local ring $\mathcal{O}_{X,P}$ also has dimension r (as $\dim \mathcal{O}_{X,P} = \operatorname{ht}\mathcal{M}$, $\mathcal{O}_{X,P} \simeq K[X]_\wp$, $K[X]_\wp/\mathcal{M} \simeq K[X]/\wp \simeq K$ and $\operatorname{ht}\mathcal{M}+\dim K = \dim K[X] = r$ by proposition 14.8). Therefore, $\mathcal{O}_{X,P}$ is regular iff $\dim_K \mathcal{M}/\mathcal{M}^2 = r$. But this is equivalent to the equality $\operatorname{rk} J = n-r$, the definition of a smooth point P. □

Exercise 7.3. Let $X = Z(y^2-x^3) \subset \mathbb{A}^2$. Show that $K[X] \simeq K[z^2,z^3]$ and that the point $(0,0)$ is singular on X. X is curve with a simplest *cuspidal* singularity.

Proposition 7.3. *If $X \subset \mathbb{A}^n$ is a smooth variety, then $I(X)$ is locally generated by $n-r$ functions.*

In particular, by the implicit function theorem, X is a complex submanifold in $\mathbb{A}^n_K$, i.e. X is a complex manifold.

Proof. Assume that $I(X) = (f_1,\ldots,f_k) \subset K[X_1,\ldots,X_n]$. Let $P \in X$ be a smooth point. Let $\wp$ be the corresponding maximal ideal in the ring $K[X]$ and $\mathcal{M}_P$ be the corresponding maximal ideal in the ring $K[X_1,\ldots,X_n]$. By Zariski theorem 7.2 we have $d = \dim X = \dim K[X] = \dim_K \wp/\wp^2 = n - \operatorname{rk}(\partial f_i/\partial X_j)(P)$. Without loss of generality we can assume that $f_1,\ldots,f_{n-d}$ form linearly independent rows in the matrix $(\partial f_i/\partial X_j)(P)$.

We claim that the ideals (f_1), $(f_1,f_2),\ldots,(f_1,\ldots,f_{n-r})$ form a chain of prime ideals in the local ring $K[X_1,\ldots,X_n]_{\mathcal{M}_P}$.

To prove this, first note that the point P is a smooth point of $\mathbb{A}^n$, as it follows from theorem 14.10. Then by the Zariski theorem the ring $K[X_1,\ldots,X_n]_{\mathcal{M}_P}$ is a regular local ring, hence it is factorial. Note that the image of the element f_1 in the ring $K[X_1,\ldots,X_n]_{\mathcal{M}_P}$ is irreducible (we'll denote this image by the same letter). Indeed, if f_1 would be a product of two elements from the maximal ideal, then all its derivatives $\partial f_1/\partial X_i(P)$ will be equal to zero, a contradiction. So, f_1 generates a *prime* ideal in the ring $K[X_1,\ldots,X_n]_{\mathcal{M}_P}$.

Now note that the ring $K[X_1,\ldots,X_n]_{\mathcal{M}_P}/(f_1)$ is again a regular local ring. To show this we'll use again the Zariski theorem. First, note that f_1 generates a prime ideal in some *finitely generated* ring $K[X_1,\ldots X_n]_h$, where h can be taken to be a product of those irreducible factors of the polynomial f_1 which does not belong to the maximal ideal $\mathcal{M}_P$. Note that $K[X_1,\ldots X_n]_h \simeq K[X_1,\ldots X_n,T]/(T\cdot h-1)$, hence the ideal $(f_1, T\cdot h-1) \subset K[X_1,\ldots X_n,T]$ is prime. The Jacoby matrix of polynomials $f_1, T\cdot h-1$ has rank two at the point $\bar{P} := (P, 1/h(P))$. Thus, this point is a smooth point of the variety $Z(f_1, T\cdot h-1) \subset \mathbb{A}^{n+1}$. Then by the Zariski theorem the local ring $(K[X_1,\ldots X_n,T]/(f_1,T\cdot h-1))_{\bar{\mathcal{M}}}$ is regular, where $\bar{\mathcal{M}}$ denotes the corresponding maximal ideal of the point $\bar{P}$. But

$$(K[X_1,\ldots X_n,T]/(f_1,T\cdot h-1))_{\bar{\mathcal{M}}} \simeq (K[X_1,\ldots X_n]_h/(f_1))_{\mathcal{M}_P} \simeq K[X_1,\ldots,X_n]_{\mathcal{M}_P}/(f_1).$$

Now we can proceed by induction: if $K[X_1,\ldots,X_n]_{\mathcal{M}_P}/(f_1,\ldots,f_{j-1})$ is a regular local ring, then the element f_j (the image of the polynomial f_j in this ring) is irreducible. For, if it is a product of two elements from the maximal ideal, then the polynomial f_j would be equal to a product of two elements from the maximal ideal $\mathcal{M}_P$ *modulo* the ideal $(f_1,\ldots,f_{j-1})$. But then the row of derivatives $\partial f_j/\partial X_i(P)$ would belong to a linear subspace generated by the rows $\partial f_k/\partial X_i(P)$, $k<j$, a contradiction with our assumption. Now by the same arguments

as above it defines a prime ideal in the ring $K[X_1,\dots,X_n]_h/(f_1,\dots,f_{j-1})$, and the variety $Z(f_1,\dots,f_j,T\cdot h-1)$ is smooth at the point $\bar{P}$.

Now from our claim it follows that $I(X)=(f_1,\dots,f_{n-r})$ in the ring $K[X_1,\dots,X_n]_{\mathcal{M}_P}$, because $\operatorname{ht} I(X)=n-r$, $\operatorname{ht}(f_1,\dots,f_{n-r})=n-r$, $I(X)\supset(f_1,\dots,f_{n-r})$ and the ideal $(f_1,\dots,f_{n-r})$ is prime.

At last, note that in fact we have shown that the ideal $(f_1,\dots,f_{n-r})$ is prime in some finitely generated ring $K[X_1,\dots X_n]_h$, i.e. $I(X)$ is locally generated by $(n-r)$ elements in some *affine* neighbourhood of P. □

Remark 7.1. Note that all arguments from the proof of the Zariski theorem remain valid even if K is not algebraically closed, but P is a smooth K-point, i.e. it has coordinates over K and the rank of the matrix $(\partial f_i/\partial x_j)(P)$ is equal to $n-r$.

Using the trick from the claim described in the proof of proposition 7.3 and standard results about integral extensions one can prove more the following generalisation of theorem 14.10.

Proposition 7.4. *Let $\wp$ be a maximal ideal of the ring $K[X_1,\dots,X_n]$, where K is not necessarily algebraically closed. Then the local ring $K[X_1,\dots,X_n]_\wp$ is regular.*

Proof. Let $\bar{K}$ be an algebraic closure of the field K. Then the ring $\bar{K}[X_1,\dots,X_n]$ is integral over $K[X_1,\dots,X_n]$ (see proposition 14.5). Let $\wp=(f_1,\dots,f_k)\subset K[X_1,\dots,X_n]$. From the Krull theorem 14.15 it follows that $k\geq n$, because $\operatorname{ht}\wp=n$ by proposition 14.8.

We claim that we can find n elements $f_1,\dots,f_n\in\wp$ such that the ideals (f_1), $(f_1,f_2),\dots,(f_1,\dots,f_n)$ form a chain of prime ideals in the local ring $K[X_1,\dots,X_n]_\wp$ and moreover, the matrix $(\partial f_i/\partial X_j)(\wp)$ has rank n, where $\partial f_i/\partial X_j(\wp)$ means the residue of the polynomial $\partial f_i/\partial X_j$ modulo $\wp$. Since $\operatorname{ht}\wp=n$ by proposition 14.8, it means that $\wp_\wp=(f_1,\dots,f_n)$ in the ring $K[X_1,\dots,X_n]_\wp$ (moreover, $\wp=(f_1,\dots,f_n)$ in the ring $K[X_1,\dots,X_n]$) and therefore the ring $K[X_1,\dots,X_n]_\wp$ is regular.

This can be shown by induction. By proposition 14.6 there exists a maximal ideal $\tilde{\wp}\subset\bar{K}[X_1,\dots,X_n]$ such that $\tilde{\wp}\cap K[X_1,\dots,X_n]=\wp$. By theorem 14.10 $\tilde{\wp}=(X_1-\alpha_1,\dots,X_n-\alpha_n)$ for some $\alpha_1,\dots,\alpha_n\in\bar{K}$.

For each α_i there is a minimal polynomial $F_i(X)\in K[X]$ such that α_i is a root of this polynomial. Note that $F_i(X_i)\in\tilde{\wp}\cap K[X_1,\dots,X_n]=\wp$ for all $1\leq i\leq n$. Consider the polynomial $F_1(X_1)$. Since F_1 is minimal, it is irreducible in the ring $K[X_1,\dots,X_n]$, and therefore generates a prime ideal of height one. Moreover, $\partial F_1/\partial X_1(\wp)\neq 0$ (as $\partial F_1/\partial X_1(\tilde{\wp})\neq 0$). Note that $K[X_1,\dots,X_n]/(F_1(X_1))\simeq L[X_2,\dots,X_n]$, where L is some extension field of K (in particular, it contains α_1). Thus, in particular, this ring is factorial.

The image of the polynomial $F_2(X_2)$ in this ring (we'll denote it by the same letter) is obviously not equal to zero. Let $\tilde{F}_2(X_2)$ be an irreducible factor of F_2 in the ring $L[X_2,\dots,X_n]$. Then again $\partial\tilde{F}_2/\partial X_2(\wp)\neq 0$ and moreover, the matrix $(\partial\tilde{F}_2/\partial X_j,\partial F_1/\partial X_j)(\wp)_{j\leq 2}$, where $\tilde{F}_2$ now denotes a preimage of $\tilde{F}_2$ in $K[X_1,\dots,X_n]$, has rank two (as the polynomials F_i depend on different variables). Besides, $\tilde{F}_2(X_2)$ generates a prime ideal of height one in the ring $L[X_2,\dots,X_n]$. Continuing this line of reasoning, we'll find n elements as desired (prime factors of the images of polynomials $F_i(X_i)$).

Note that, in particular, we obtain that the map $\theta':\wp/\wp^2\to K(\wp)^n$ defined in the same way as in the proof of the Zariski theorem, is an isomorphism of $K(\wp)$-vector spaces, where $K(\wp)=K[X_1,\dots,X_n]/\wp$ is the residue field. □

Remark 7.2. Note that all arguments from the proof of the Zariski theorem remain valid even if K is not algebraically closed, and P is not a smooth K-point (cf. remark 7.1), but just a smooth point in the following sense. Let's say that the maximal ideal $\wp\subset K[x]=K[X_1,\dots,X_n]/(f_1,\dots,f_k)$ defines a smooth point (cf. the next section) if the rank of the matrix $(\partial f_i/\partial X_j)(P)$ is equal to $n-r$, where r is the dimension of X.

Now the Zariski theorem should sound as follows. Let $X \subset \mathbb{A}^n_K$ be an affine variety. It is smooth at $\wp \in K[X]$ if and only if the local ring $K[X]_\wp$ is regular.

Indeed, as we already know from proposition 7.4, the corresponding maximal ideal in the ring $K[X_1, \ldots, X_n]$ defines a smooth point in the affine space $\mathbb{A}^n_K$, and the map θ' is an isomorphism. All other arguments remain the same.

Exercise 7.4. Let $X \subset \mathbb{A}^n$ be an affine variety defined over a not algebraically closed field K, i.e. it is defined by the prime ideal $I(X) = (f_1, \ldots, f_k) \subset K[X_1, \ldots, X_n]$ or $I(X) = (f_1, \ldots, f_k) \subset \bar{K}[X_1, \ldots, X_n]$, but $f_1, \ldots, f_k \in K[X_1, \ldots, X_n]$. Let $P \in X$ be a smooth K-point. Extend the proof of proposition 7.3 to this case: show that $I(X)$ is locally generated by $(n-r)$ elements.

8 Necessary facts and constructions about sheaves and schemes

In this chapter we recall basic definitions and constructions from the theory of sheaves and schemes. The main reference for this section is [32, Ch.2, §1,2, 5]. We recommend to continue to study more about sheaves and schemes from this book.

Definition 8.1. A sheaf (of abelian groups or rings of modules, etc.) on algebraic variety (or more generally, on a topological space X) consists of

a) an abelian group (or a ring or a module) $\mathcal{F}(U)$ (its elements are called *sections* of the sheaf over U) for any open subset $U \subset X$,

b) a homomorphism of abelian groups (rings, modules) $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ (called restriction maps) for any open $V \subset U \subset X$ such that

1. $\mathcal{F}(\emptyset) = 0$
2. $\rho_{U,U} = id$
3. if $W \subset V \subset U$, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$
4. if $U = \cup_i V_i$ and $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0$ for any i, then $s = 0$
5. if for any i there are given sections $s_i \in \mathcal{F}(V_i)$ such that $\rho_{V_i, V_i \cap V_j}(s_i) = \rho_{V_j, V_i \cap V_j}(s_j)$ for any i, j, then there exists a section $s \in \mathcal{F}(U)$ (which is unique by item 4) such that $\rho_{U,V_i}(s) = s_i$ for any i.

A collection $\mathcal{F}(U)$ together with the homomorphisms $\rho_{U,V}$ which satisfy conditions 1-3 is called a *pre-sheaf.*

A *morphism* of sheaves is a collection of groups (rings, modules) homomorphisms $\mathcal{F}(U) \to \mathcal{F}'(U)$ compatible with the restriction maps.

Example 8.1. 1) If we set $\mathcal{F}(U) = K[U]$ for any open U of an algebraic variety X, we get a *sheaf of rings* called the *structure sheaf* and denoted by $\mathcal{O}_X$.

2) If $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module for any open U of a variety X and the maps $\rho_{U,V}$ are compatible with the module structure, then $\mathcal{F}$ is called *a sheaf of* $\mathcal{O}_X$ *-modules.*

Definition 8.2. For any pre-sheaf we can define the notion of a *stalk* analogous to the notion of the stalk of the ring of regular functions:

$$\mathcal{F}_{X,p} = \{(f, U) | \quad f \in \mathcal{F}(U),\, U\text{-open}, p \in U\} / \sim,$$

where two pairs are equivalent $(f, U) \sim (g, V)$, if there exists an open subset $p \in W \subset U \cap V$ such that $\rho_{U,W}(f) = \rho_{V,W}(g)$.

The stalk of a sheaf is defined in the same way, i.e. the stalk of the sheaf is just the stalk of its pre-sheaf.

8.1 Basic sheaves constructions

The most basic construction which is used (with a few variations) in many other sheaves constructions is the construction of the sheaf associated to a pre-sheaf.

Definition 8.3. For any pre-sheaf $\mathcal{F}$ there exists a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying the following property: for any sheaf $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. The pair $(\mathcal{F}^+, \theta)$ is unique up to a uniquely defined isomorphism. The sheaf $\mathcal{F}^+$ is called *the sheaf associated to a pre-sheaf* $\mathcal{F}$.

The sheaf $\mathcal{F}^+$ is defined as follows:

$$\begin{aligned} \mathcal{F}^+(U) = \{\text{set of maps} \quad & s : U \to \cup_{P\in U}\mathcal{F}_P \quad \text{s.t. } s(P) \in \mathcal{F}_P \ \forall P \in U \text{ and } \forall P \in U \\ & \exists V \subset U,\, V \ni P \text{ and } \exists t \in \mathcal{F}(V) \text{ s.t. } s(q) = t_q \in \mathcal{F}_q \ \forall q \in V\} \end{aligned} \tag{14}$$

(here t_q denotes the stalk of the element t at q), and the morphism $\theta(U)$ sends an element $f \in \mathcal{F}(U)$ to the map $s : P \mapsto f_P \in \mathcal{F}_P$.

Exercise 8.1. Check that $\mathcal{F}^+$ is a sheaf, θ is a morphism of sheaves and that the property from definition holds.

Definition 8.4. Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ we define the *direct image* $f_*\mathcal{F}$ as $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for every open subset $V \subset Y$.

For any sheaf $\mathcal{G}$ on Y we define its *inverse image* $f^{-1}\mathcal{G}$ on X as the sheaf associated to to a pre-sheaf $U \mapsto \varinjlim_{V\supset f(U)} \mathcal{G}(V)$, where U is an open subset of X.

Definition 8.5. Let $Z \subset X$ be a subset of the topological space X endowed with the induced topology, $i : Z \to X$ be an embedding and $\mathcal{F}$ be a sheaf on X. We define the *restriction* of $\mathcal{F}$ on Z as $\mathcal{F}|_Z := i^{-1}\mathcal{F}$.

Note that $(\mathcal{F}|_Z)_P = \mathcal{F}_P$ for any $P \in Z$.

Definition 8.6. A *subsheaf* $\mathcal{F}'$ of the sheaf $\mathcal{F}$ is a sheaf such that $\mathcal{F}'(U)$ is a subgroup (subring, submodule, etc.) of the group $\mathcal{F}(U)$ for any open $U \subset X$ and the restriction maps of the sheaf $\mathcal{F}'$ are induced by the restriction maps of the sheaf $\mathcal{F}$.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves , we define the *kernel* of φ as a sheaf $(\ker\varphi)(U) = \ker(\varphi(U))$. Obviously, this is a subsheaf of $\mathcal{F}$.

The *image* of φ is a sheaf associated to a pre-sheaf $im(\varphi)(U) = im(\varphi(U))$. By the universal property of the sheaf associated to a pre-sheaf there is a natural morphism $i : im(\varphi) \to \mathcal{G}$ which is injective, i.e. $im(\varphi)$ gives a subsheaf of $\mathcal{G}$.

The morphism φ of sheaves is called *injective* if $\ker\varphi = 0$; it is called *surjective* if $im(\varphi) = \mathcal{G}$.

If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ then the *factor sheaf* $\mathcal{F}/\mathcal{F}'$ is the sheaf associated to a pre-sheaf $U \mapsto \mathcal{F}(U)/\mathcal{F}'(U)$. Note that $(\mathcal{F}/\mathcal{F}')_P \simeq \mathcal{F}_P/\mathcal{F}'_P$.

Now let X be a topological space endowed with a sheaf of rings $\mathcal{O}_X$. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules on X. We'll denote by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the group of morphisms of sheaves. Note that all constructions described above will give again sheaves of $\mathcal{O}_X$-modules.

Definition 8.7. We define the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ as

$$U \mapsto \underset{\mathcal{O}_X|_U}{\mathcal{H}\mathrm{om}}(\mathcal{F}, \mathcal{G}).$$

We define the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ as the sheaf associated to a pre-sheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Exercise 8.2. Check that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is indeed a sheaf of $\mathcal{O}_X$-modules.

Exercise 8.3. Show that the pre-sheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

is not necessary a sheaf.

Next construction we'll need is a construction of a pull-back of a sheaf of modules. Let $f : X \to Y$ be a morphism of affine varieties (or, more generally, of ringed spaces, see below). In this case by proposition 7.2 we have for any open $U \subset Y$ the induced homomorphisms of rings $f_U^* : K[U] \to K[f^{-1}(U)]$. Clearly, these homomorphisms are compatible with the restriction homomorphisms of the structure sheaves on X and Y, and therefore they define a morphism of sheaves of rings $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, which endows the sheaf $f_*\mathcal{O}_X$ the structure of a sheaf of $\mathcal{O}_Y$-modules.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}_X$-modules and therefore $f_*\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathcal{G}$ is a sheaf of $f^{-1}\mathcal{O}_Y$-modules. Then it can be checked directly that

$$\operatorname*{Hom}_X(f^{-1}\mathcal{O}_Y, \mathcal{O}_X) = \operatorname*{Hom}_Y(\mathcal{O}_Y, f_*\mathcal{O}_X),$$

where the correspondence between the right and left group is given as follows: if $\varphi \in \mathrm{Hom}_X(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$, then the corresponding morphism $\psi \in \mathrm{Hom}_Y(\mathcal{O}_Y, f_*\mathcal{O}_X)$ is defined as $\psi(U) := \varphi(f^{-1}(U)) : \mathcal{O}_Y(U) = f^{-1}\mathcal{O}_Y(f^{-1}(U)) \to \mathcal{O}_X(f^{-1}(U))$. Vice versa, if $\psi \in \mathrm{Hom}_Y(\mathcal{O}_Y, f_*\mathcal{O}_X)$, then the corresponding morphism $\varphi \in \mathrm{Hom}_X(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$ is defined for each open $U \subset X$ as a composition of homomorphisms

$$\varphi(U) : f^{-1}\mathcal{O}_Y(U) = \varinjlim_{V \supset f(U)} \mathcal{O}_Y(V) \xrightarrow{\psi(V)} \varinjlim_{V \supset f(U)} \mathcal{O}_X(f^{-1}(V)) \xrightarrow{\rho_{f^{-1}(V),U}} \mathcal{O}_X(U)$$

Exercise 8.4. Check that these correspondences give the equality of groups.

So, by this equality the morphism $f^\sharp$ corresponds to a morphism $f^\natural : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Definition 8.8. Let $f : X \to Y$ be a morphism of affine varieties (or, more generally, of ringed spaces) and $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules.

We define the *pull-back* sheaf as $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Note that it is a sheaf of $\mathcal{O}_X$-modules.

Exercise 8.5. Show that there is a canonical isomorphism of groups $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$.

As an application of all these constructions we define the fiber of a sheaf. Let X be an affine variety, $P \in X$ be a point, $K(P)$ be its residue field, $\mathcal{M}_P$ be the corresponding maximal ideal in $K[X]$ or in $\mathcal{O}_{X,P}$, $\chi : P \hookrightarrow X$ be the embedding of affine varieties, $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *fiber* as $\mathcal{F}|_P := \chi^*\mathcal{F}$.

Note that for a non-empty U $(= P)$ we have $\chi^{-1}\mathcal{O}_X(U) = \varinjlim_{U \supset P} \mathcal{O}_X(U) = \mathcal{O}_{X,P}$, $\chi^{-1}\mathcal{F}(U) = \mathcal{F}_P$, the structure sheaf of P is $\mathcal{O}_P = K[X]/\mathcal{M}_P \simeq \mathcal{O}_{X,P}/\mathcal{M}_P = K(P)$ and it is also a $\mathcal{O}_{X,P}$-algebra. So, $\chi^*\mathcal{F} = \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_P$ is a $K(P)$-vector space!

8.2 Affine schemes

Let A be a commutative ring. Define the topological space $\underline{Spec}A := \{\wp \subset A,\ \wp \text{ — prime ideal}\}$. Define the Zariski topology on it by defining closed subsets as $V(\mathfrak{a}) = \{\wp \supset \mathfrak{a}\}$, where $\mathfrak{a}$ is an arbitrary ideal of A.

Exercise 8.6. Check that this is really a topology. Hint: use the weak form of the prime decomposition: $\sqrt{\mathfrak{a}} = \bigcap\limits_{\wp \supset \mathfrak{a}} \wp$.

Example 8.2. Note that there are closed points in $\underline{Spec}A$ (maximal ideals) and not closed points (other prime ideals). If A is an integral domain, then the point $(0) \in \underline{Spec}A$ is called the *generic point*. In general case the set of minimal primes of a Noetherian ring A form generic points of irreducible components of $\underline{Spec}A$.

Now let's define a *structure sheaf* of rings on $\underline{Spec}A$: for any open $U \subset \underline{Spec}A$

$$\mathcal{O}(U) := \{s : U \to \coprod_{p \in U} A_p| \quad s(p) \in A_p \quad \forall p \quad \text{and} \quad \forall p \in U \quad \exists p \in V \subset U :$$

$$\exists a, f \in A \quad \text{such that} \quad s(q) = \frac{a}{f} \quad \forall q \in V, \quad f \notin q\}$$

(note: this is an analogue of the sheaf of regular functions on affine varieties). The restriction maps $\rho_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ are clearly the homomorphisms of rings, and it is obvious that $\mathcal{O}$ is a presheaf. Almost obvious that $\mathcal{O}$ is a sheaf of rings.

Definition 8.9. For a ring A we define $\operatorname{Spec} A$ as a topological space $\underline{Spec}A$ with the sheaf $\mathcal{O}$: $\operatorname{Spec} A = (\underline{Spec}A, \mathcal{O})$.

For any $f \in A$ denote by $D(f) = \underline{Spec}A \backslash V((f))$. It is not difficult to see that sets $D(f)$ form a base of topology on $\underline{Spec}A$.

Proposition 8.1. *[32, Prop. 2.2] There are the following properties:*

1. *for any $p \in \underline{Spec}A$ we have $\mathcal{O}_p \simeq A_p$;*
2. *for any $f \in A$ we have $\mathcal{O}(D(f)) \simeq A_f$;*
3. *in particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}) := \mathcal{O}(\operatorname{Spec} A) \simeq A$.*

Proof. We give here only a sketch of the proof (see the reference for all details).

1) Define a homomorphism $\varphi : \mathcal{O}_p \to A_p$, $[(U,s)] \mapsto s(p)$ (check it does not depend on the choice of a pair (U,s).

It is surjective, since any element from A_p has the form a/f, where $f \in p$, whence $p \in D(f)$ and a/f defines a section of $\mathcal{O}(D(f))$, whose value at p is a/f.

It is injective: for, let $p \in U$, $s, t \in \mathcal{O}(U)$ be such that $s(p) = t(p)$. Then for some smaller $V \subset U$ $s = a/f$, $t = b/g$ with $f, g \notin p$ and there exists $h \notin p$ such that $h(ga - bf) = 0$. Therefore, $a/f = b/g$ in any ring A_q for $q \in V$.

2) Define a homomorphism $\psi : A_f \to \mathcal{O}(D(f))$, $a/f^n \mapsto s : p \mapsto \bar{a}/\bar{f}^n \in A_p$.

It is injective. Indeed, if $\psi(a/f^n) = \psi(b/f^m)$, then for any $p \in D(f)$ $a/f^n = b/f^m$ in A_p. Then there exists $h \notin p$ with $h(f^m a - f^n b) = 0$. If $\mathfrak{a} = Ann(f^m a - f^n b)$, then $h \in \mathfrak{a}$, $h \notin p$. Therefore, $\mathfrak{a} \not\subseteq p$ and $V(\mathfrak{a}) \cap D(f) = \emptyset$. Hence $f \in \sqrt{\mathfrak{a}}$, $f^l \in \mathfrak{a}$ and $f^l(f^m a - f^n b) = 0$, where from $a/f^n = b/f^m$ in A_f.

The proof that ψ is surjective is more difficult and is based on the following 3 steps:
a) show that $D(f) = \cup D(h_i)$ and $s|_{D(h_i)} = a_i/h_i$;
b) show that this covering is finite;
c) show that $f^n = \sum b_i h_i$, then set $a = \sum b_i a_i$. Then $h_j a = f^n a_j$ and $a/f^n = a_j/h_j$ for any j on $D(h_j)$. So, by the glueing axiom of a sheaf $\psi(a/f^n) = s$. $\square$

Exercise 8.7. Let $U \subset \underline{Spec} A$ be an open subset. Prove that $\mathcal{O}(U) \simeq S^{-1}A$, where $S = A \setminus \cup_{p \in U} p$

Definition 8.10. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings. A *morphism* of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map, $f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_Y$ is a morphism of sheaves of rings.

The ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space*, if for any $p \in X$ the stalk $\mathcal{O}_{X,p}$ is a local ring. *Morphism* of locally ringed spaces is a morphism of ringed spaces such that for any $p \in X$ the induced map $f^\sharp_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ (obtained as a composition of maps

$$\mathcal{O}_{Y,f(p)} = \varinjlim_{V \supset f(p)} \mathcal{O}_Y(V) \overset{f^\sharp_p}{\to} \varinjlim_{V \supset f(p)} \to \varinjlim_{U \supset p} \mathcal{O}_X(U) = \mathcal{O}_{X,p})$$

is a *local homomorphism* of local rings, i.e. $(f^\sharp_p)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{f(p)}$ (here $\mathfrak{m}$ denotes the maximal ideal of a local ring).

An *isomorphism* of locally ringed spaces is a pair $(f, f^\sharp)$, where f is a homeomorphism and $f^\sharp$ is an isomorphism of sheaves.

The following proposition describes basic properties of Spec.

Proposition 8.2. *[32, Ch. 2, Prop.2.3]*

1. $(\underline{\mathrm{Spec}} A, \mathcal{O})$ *is a locally ringed space;*

2. *if* $\varphi : A \to B$ *is a homomorphism of rings, then* φ *induces a morphism of locally ringed spaces*
$$(f, f^\sharp) : (\underline{\mathrm{Spec}} B, \mathcal{O}_{\mathrm{Spec}\, B}) \to (\underline{\mathrm{Spec}} A, \mathcal{O}_{\mathrm{Spec}\, A}),$$

3. *any morphism of locally ringed spaces* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *is induced by a homomorphism of rings* $\varphi : A \to B$.

Definition 8.11. An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\mathrm{Spec}\, A$ for some ring A.

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme, i.e. for any $P \in X$ there exists an open neighbourhood $P \in U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

There is the following clear connection of affine schemes with affine varieties. Let $K = \bar{K}$ and X be an affine variety over K. Then schemes over K and varieties over K form *categories* $Sch(K)$ and $Var(K)$ correspondingly.

Proposition 8.3. *[32, Ch. 2, prop. 2.6] There exists a fully faithful functor* $t : Var(K) \to Sch(K)$, $X \mapsto \mathrm{Spec}(K[X])$ *such that the topological space* $\underline{X}$ *is homeomorphic to the topological space* $sp(t(X))$ *(closed points of the scheme* $t(X)$ *with the induced Zariski topology), and the sheaf of regular functions is the restriction of the structure sheaf of the scheme* $t(X)$ *via this homeomorphism.*

In fact, this assertion is true not only for affine varieties and schemes (see loc. cit. and section 9.2 below).

8.3 Affine spectral data

Now we are ready to define the *affine spectral data*. For a given elliptic commutative ring $B \subset D$ we associate

Definition 8.12. For a given elliptic commutative ring $B \subset D$ the affine spectral curve is $C_0 = \operatorname{Spec} B$.

Later we will see that the affine curve can be completed by one smooth point. So, if the curve is smooth, then the completion will be a Riemann surface of finite genus.

Example 8.3. Consider the following example of Wallenberg:

$$P = \partial^2 - \frac{2}{(x+1)^2}, \quad Q = 2\partial^3 - \frac{6}{(x+1)^2}\partial + \frac{6}{(x+1)^3}$$

commute. The ring $B = \mathbb{C}[P, Q] \simeq \mathbb{C}[T_1, T_2]/(f)$, where $f = T_2^2 - 4T_1^3$. Thus the spectral curve in this case is a plane curve defined by the equation $f(T_1, T_2) = 0$.

The *spectral sheaf* is a sheaf of $\mathcal{O}_{C_0}$-modules. The standard *construction* of a sheaf associated with module is as follows:

Definition 8.13. Let M be a A-module, where A is a commutative ring. Construct the *associated sheaf of $\mathcal{O}_X$-modules $M^{\sim}$* on the affine scheme $X = \operatorname{Spec} A$ as follows:

$$\begin{aligned} M^{\sim}(U) = \{ &\text{set of maps} \quad s: U \to \amalg_{P \in U} M_P \quad \text{s.t. } s(P) \in M_P \ \forall P \in U \text{ and } \forall P \in U \\ &\exists V \subset U,\, V \ni P \text{ and } \exists f \in M,\, b \in A \text{ s.t. } s(q) = f/b \in M_q \text{ and } b \notin q \ \forall q \in V\} \end{aligned} \tag{15}$$

with the obvious module structure (componentwise multiplication) on the maps.

Definition 8.14. A sheaf $\mathcal{F}$ on a scheme X is called *quasi-coherent* if there is a covering $X = \cup_i U_i$ by affine open subsets such that $\mathcal{F}|_{U_i} \simeq M_i^{\sim}$, where M_i is a $\mathcal{O}_X(U_i)$-module.

A sheaf $\mathcal{F}$ is called *coherent* if there is a covering $X = \cup_i U_i$ by affine open subsets such that $\mathcal{F}|_{U_i} \simeq M_i^{\sim}$, where M_i is a *finitely generated* $\mathcal{O}_X(U_i)$-module.

Definition 8.15. The sheaf $F^{\sim}$ associated to the spectral module F is called the *affine spectral sheaf*. This sheaf is coherent by theorem 6.6.

Note that $F^{\sim}$ is *a torsion free sheaf*, i.e. for any U $F^{\sim}(U)$ is a torsion free $\mathcal{O}_{C_0}(U) = K[U]$-module (i.e. for any $0 \neq m \in F^{\sim}(U)$ and for any $0 \neq a \in K[U]$ $ma \neq 0$). In particular, $\mathcal{F}_p$ is a torsion free $\mathcal{O}_{C_0,p}$-module.

Exercise 8.8. Check that modules $F^{\sim}(U)$ with usual restriction maps form a sheaf of $\mathcal{O}_{C_0}$-modules.

The following properties can be derived easily from definition:

Proposition 8.4. *[32, Ch.2, §5] In the notation of definition 8.13 we have the following properties:*

1. *$M^{\sim}$ is a sheaf of $\mathcal{O}_X$-modules;*

2. *for any point $p \in X$ we have $(M^{\sim})_p \simeq M_p$;*

3. *for any $b \in A$ we have $M^{\sim}(D(b)) \simeq M_b$, in particular, $M^{\sim}(X) \simeq M$;*

Definition 8.16. The sheaf $\mathcal{F}$ is *locally free*, if there exists a covering $X = \cup_i U_i$ such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module (in particular, for any point $p \in X$ $\mathcal{F}_p$ is a free $\mathcal{O}_{X,p}$-module, i.e. $\mathcal{F}_p \simeq \mathcal{O}_{X,p}^{\oplus \dots}$).

If the sheaf is coherent (as we'll consider here), then these free modules are of finite rank.

We'll need one more fact from commutative algebra:

Proposition 8.5. *A torsion free finitely generated module over a Noetherian regular local ring of dimension one (i.e. over an DVR) is free.*

Proof. Let $m_1, \dots, m_s$ be a minimal set of generators of a module M over a ring A from proposition. If M is not free, there is a relation $a_1 m_1 + \dots + a_s m_s = 0$ for some $a_1, \dots, a_s \in A$, not all of them are zero. If one of elements a_i is a unit, then m_i belongs to a submodule generated by all other m_j, i.e. $m_1, \dots, m_s$ is not a minimal set of generators. If there are no units, all elements a_i belong to the maximal ideal. Since A is a local ring, this ideal is principal, i.e. for any i we have $a_i = t a_i'$, where t is a generator of the maximal ideal, $a_i' \in A$. Then $a_1 m_1 + \dots + a_s m_s = t(a_1' m_1 + \dots + a_s' m_s) = 0$, and, since M is torsion free, we must have $a_1' m_1 + \dots + a_s' m_s = 0$. Repeating our arguments again, we will come to a relation with a unit as one of its coefficients, a contradiction (by lemma 14.4 we will need only finite number of repetitions). $\square$

Corollary 8.1. *If $P \in C_0$ is a smooth point (i.e. $\mathcal{O}_{C_0,P}$ is regular of dimension one), then $(F^{\sim})_P \simeq \mathcal{O}_{C_0,P}^{\oplus r}$.*

Proof. By proposition, the stalk $(F^{\sim})_P$ is a free $\mathcal{O}_{C_0,P}$-module. Since the spectral sheaf is coherent, $(F^{\sim})_P \simeq \mathcal{O}_{C_0,P}^{\oplus l}$ for some $l < \infty$. By proposition 8.4 $(F^{\sim})_P \simeq F_P$, and by exercise 6.5 $F_P \cdot \mathrm{Quot}(B) = F \cdot \mathrm{Quot}(B) \simeq (\mathrm{Quot}(B))^{\oplus r}$. By the same proposition $\mathcal{O}_{C_0,P}^{\oplus l} \cdot \mathrm{Quot}(B) \simeq (\mathrm{Quot}(B))^{\oplus l}$ and therefore we must have $l = r$. $\square$

Remark 8.1. We would like to emphasize that the spectral sheaf is not, in general, locally free. It is locally free only over a smooth locus of the spectral curve. On the other hand, locally free sheaves are closely related to *geometric* objects — vector bundles, see the section 9.4.

8.4 Geometric and analytical meaning of the affine spectral sheaf

Definition 8.17. Let $B \subset D$ be a commutative elliptic ring. We define the *rank* of its affine spectral sheaf $F^{\sim}$ as the rank of the stalk at the generic point : $\mathrm{rk}\, F^{\sim} := \dim_{\mathrm{Quot}(B)} (F^{\sim})_{(0)}$.

Recall that by proposition 8.4 and exercise 6.5 $(F^{\sim})_{(0)} = F \cdot \mathrm{Quot}(B) \simeq (\mathrm{Quot}(B))^{\oplus r}$, i.e. $\mathrm{rk}\, F^{\sim} = r = \mathrm{rk}(B)$.

Comment 8.1. This definition is usually accepted in a more general situation of a coherent sheaf on a connected Noetherian scheme, cf. [32, Ch.2, §3].

Below we will need the following fact from commutative algebra.

Proposition 8.6. *Let $\mathcal{F}$ be a coherent torsion free sheaf of rank r on the affine scheme $X = \mathrm{Spec}\, A$, where A is an integral domain. Then $\dim_{K(p)}(\mathcal{F}|_p) \geq r$, where $p \in \underline{\mathrm{Spec} A}$ is any point, and $\mathcal{F}$ is locally free at p iff $\dim_{K(p)}(\mathcal{F}|_p) = r$ (the fiber of a sheaf at any point of a scheme is defined in the same way as at the end of section 8.1).*

Proof. Recall that $\mathcal{F}|_p \simeq K(p) \otimes_{\mathcal{O}_{X,p}} \mathcal{F}_p$. If $\mathcal{F}$ is locally free at p, then by the same arguments as in the proof of corollary 8.1 $\mathcal{F}_p \simeq \mathcal{O}_{X,p}^{\oplus r}$ and therefore $\dim_{K(p)}(\mathcal{F}|_p) = r$.

Conversely, assume that $\dim_{K(p)}(\mathcal{F}|_p) = r$. Choose a $K(p)$-basis $\{f_1, \dots, f_r\}$ of the vector space $\mathcal{F}|_p$. Note that $f_i = 1 \otimes \tilde{f}_i$ for some $\tilde{f}_i = [(U_i, \tilde{f}_i \in \mathcal{F}(U_i))] \in \mathcal{F}_p$, $i = 1, \dots, r$. Now consider the neighbourhood $U = \cap_{i=1}^r U_i \ni p$ and consider the following exact sequence of $\mathcal{O}_X(U)$-modules

$$\mathcal{O}_X(U)^{\oplus r} \to \mathcal{F}(U) \to M := \mathcal{F}(U)/\mathcal{O}_X(U)^{\oplus r} \to 0,$$

where the first map is $(w_1, \ldots, w_r) \mapsto w_1\tilde{f}_1 + \ldots + w_r\tilde{f}_r$. Recall that $K(p) \simeq \mathcal{O}_X(U)/\mathfrak{m}_p$, where $\mathfrak{m}_p$ is the ideal in the ring $\mathcal{O}_X(U)$ corresponding to the point p, and therefore has a natural structure of $\mathcal{O}_X(U)$-module. Note that $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} K(p) \simeq \mathcal{F}|_p$. So, by tensoring the last sequence by $K(p)$, we obtain the exact sequence

$$K(p)^{\oplus r} \to \mathcal{F}|_p \to M \otimes_{\mathcal{O}_X(U)} K(p) \to 0,$$

where the first map is surjective since the image contains the basis $\{f_1, \ldots, f_r\}$. Then by the Nakayama lemma [4, Prop. 2.6] $M = 0$, i.e. the first homomorphism of the first sequence is surjective. Thus, we obtain the exact sequence

$$0 \to \ker\varphi \to \mathcal{O}_X(U)^{\oplus r} \xrightarrow{\varphi} \mathcal{F}(U) \to 0.$$

Since the ring A is an integral domain, $K(X)$ is a flat $\mathcal{O}_X(U)$- module, and therefore the sequence

$$0 \to L \otimes_{\mathcal{O}_X(U)} K(X) \to K(X)^{\oplus r} \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} K(X) \simeq \mathcal{F}_{(0)} \to 0$$

is exact too, where the last isomorphism follows from proposition 8.4 and generic properties of the coherent sheaf, cf. [32, Ch.2, Prop. 5.4]. Now note that the last non-trivial map in this sequence is an isomorphism, and therefore $L \otimes_{\mathcal{O}_X(U)} K(X) = 0$. Since L is torsion free as a submodule of a torsion free module, this means $L = 0$, i.e. $\mathcal{F}(U)$ is a free module.

Finally, the same arguments show that $\dim_{K(p)}(\mathcal{F}|_p) \geq r$. □

All closed points of the affine variety (scheme) $C_0 = \mathrm{Spec}(B)$ are in one to one correspondence with maximal ideals of the ring B. Any maximal ideal $q \subset B$ gives a K-algebra homomorphism $\chi_q : B \to B/q \simeq K_q$, where $K_q \supset K$ is a finite algebraic extension, and vice versa.

Definition 8.18. Let $q \in C_0$ be any closed point and $\chi = \chi_q : B \to K_q$ the corresponding character. We call the K_q-vector space

$$\mathsf{Sol}\big(B, \chi_q\big) := \big\{f \in K_q[[x]] \big| P(f) = \chi_q(P)f \text{ for all } P \in B\big\} \tag{16}$$

the *solution space* of the algebra B at the point q.

Observe, that $\mathsf{Sol}\big(B, \chi\big)$ has a natural B–module structure: $f \in \mathsf{Sol}\big(B, \chi\big) \Rightarrow \forall Q \in B$ $Q(f) \in \mathsf{Sol}\big(B, \chi\big)$. Recall that by exercise 6.3 all eigenfunctions of operators from B belong to $K_q[[x]]$ (as B is elliptic).

The geometric meaning of the spectral sheaf $F^{\sim}$ is explained by the next result.

Theorem 8.1. *The following K–linear map*

$$F \overset{\eta_{\chi_q}}{\to} \mathsf{Sol}\big(B, \chi_q\big)^*, \quad \partial^i \mapsto \Big(f \mapsto \frac{1}{i!} f^{(i)}(0)\Big) \tag{17}$$

is also B–linear, where $\mathsf{Sol}\big(B, \chi_q\big)^* = \mathrm{Hom}_{K_q}\big(\mathsf{Sol}\big(B, \chi_q\big), K_q\big)$ is the vector space dual of the solution space. Moreover, the induced map

$$(F^{\sim})|_q \simeq B/\ker(\chi_q) \otimes_B F \overset{\bar{\eta}_{\chi_q}}{\to} \mathsf{Sol}\big(B, \chi_q\big)^* \tag{18}$$

is an isomorphism of B–modules.

Proof. First note that the following map

$$\mathrm{Hom}_{K_q}\big(F, K_q\big) \stackrel{\Phi}{\to} K_q[[x]], \quad \lambda \mapsto \sum_{p=0}^{\infty} \frac{1}{p!}\lambda(\partial^p)x^p \tag{19}$$

is an isomorphism of left D–modules, where the D-module structure on Hom_{K_q} is given, as usual, by the rule $(P \cdot \lambda)(-) := \lambda(- \cdot P)$ and D acts on $K_q[[x]]$ by the usual application of a differential operator on a function.

Exercise 8.9. Check that Φ is an isomorphism of D-modules.

Let $B \stackrel{\chi_q}{\to} K_q$ be a character, then $K_q = B/\ker(\chi_q)$ is a left B–module. We obtain a B–linear map

$$\Psi : \mathrm{Hom}_B(F, K_q) \stackrel{I}{\to} \mathrm{Hom}_{K_q}(F, K_q) \stackrel{\Phi}{\to} K_q[[x]], \tag{20}$$

where I is the forgetful map. The image of I consists of those K_q-linear functionals, which are also B–linear, i.e.

$$\mathsf{Im}(I) = \big\{\lambda \in \mathrm{Hom}_{K_q}(F, K_q) \,\big|\, \lambda(-\cdot P) = \chi_q(P)\cdot\lambda(-) \text{ for all } P \in B\big\}.$$

This implies that $\mathsf{Im}(\Psi) = \mathsf{Sol}(B, \chi_q)$. Next, we have a canonical isomorphism of B–modules (check it!): $\mathrm{Hom}_B(F, K_q) \cong \mathrm{Hom}_{K_q}\big(B/\ker(\chi_q) \otimes_B F, K_q\big)$. Dualizing again, we get an isomorphism of vector spaces

$$\Psi^* : \mathsf{Sol}(B, \chi_q)^* \to \big(B/\ker(\chi_q) \otimes_B F\big)^{**} \cong B/\ker(\chi_q) \otimes_B F.$$

It remains to observe that Ψ^* is also B–linear and $\big(\Psi^*\big)^{-1} = \bar{\eta}_{\chi_q}$. □

Corollary 8.2. *Let $B \subset D$ be a commutative elliptic subring of rank r. Then for any character $B \stackrel{\chi_q}{\to} K_q$ we have: $r \le \dim_{K_q}\big(\mathsf{Sol}(B, \chi_q)\big) < \infty$. Moreover, $\dim_{K_q}\big(\mathsf{Sol}(B, \chi_q)\big) \ge r + 1$ if only if χ_q defines a* singular point $q \in C_0$ and $F^\sim$ *is* not locally free *at* q.

The corollary directly follows from theorem 8.1 and propositions 8.6 and 8.5.

Example 8.4. Let's illustrate this corollary.

Consider the ring $B = K[\partial^2, \partial^3] \subset D$. Then $\mathrm{Spec}\, B$ is an affine curve given by the equation $y^2 = x^3$, which has a so called *ordinary cuspidal singularity* at the point $(0,0)$. Let $q = (\partial^2, \partial^3)$ be the maximal ideal of this point, and $\chi_q : B \to K$ be the corresponding character. Then it is not difficult to see that $\mathsf{Sol}(B, \chi_q) = \{f \in K[[x]] | \quad \partial^2(f) = \chi_q(\partial^2)f = 0, \partial^3(f) = \chi_q(\partial^3)f = 0\} = \langle 1, x\rangle$.

On the other hand, $\mathrm{rk}\, F = 1$, because $F \cdot \mathrm{Quot}(B) \simeq K(t)$. Thus, the spectral sheaf is not locally free at $(0,0)$.

For any character $B \stackrel{\chi_q}{\to} K_q$ consider the K_q–vector space

$$\mathsf{Sol}'\big(B, \chi_q\big) := \big\{f \in K_q[[x]] \,\big|\, P(f) = \chi_q(P)f \text{ for all } P \in B\big\}. \tag{21}$$

Obviously, $\mathsf{Sol}\big(B, \chi_q\big) \subseteq \mathsf{Sol}'\big(B, \chi_q\big)$. However, the following result is true.

Theorem 8.2. *Let $B \subset D$ be a commutative elliptic subring of rank r and $B \stackrel{\chi_q}{\to} K_q$ a character. Then we have: $\mathsf{Sol}\big(B, \chi_q\big) = \mathsf{Sol}'\big(B, \chi_q\big)$ and there exists a uniquely determined*

$$R_{\chi_q} = \partial^m + c_1\partial^{m-1} + \cdots + c_m \in \widetilde{D} = K_q((x))[\partial] \tag{22}$$

such that $\ker(R_{\chi_q}) = \mathsf{Sol}'\big(B, \chi_q\big)$. Moreover, $m \ge r$ and $m = r$ if and only if $\mathcal{F} = F^\sim$ is locally free at the point $q \in C_0$ corresponding to χ_q.

Proof. Let $P = \partial^n + a_1\partial^{n-1} + \cdots + a_n \in K_q[[x]][\partial]$. Then the dimension of the K_q –vector space $\ker(P) \subset K_q[[x]]$ is n and $\ker(P) \subset K_q[[x]]$ (see exercise 6.3). This implies that $\mathsf{Sol}\big(B, \chi_q\big) = \mathsf{Sol}'\big(B, \chi_q\big)$.

For any differential operators $Q_1, \ldots, Q_l \in \widetilde{D}$ we denote by $\langle Q_1, \ldots, Q_l \rangle \subsetneqq \widetilde{D}$ the left ideal generated by these elements. Recall that by theorem 4.2 any left ideal $J \subseteq \widetilde{D}$ is principal. Let $P_1, \ldots, P_n \in B$ be the algebra generators of B (i.e. $B = K[P_1, \ldots, P_n]$) and $\alpha_i = \chi_q(P_i)$ for all $1 \le i \le n$. Then there exists a uniquely determined $R_{\chi_q} \in \widetilde{D}$ as in (22) such that

$$\big\langle P - \chi_q(P)1 \bigm| P \in B \big\rangle = \big\langle P_1 - \alpha_1, \ldots, P_n - \alpha_n \big\rangle = \langle R_{\chi_q} \rangle. \tag{23}$$

Now let's use Differential Galois Theory: by the Kolchin-Ritt theorem 6.1 there is the universal Picard–Vessiot extension $PV(K_q((x)))$ of $K_q((x))$ (cf. [87, Section 3.2]), where R_{χ_q} (actually any differential operator of order m) from $\widetilde{D}$ has exactly m linearly independent solutions with values in $PV(K_q((x)))$.

Obviously, $\ker(R_{\chi_q}) = \mathsf{Sol}'\big(B, \chi_q\big) = \mathsf{Sol}\big(B, \chi_q\big)$ viewed as subspaces of $PV(K_q((x)))$. Moreover, $\dim_{K_q}\big(\ker(R_{\chi_q})\big) = \mathsf{ord}(R_{\chi_q})$. In virtue of Corollary 8.2, we get the statement about the order of R_{χ_q}. $\square$

9 Projective varieties, schemes and their basic properties

In this section we collect necessary basic facts about projective varieties and schemes. The references for this section are the same as in the previous section.

9.1 Projective varieties

From the affine spectral data $(C_0 = \mathrm{Spec}(B), F^{\sim})$ we can get a natural completion $(C, \mathcal{F})$, where C is a projective curve and $\mathcal{F}$ is a coherent sheaf on C. In order to explain this we need to give a short introduction to the *projective algebraic geometry.*

It studies algebraic subsets in the *projective space*. The projective space $\mathbb{P}^n_K$ is the set of equivalence classes of points of $\mathbb{A}^{n+1}_K \backslash \{(0, \ldots, 0)\}$, where two points are equivalent if they differ by a common non-zero multiple. The equivalence class of $(x_0, x_1, \ldots, x_n)$ is denoted by $(x_0 : x_1 : \ldots : x_n)$.

The *Zariski topology* on the projective space is defined with the help of *homogeneous* polynomials. The zeros of homogeneous polynomials, denoted also as $Z(T)$, are *closed projective sets*. The condition $x_i \neq 0$ defines an open subset of $\mathbb{P}^n_K$ isomorphic to the affine space $\mathbb{A}^n_K$ with coordinates $x_0/x_i, \ldots, x_n/x_i$. We get $n+1$ affine spaces which provide an open covering of $\mathbb{P}^n_K$.

For any affine variety there is a "usual" projective closure defined with the help of *homogenisation* procedure of polynomials. Let $f(T_1, \ldots, T_n)$ be a polynomial of degree d. It can be written as the sum $f = f_0 + \ldots + f_d$, where f_i is a form of degree i. The homogenization of f is the form of degree d in $n+1$ variables given by

$$F(T_0, T_1, \ldots, T_n) = F = T_0^d f_0 + T_0^{d-1} f_1 + \ldots + f_d.$$

Now if $X \subset \mathbb{A}^n_K$ is a closed affine set, then associating to polynomials in the ideal of X their homogenizations defines the projective closure of X.

For closed projective sets there is an analogous notion of irreducible set. The irreducible projective sets are called *projective varieties. Quasi-projective varieties* are dense open subsets of projective varieties. Projective varieties are simpler than affine, since they are *compact* (also in the usual complex topology if $K = \mathbb{C}$). Geometry of projective varieties is based on the commutative algebra of *graded rings* and *homogeneous ideals.*

Definition 9.1. A ring R is graded if $R = \oplus_{d\geq 0} R_d$, where R_d are abelian groups (called group of homogeneous elements) and $R_d \cdot R_e \subset R_{d+e}$.

Analogously, a *graded* R*-module*, where R is a graded ring, is a module $M = \oplus_{d\geq 0} M_d$ (we can take also the sum for $d \in \mathbb{Z}$) with $M_d \cdot R_e \subset M_{d+e}$.

An ideal $J \subset R$ is *homogeneous* if $J = \oplus_{d\geq 0}(J \cap R_d)$.

Example 9.1. The ring of polynomials $K[T_0, \ldots, T_n]$ can be considered as graded: $K[T_0, \ldots, T_n] = \oplus_{d\geq 0} R_d$, where R_d is the group of homogeneous polynomials of degree d. An ideal $J \subset K[T_0, \ldots, T_n]$ is homogeneous if whenever $f \in J$, its homogeneous part $f_i \in J$.

Exercise 9.1. Prove the projective Nullstellensatz: if J is a homogeneous ideal, then

(1) $Z(J) = \emptyset$ if the radical of J contains the ideal $(T_0, \ldots, T_n)$ (the maximal ideal of the zero point in $\mathbb{A}_K^{n+1}$),

(2) if $Z(J) \neq \emptyset$, then $I(Z(J)) = \sqrt{J}$.

For graded rings we have special localisations:

Localisation. If T is a multiplicative system of *homogeneous* elements in a graded ring R, then the localisation with respect to T is

$$T^{-1}R = \{\frac{a}{b}, \quad a, b \in R_d \text{ for some } d \text{ and } b \in T\}.$$

As in the case of usual localisation $T^{-1}R$ is a ring (but not necessarily a graded ring).

Analogously, if M is a graded R-module, then

$$T^{-1}M = \{\frac{m}{b}, \quad b \in R_d,\ m \in M_d \text{ for some } d \in \mathbb{N} \text{ and } b \in T\}.$$

is a $T^{-1}R$-module.

Example 9.2. 1) If $\wp \subset R$, $\wp \not\supset R_+ = \oplus_{d>0} R_d$, then we can take set T to be equal to the set of homogeneous elements in $R \backslash \wp$. Denote by $R_{(\wp)} := T^{-1}R$. This is a local ring (analogue of $R_\wp$ for ordinary rings).

2) If a is a homogeneous element from R, let's take $T = \{a^n\}$, $n \geq 0$ and denote by $R_{(a)} = T^{-1}R$. This ring is an analogue of the ring R_a in affine geometry. In particular, if $a = T_i$ in $R = K[T_0, \ldots, T_n]$, then $R_{(a)} \simeq K[x_1, \ldots, x_n]$.

For projective varieties we have also a dictionary between geometry and commutative algebra, which is analogous to the dictionary in affine geometry.

- Projective variety X defined by homogeneous polynomials $f_1, \ldots, f_k$ corresponds to the graded ring $R = K[T_0, \ldots, T_n]/I$, where $I = (f_1, \ldots, f_k)$ is a prime homogeneous ideal. We'll denote this variety by $\operatorname{Proj} R$.

- $\dim X = \dim R - 1 = trdeg R_{((0))}$

- Open subset of X defined by condition $h \neq 0$, where h is a homogeneous element, corresponds to the ring $R_{(h)}$

- closed subsets are defined by homogeneous radical ideals in R

- Rational functions $K(X)$ on X are defined as elements of the ring $R_{((0))}$, and stalks of regular functions (defined in the same way as for affine varieties) are isomorphic to the local rings $R_{(\wp)}$.

Remark 9.1. Note that the graded ring $K[T_0,\dots,T_n]/I$ from the list above is finitely generated over K by the set from its first homogeneous component (by the images of the elements $T_0,\dots,T_n$). Clearly, if we have a graded ring which is finitely generated by its first homogeneous component over its zero component equal to K, then such a ring is isomorphic to the image of the graded ring $K[T_0,\dots,T_m]$ for some m, i.e. to the ring $K[T_0,\dots,T_m]/J$ for some homogeneous ideal J. In particular, if this ring is integral, then it determines a projective variety. We will see below that this fact can be extended for all graded rings finitely generated over its zero component equal to K.

Morphisms. A *rational map* $f : X \dashrightarrow \mathbb{P}^n_K$ is (a not necessarily everywhere defined function) given by $(F_0,\dots,F_n)$, where $F_i \in K(X)^*$, defined up to an overall multiple from $K(X)^*$. A rational map f is *regular* at $P \in X$ if there exists a representative $(F_0,\dots,F_n)$, such that all the F_i's are regular at P, and $(F_0(P),\dots,F_n(P)) \neq (0,\dots,0)$. A *morphism* is an everywhere regular rational map.

9.2 Projective schemes

Let $S = \oplus_{d\geq 0} S_d$ be a graded ring. First we define the topological space

$$\underline{\mathrm{Proj}} S = \{\text{homogeneous prime ideals } \wp \subset S\,,\ \wp \not\supseteq S_+\,\}$$

with the topology defined by closed subsets $V(\eth) = \{\wp \in \underline{\mathrm{Proj}} S | \quad \wp \supset \eth\}$.

Exercise 9.2. 1) Prove that a homogeneous ideal $\wp \subset S$ is prime iff for any *homogeneous* elements $a, b \in \wp \ \Rightarrow \ a \in \wp$ or $b \in \wp$.

2) Let $\eth_1, \eth_2$ be two homogeneous ideals in S. Then $V(\eth_1\eth_2) = V(\eth_1) \cup V(\eth_2)$.

3) Let $\{\eth_i\}$ be a family of homogeneous ideals in S. Then $V(\sum \eth_i) = \cap V(\eth_i)$.

Now let's define the structure sheaf $\mathcal{O}$: for any open $U \subset \underline{\mathrm{Proj}} S$

$$\mathcal{O}(U) := \{\text{set of maps} \quad s : U \to \coprod_{P\in U} S_{(P)} \quad \text{s.t. } s(P) \in S_{(P)}\ \forall P \in U \text{ and } \forall P \in U$$
$$\exists V \subset U,\, V \ni P \text{ and } \exists a, f \in S_d, \text{ s.t. } s(q) = a/f \in S_{(q)} \text{ and } f \notin q\ \forall q \in V\}$$

As we have seen many times before, it is not difficult to show that it is really a sheaf of rings. Now we define the ringed space $\mathrm{Proj}\, S := (\underline{\mathrm{Proj}} S, \mathcal{O})$.

Proposition 9.1. *[32, Ch. 2, Prop. 2.5]*

1. *For any point $p \in \underline{\mathrm{Proj}} S$ we have $\mathcal{O}_p \simeq S_{(p)}$. So, $\mathrm{Proj}\, S$ is a locally ringed space;*
2. *for any homogeneous $f \in S_+$ set $D_+(f) = \{p \in \underline{\mathrm{Proj}} S \quad | \quad f \notin p\}$. Then $D_+(f)$ is open in $\underline{\mathrm{Proj}} S$ and for different f these sets form a base of topology on $\underline{\mathrm{Proj}} S$. Moreover, $(D_+(f), \mathcal{O}|_{D_+(f)}) \simeq \mathrm{Spec}\, S_{(f)}$, i.e. $\mathrm{Proj}\, S$ is a scheme.*

Example 9.3. The typical example of a projective scheme is the scheme $\mathbb{P}^n_A := \mathrm{Proj}\, A[T_0,\dots,T_n]$ — a projective space over a ring A.

For any graded ring S denote by $S^{(d)} := \oplus_{n\geq 0} S_{nd}$ the "Veronese ring". The following theorem from [30, Prop. 2.4.7] will be important for us:

Theorem 9.1. *Let $S = \oplus_{d\geq 0} S_d$ be a graded ring. For all $d > 0$ there exists a canonical isomorphism* $\mathrm{Proj}\, S \simeq \mathrm{Proj}\, S^{(d)}$.

Proof. First let's show that the map $\wp \mapsto \wp \cap S^{(d)}$ induce a bijection $\underline{\mathrm{Proj}} S \to \underline{\mathrm{Proj}} S^{(d)}$. Indeed, let $\wp'$ be a homogeneous prime ideal of $S^{(d)}$ and put $\wp_{nd} := \wp' \cap S_{nd}$ for all $n \geq 0$. For all $n > 0$ not divisible by d define $\wp_n := \{x \in S_n | \quad x^d \in \wp_{nd}\}$. If $x, y \in \wp_n$, then $(x+y)^{2d} \in \wp_{2nd}$, hence $(x+y)^d \in \wp_{nd}$, since $\wp'$ is prime. So, $\wp_n$ are additive subgroups of S_n. They form a homogeneous prime ideal in S, as it follows from the next lemma.

Lemma 9.1. *Let n_0 be a positive integer. For all $n \geq n_0$ let $\wp_n$ be a subgroup of S_n. Then there exists a homogeneous prime ideal $\wp \in \underline{\mathrm{Proj}} S$ such that $\wp \cap S_n = \wp_n$ for all $n \geq n_0$ if and only if the following conditions hold:*

1. *$S_m \wp_n \subset \wp_{n+m}$ fro all $m \geq 0$ and $n \geq n_0$;*
2. *for $m, n \geq n_0$, $f \in S_m$, $g \in S_n$ the relation $fg \in \wp_{n+m}$ imply $f \in \wp_m$ or $g \in \wp_n$;*
3. *for $n \geq n_0$ $\wp_n \neq S_n$.*

Besides, if such an ideal exists, then it is unique.

Proof. Obviously, conditions from items 1 and 2 are necessary. Besides, since $\wp \not\supseteq S_+$, there exists $k > 0$ such that $\wp \cap S_k \neq S_k$. But then the relation $\wp \cap S_n = S_n$ implies $\wp \cap S_{n-mk} = S_{n-mk}$ (indeed, if $f \in S_k$, $f \notin \wp$, and $g \in S_{n-mk}$, $g \notin \wp$, then $gf^m \in \wp$, a contradiction). Therefore, the relation $\wp \cap S_n = S_n$ at least for one n imply $\wp \supset S_+$, which contradicts condition from item 3, i.e. this condition is also necessary.

Conversely, assume the conditions 1,2,3 hold. Note that, if $\wp$ exists, and $f \in S_d$, $d \geq n_0$ is such that $f \notin \wp$, then for all $m < n_0$ the homogeneous component $\wp_m$ consists of elements $x \in S_m$ such that $f^r x \in \wp_{m+rd}$ for almost all $r \geq 0$. This already proves that $\wp$ is unique if it exists. Let's prove that $\wp = \sum_{i=0}^{\infty} \wp_i$, where $\wp_m$ with $m < n_0$ are defined as above, is a homogeneous prime ideal.

Note that from item 2 it follows that for $m \geq n_0$ $\wp_m$ can be also defined as the set of $x \in S_m$ such that $f^r x \in \wp_{m+rd}$ for almost all $r \geq 0$. Then, if $g \in S_m$, $x \in \wp_n$, we have $f^r gx \in \wp_{m+n+rd}$ for almost all $r \geq 0$, whence $xg \in \wp_{m+n}$, i.e. $\wp$ is a homogeneous ideal. To prove that $\wp$ is prime, or equivalently that the graded ring $S/\wp = \oplus_{i \geq 0} S_i/\wp_i$ is an integral domain, it suffices to prove that if $x \in S_m$, $y \in S_n$ are such that $x \notin \wp_m$, $y \notin \wp_n$, then $xy \notin \wp_{n+m}$. Assume the converse. Then for r big enough $f^{2r} xy \in \wp_{n+m+2rd}$. But $f^r y \notin \wp_{n+rd}$ for all r, therefore from item 2 we have: $f^r x \in \wp_{m+rd}$ for almost all $r \geq 0$, i.e. $x \in \wp_m$, a contradiction. □

Since for all homogeneous $f \in S_+$ we have $D_+(f) = D_+(f^d)$ the bijection $\wp \mapsto \wp \cap S^{(d)}$ is a *homeomorphism* of topological spaces $\underline{\mathrm{Proj}} S \simeq \underline{\mathrm{Proj}} S^{(d)}$. Finally, the rings $S_{(f)}$ and $S_{(f^d)}$ can be canonically identified, where from $\mathrm{Proj}\, S$ and $\mathrm{Proj}\, S^{(d)}$ can be canonically identifeid as *schemes.* □

At last, the following result from commutative algebra ([7, Ch.III, § 1.3, prop. 3]) will be important for us:

Proposition 9.2. *Let R be a graded ring finitely generated over $K = R_0$. Then there exists $d > 0$ such that the graded ring $R^{(d)} := \oplus_{i \geq 0} R_{id} \subset R$ is finitely generated by its first graded component $R_1^{(d)} = R_d$ as a $R_0^{(d)} = K$-algebra. More precisely, there exists an integer $e \geq 1$ such that $R^{(me)} = R_0[R_{me}]$ for any $m \geq 1$.*

Proof. Let $\{x_j\}_{1 \leq j \leq s}$ be a system of homogeneous generators of the R_0-algebra R (with the degrees ≥ 1). Set $h_j = \deg(x_j)$ and let q be the common multiple of the numbers h_j and set $q_j := q/h_j$, $1 \leq j \leq s$. Then $\deg(x_j^{q_j}) = q$. Let B be a R_0-subalgebra of the ring R generated by the elements $x_j^{q_j}$; this is a graded subalgebra in R and $B_i = 0$ if i is not divisible by q. Let

A (correspondingly S) be a graded ring, which coincides with B (correspondingly $R^{(q)}$) as a usual ring, and the grading is given by $A_i = B_{iq}$ (correspondingly $S_i = R_{iq}$). By definition of B we have $A = A_0[A_1]$. Consider finite number of elements from R of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$, where $0 \le \alpha_j < q_j$ and $\alpha_1 h_1 + \ldots + \alpha_s h_s = 0 \mod q$; we claim they generate the B-module $R^{(q)}$. It suffices to prove that any element from $R^{(q)}$ of the form $x_1^{n_1} x_2^{n_2} \ldots x_s^{n_s}$ is a B-linear combination of the aforementioned elements. Indeed, there are positive integers k_j, r_j that satisfy the equalities $n_j = k_j q_j + r_j$ and $r_j < q_j$, $1 \le j \le s$; therefore

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s} = (x_1^{q_1})^{k_1} (x_2^{q_2})^{k_2} \cdots (x_s^{q_s})^{k_s} \cdot (x_1^{r_1} x_2^{r_2} \cdots x_s^{r_s})$$

and, according to our assumption, $\sum n_j h_j = 0 \mod q$, whence $\sum r_j h_j = 0 \mod q$. Since the elements $x_j^{q_j}$ belong to the ring B, our claim is proved. The rest of the proof follows from the claim:

Claim. There exists $n_0 >$ such that for $d \ge n_0$ the equality $S^{(d)} = S_0[S_d]$ holds (hence, $R^{(qd)} = R_0[R_{qd}]$ for $d \ge n_0$ and we can set $e = qn_0$).

It suffice to prove that for all $n \ge n_0$ and $k \ge 0$ $S^{(d)}_{n+k} = A_k S^{(d)}_n$. Indeed, in this case evidently $S^{(d)}_{n+k} = S^{(d)}_k S^{(d)}_n$. For $l \ge n_0$ and $m > 0$ by induction we then have $S^{(d)}_{ml} = (S^{(d)}_l)^m$.

Since $S^{(d)}$ is an A-module of finite type, there exists n_0 such that all degrees of homogeneous generators $\{y_j\}$ of $S^{(d)}$ over A are less that n_0. Let $n \ge n_0$ and $x \in S^{(d)}_{n+1}$. Since the elements y_j generate $S^{(d)}$, there exists a set $\{a_j\}$ of homogeneous elements from A with $\deg a_j = n+1-\deg(y_j)$ such that $x = \sum a_j y_j$. Since $A = A_0[A_1]$ and $\deg a_j > 0$, any element a_j is a sum of elements of the form bb', where $b \in A_1$ and $b' \in A$, where from $x \in A_1 S^{(d)}_n$. Therefore, $S^{(d)}_{n+1} = A_1 S^{(d)}_n$ and $S^{(d)}_{n+k} = A_k S^{(d)}_n$ by induction. □

Corollary 9.1. *Let R be a graded integral domain as in proposition 9.2 with $R_0 = K$. Then* $\operatorname{Proj} R$ *defines a projective variety.*

Proof. The proof immediately follows from theorem 9.1 and remark 9.1. □

Remark 9.2. The general proposition 8.3 holds also for projective schemes.

Exercise 9.3. 1) Let S be a graded ring. Show that $\operatorname{Proj} S = \emptyset$ iff all elements from S_+ are nilpotent.

2) Consider two graded rings R_1, R_2. Let $\varphi : R_2 \to R_1$ be a homomorphism of graded rings (i.e. $\varphi(R_{2,d}) \subseteq R_{1,d}$). Set $U = \{\wp \in \operatorname{Proj} R_1 | \quad \wp \not\supseteq \varphi((R_2)_+)\}$. Show that U is an open subset in $\operatorname{Proj} R_1$ and that φ defines a morphism of schemes $\varphi^* : U \to \operatorname{Proj} R_2$.

Analogously to the case of affine varieties and schemes, we can define a sheaf associated with a graded module on a projective scheme.

Definition 9.2. Let S be a graded ring and M be a graded S-module. Define a sheaf $\operatorname{Proj} M$ on $\operatorname{Proj} S$ *associated with a graded module* M as follows:

$$\begin{aligned}(\operatorname{Proj} M)(U) := \{&\text{set of maps} \quad s : U \to \coprod_{P \in U} M_{(P)} \quad \text{s.t. } s(P) \in M_{(P)}\ \forall P \in U \text{ and } \forall P \in U \\ &\exists V \subset U,\ V \ni P \text{ and } \exists f \in M_d,\ b \in M_d \text{ s.t. } s(q) = f/b \in M_{(q)} \text{ and } b \notin q\ \forall q \in V\}.\end{aligned}$$

Such a sheaf has similar properties:

Proposition 9.3. *[32, Ch.2, prop. 5.11] Let S be a graded ring and M be a graded S-module. Let $X = \operatorname{Proj} S$. Then*

1. *for any $p \in X$ the stalk $(\operatorname{Proj} M)_p$ is equal to $M_{(p)}$;*

2. *for any homogeneous element $f \in S_+$ there is an isomorphism* $(\operatorname{Proj} M)|_{D_+(f)} \simeq (M_{(f)})^{\sim}$ *which is constructed with the help of the isomorphism* $D_+(f) \simeq \operatorname{Spec} S_{(f)}$ *;*

3. $\operatorname{Proj} M$ *is a quasi-coherent* $\mathcal{O}_X$ *-module. If* S *is noetherian and* M *is finitely generated, then the sheaf* $\operatorname{Proj} M$ *is coherent.*

Exercise 9.4. Prove that an analogue of theorem 9.1 hold for sheaves $\operatorname{Proj} M$. Namely, let $S = \oplus_{d\geq 0} S_d$ be a graded ring, M be a graded S-module. Then for all $d > 0$ $(i_d)_*(\operatorname{Proj} M^{(d)}) \simeq \operatorname{Proj} M$, where i_d is the isomorphism of schemes $\operatorname{Proj} S^{(d)} \simeq \operatorname{Proj} S$.

At last, we will need one more result about associated sheaves. To formulate it, let's first define the twisting sheaves of Serre.

Definition 9.3. Let S be a graded ring and $X = \operatorname{Proj} S$. For any $n \in \mathbb{Z}$ we define the sheaf $\mathcal{O}_X(n)$ as $\operatorname{Proj} S(n)$, where $S(n)$ is a graded S-module with the following grading: $S(n)_k := S_{n+k}$ (we set to 0 all undefined components). For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we'll denote by $\mathcal{F}(n)$ the twisted sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 9.4. *[32, Ch. 2, prop. 5.12] Let* S *be a graded ring and* $X = \operatorname{Proj} S$*. Assume that* S *is generated by* S_1 *as a* S_0*-algebra. Then*

1. *the sheaf* $\mathcal{O}_X(n)$ *is an invertible sheaf on* X*, i.e. it is locally free of rank one at each point;*

2. *for any graded* S*-module* M *we have* $(\operatorname{Proj} M)(n) \simeq \operatorname{Proj}(M(n))$*, in particular* $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$*;*

3. *let* T *be another graded ring generated by* T_1 *as* T_0*-algebra,* $\varphi : S \to T$ *is a homogeneous homomorphism of graded rings,* $U \subset Y = \operatorname{Proj} T$ *is an open subset and* $f : U \to X$ *is a morphism defined by the homomorphism* φ*. Then* $f^*(\mathcal{O}_X(n)) \simeq \mathcal{O}_Y(n)|_U$ *and* $f_+(\mathcal{O}_Y(n)|_U) \simeq (f_*\mathcal{O}_U)(n)$*.*

The operation of twisting permits to define a graded S-module associated with any sheaf of modules on a scheme $X = \operatorname{Proj} S$, which is closely related with the original sheaf.

Definition 9.4. Let S be a graded ring, $X = \operatorname{Proj} S$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Define a *graded* S*-module associated with* $\mathcal{F}$ as the group $\Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$, where $\Gamma(X, \mathcal{F}(n)) := \mathcal{F}(n)(X)$. The structure of a graded S module is given as follows. Each element $s \in S_d$ naturally defines a section $s \in \Gamma(X, \mathcal{O}_X(d))$. Then for any $t \in \Gamma(X, \mathcal{F}(n))$ the product $s \cdot t \in \Gamma(X, \mathcal{F}(n+d))$ is defined as a tensor product $s \otimes t$ via the isomorphism $\mathcal{F}(n) \otimes \mathcal{O}_X(d) \simeq \mathcal{F}(n+d)$.

Proposition 9.5. *[32, Ch.2, prop. 5.15] Let* S *be a graded ring finitely generated by* S_1 *as a* S_0*-algebra. Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* $X = \operatorname{Proj} S$*. Then there exists a natural isomorphism* $\beta : \operatorname{Proj}(\Gamma_*(\mathcal{F})) \to \mathcal{F}$*.*

9.3 Some extra properties of schemes

Of course, there is a rich theory of schemes which is not presented in these notes. We strongly recommend to learn about it at least from the book [32]. Here we present, for a completeness of the picture, some extra properties (some of them we'll need further).

Definition 9.5. A scheme is called *connected* if its topological space is connected. A scheme is called *irreducible* if its topological space is irreducible.

Definition 9.6. A scheme X is called *reduced* if for any open subset U the ring $\mathcal{O}_X(U)$ has no non-zero nilpotents.

A scheme X is called *integral* if if for any open subset U the ring $\mathcal{O}_X(U)$ is an integral domain.

Proposition 9.6. *[32, Ch. II, Prop. 3.1] A scheme X is integral iff it is reduced and irreducible.*

Definition 9.7. A scheme X is called *locally noetherian* if there is a covering by open affine subsets $\operatorname{Spec} A_i$, where A_i are noetherian rings. A scheme X is called *noetherian* if it is locally noetherian and quasicompact.

Remark 9.3. If X is a noetherian scheme, then its topological space is also noetherian, i.e. for any chain of closed subspaces $Y_1 \supset Y_2 \supset \dots$ there exists $r \in \mathbb{N}$ such that $Y_r = Y_{r+1} = \dots$. The converse is not true, see [32, Ch. 2, §3].

Definition 9.8. A morphism of schemes $f : X \to Y$ is called *a morphism of locally finite type*, if there is a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for any set $f^{-1}(V_i)$ there is a covering by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where A_{ij} are finitely generated B_i-algebras. The morphism f is called *a morphism of finite type* if $f^{-1}(V_i)$ can be covered by a *finite number* of subsets U_{ij}.

Definition 9.9. An *open subscheme* of a scheme X is a scheme U whose topological space is an open subset in X and $\mathcal{O}_U \simeq \mathcal{O}_X|_U$. An *open immersion* is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme in Y.

Each open subset of a scheme can endowed by a unique structure of an open subscheme (cf. [32, Ch. 2, §3]).

Definition 9.10. A *closed subscheme* of a scheme X is an *equivalence class* of pairs (Y, i), where Y is a scheme and $i : Y \to X$ is an embedding such that the image of the topological space Y is closed in X, and the induced morphism of sheaves $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective. Two pairs (Y, i), (Y', i') are equivalent, if there exists an isomorphism $f : Y \simeq Y'$ such that $i = i' \circ f$. A *closed immersion* is a morphism $f : Y \to X$ that induces an isomorphism of Y on a closed subscheme in X.

Definition 9.11. A *dimension* of a scheme is the dimension of its topological space (see definition 7.3).

Definition 9.12. Let S be a scheme and X, Y be schemes over S, i.e. there are morphisms to S. A *fibred product* of X and Y over S is a scheme $X \times_S Y$ together with morphisms $p_1 : X \times_S Y \to X$, $p_2 : X \times_S Y \to Y$ such that for any scheme Z over S and morphisms $f : Z \to X$, $g : Z \to Y$ compatible with morphisms $X \to S$, $Y \to S$ (i.e. the diagram of morphisms is commutative), there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$.

Theorem 9.2. *[32, Ch. 2, T. 3.3.] For any two schemes X and Y over S the fibred product $X \times_S Y$ exists and is unique up to an isomorphism.*

Definition 9.13. Let $f : X \to Y$ be a morphism of schemes. A *diagonal morphism* is a uniquely determined morphism $\triangle : X \to X \times_Y X$ such that its composition with each projection $p_i : X \times_Y X \to X$ is the identity morphism $X \to X$. The morphism f is called *separated* if the diagonal morphism $\triangle$ is a closed embedding. In such case we'll say that the scheme X is *separated over Y*.

Proposition 9.7. *[32, Ch.2, prop. 4.1, cor. 4.6] Let $f : X \to Y$ be a morphism of affine schemes or an open or a closed immersion. Then f is separated.*

For affine schemes this proposition is rather easy: we have $X\times_Y X = \operatorname{Spec} A\otimes_B A$, where $X=\operatorname{Spec} A$, $Y=\operatorname{Spec} B$, and the diagonal morphism is defined by the surjective homomorphism of multiplication $f: A\otimes_B A\to A$, $a\otimes a'\mapsto aa'$. Thus, $X\simeq \operatorname{Spec}(A\otimes_B A/\ker f)$ and $\triangle$ is a closed embedding, because $\triangle(X)=V(\ker f)$ is a closed subset.

Exercise 9.5. Let X be a separated scheme. Show that $U_1\cap U_2$ is affine if U_1, U_2 are affine open subsets. Hint: use that $\triangle$ is a closed embedding and that a closed subset of an affine scheme is affine.

Remark 9.4. It is more convenient to replace the definition of algebraic variety by the following definition. An (abstract) algebraic variety is an *integral separated scheme over a field k of finite type.*

This definition is in a sense analogous to the definition of an abstract smooth manifold. It is convenient, in particular, because we can glue several such schemes. The definition of glueing schemes can be given as follows.

Let X_1, X_2 be two schemes. Assume that $U_1\subset X_1$, $U_2\subset X_2$ are two open subsets endowed with the induced scheme structure, such that $U_1\simeq U_2$. Then there exists a new scheme X whose topological space is a space glued from X_1, X_2, $X=X_1\cup_{U_1\sim U_2} X_2$, and there are two open embeddings $i_{1,2}: X_i\hookrightarrow X$ such that $X=i_1(X_1)\cup i_2(X_2)$, $i_j(U_j)=i_1(X_1)\cap i_2(X_2)$.

Exercise 9.6. Show that such a scheme X exists.

Remark 9.5. Another useful notion in algebraic geometry is a notion of a *relatively very ample sheaf.* Namely, let X be a scheme over another scheme Y. An invertible sheaf $\mathcal{L}$ on X is called very ample with respect to Y if there exists an immersion $i: X\hookrightarrow \mathbb{P}^r_Y$ for some r (X is called *projective* over Y if i is a closed immersion) such that $i^*\mathcal{O}_{\mathbb{P}}(1)\simeq\mathcal{L}$.

In particular, if $X=\operatorname{Proj} S$ is a projective variety over K, then the sheaf $\mathcal{O}_X(1)$ is very ample sheaf with respect to $\operatorname{Spec} K$. In general, if S is a finitely generated graded ring over $K=S_0$ and $S^{(d)}$ is a Veronese ring from proposition 9.2, then the sheaf $\mathcal{O}_X(d)$, where $X=\operatorname{Proj} S$, is a very ample sheaf with respect to $\operatorname{Spec} K$. Note that $\mathcal{O}_X(d)\simeq (i_d)_*(\mathcal{O}_{X'}(1))$, where $X'=\operatorname{Proj} S^{(d)}$ and i_d is the isomorphism $\operatorname{Proj} S^{(d)}\simeq\operatorname{Proj} S$, cf. exercise 9.4

9.4 Locally free sheaves and vector bundles

In this subsection we would like to explain the relation between locally free sheaves and vector bundles over algebraic varieties. Namely, the isomorphism classes of locally free sheaves are in one to one correspondence with the isomorphism classes of vector bundles, i.e. we can "identify" them.

Definition 9.14. Let X be an algebraic variety. A *vector bundle of rank n* over X is a variety Y together with a morphism $f: Y\to X$ and with the following additional structure: there is an open covering $X=\cup_i U_i$ and isomorphisms $\psi_i: f^{-1}(U_i)\to\mathbb{A}^n_K\times U_i$ such that for any i,j and any open affine subset $V=\operatorname{Spec}(A)\subset U_i\cap U_j$ the automorphism $\psi=\psi_j\circ\psi_i^{-1}$ of the space $\mathbb{A}^n_K\times V=\operatorname{Spec} A[x_1,\dots,x_n]$ is given by a linear automorphism θ of the algebra $A[x_1,\dots,x_n]$, i.e. $\theta(a)=a$ for any $a\in A$ and $\theta(x_i)=\sum a_{ij}x_j$, $a_{ij}\in A$.

An isomorphism $g:(Y,f,\{U_i\},\{\psi_i\})\to(Y',f',\{U'_i\},\{\psi'_i\})$ of vector bundles of rank n is an isomorphism $g: Y\to Y'$ such that $f=f'\circ g$ and Y, f together with the covering of X consisting of all open U_i, U'_i and isomorphisms ψ_i, $\psi'_i\circ g$ also define a structure of a vector bundle on Y.

Construction. Let $\mathcal{F}$ be a locally free sheaf of rank r and let $\{U_i\}$ be a covering $X=\cup_i U_i$ such that $\mathcal{F}(U_i)\simeq\mathcal{O}_X(U_i)^{\oplus r}$. Consider the symmetric algebra $S(\mathcal{F}(U_i))=(\oplus_{n\geq 0}\mathcal{F}(U_i)\otimes_{\mathcal{O}_X(U_i)}$

$\ldots \otimes_{\mathcal{O}_X(U_i)} \mathcal{F}(U_i))/(x\otimes y - y\otimes x)$ — this is an $\mathcal{O}_X(U_i)$-algebra. If we choose a basis $\{x_1,\ldots,x_n\}$ of $\mathcal{F}(U_i)$ over $\mathcal{O}_X(U_i)$, then there is a natural isomorphism

$$S(\mathcal{F}(U_i)) \simeq \mathcal{O}_X(U_i)[x_1,\ldots,x_n].$$

Obvious homomorphisms of rings $f^*_{U_i} : \mathcal{O}_X(U_i) \to S(\mathcal{F}(U_i))$ define the morphisms of corresponding affine varieties $f_{U_i} : \operatorname{Spec} S(\mathcal{F}(U_i)) \to U_i$, and we have natural isomorphisms $\psi_i : \mathbb{A}^n_K \times U_i = \operatorname{Spec} \mathcal{O}_X(U_i) \otimes_K K[x_1,\ldots,x_n] \to \operatorname{Spec} S(\mathcal{F}(U_i))$.

Symmetric algebras $S(\mathcal{F}(U_i))$ form a sheaf $S(\mathcal{F})$ of $\mathcal{O}_X$-algebras, and varieties $\operatorname{Spec}(\mathcal{F}(U_i))$ can be glued together to form a variety E (it can be not affine, but projective or quasiprojective) with a morphism $E \to X$ — a vector bundle $V(\mathcal{F})$ (all other data are already defined).

Vice versa: If $f : E \to X$ is a vector bundle of rank n, then we can construct the sheaf of sections

$$U \mapsto \{\text{the set of sections of } f \text{ over } U\text{, i.e. morphisms } s : U \to E \text{ s.t. } f\circ s = id_U\}.$$

Direct check shows that it is a sheaf of $\mathcal{O}_X$-modules, called $\mathcal{F}(E)$, which is locally free of rank n.

Remark 9.6. We would like to emphasize that $\mathcal{F}(V(\mathcal{F})) \simeq \mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$ (not $\mathcal{F}$ itself!). The sheaf $\mathcal{F}^\vee$ is defined as

$$U \mapsto Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U)$$

(and it is easy to see that $Hom_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{O}_X|_U)$ are $\mathcal{O}_X(U)$-modules). If we are given $s \in \mathcal{F}^\vee(V)$ for an open affine V, then s defines a homomorphism of $\mathcal{O}_V$-algebras $S(\mathcal{F}|_V) \to \mathcal{O}_V$ which defines a morphism of affine varieties $V \to f^{-1}(V) = \operatorname{Spec} S(\mathcal{F}|_V)$, i.e. a section of the vector bundle $V(\mathcal{F}) \to X$. This construction establishes the isomorphism $\mathcal{F}(V(\mathcal{F})) \simeq \mathcal{F}^\vee$.

One important special case of locally free sheaves are invertible sheaves, i.e. l.f. sheaves of rank one. These sheaves form a group with respect to the tensor product operation: the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is again an invertible sheaf for any invertible $\mathcal{F},\mathcal{G}$; for any invertible $\mathcal{F}$ its inverse is $\mathcal{F}^{-1} := \mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$ and the unity is, clearly, the sheaf $\mathcal{O}_X$. All these properties can be easily seen from standard sheaves properties and from the following exercise.

Exercise 9.7. Let $(X,\mathcal{O}_X)$ be a ringed space, $\mathcal{E}$ is a locally free sheaf on X.

1. Show that $(\mathcal{E}^\vee)^\vee \simeq \mathcal{E}$ and $\mathcal{O}_X \simeq \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{O}_X)$;
2. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \simeq \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$;
3. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}\otimes\mathcal{F},\mathcal{G}) \simeq \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E},\mathcal{G}))$;
4. If $f : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a morphism of ringed spaces, then $f_*(\mathcal{F}\otimes_{\mathcal{O}_X} f^*\mathcal{E}) \simeq f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}$.

Definition 9.15. For any ringed space X the *Picard group* $\operatorname{Pic} X$ is a group of isomorphism classes of invertible sheaves.

The Picard group is closely related with another important group of *Cartier divisors.*

Definition 9.16. *A Cartier divisor* on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*_X$, where $\mathcal{O}^*_X$ denotes the sheaf of groups of invertible invertible functions, $\mathcal{K}$ denotes the constant sheaf of rational functions on X (i.e. for any affine $U = \operatorname{Spec} A \subset X$ $\mathcal{K}(U) = A\cdot S^{-1}$, where S is the set of all non-zero divisors in A) and $\mathcal{K}^*$ denotes the sheaf of groups of invertible elements of the sheaf $\mathcal{K}$.

In other words, a Cartier divisor is given by the following data: an open covering $\{U_i\}$ of X and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ for any i, j. A Cartier divisor is *principal* if it it belongs to the image of the natural map $\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$. Two Cartier divisors are called *linearly equivalent*, if their difference is a principal Cartier divisor (note that the group structure on $\mathcal{K}^*/\mathcal{O}_X^*$ is multiplicative).

Definition 9.17. Let D be a Cartier divisor on a scheme X, represented by a system $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{L}(D) \subset \mathcal{K}$ as an $\mathcal{O}_X$-submodule in $\mathcal{K}$ generated by the functions f_i^{-1} over U_i. Since the functions f_i/f_j are invertible on $U_i \cap U_j$, the functions f_i^{-1}, f_j^{-1} generate the same $\mathcal{O}_X(U_i \cap U_j)$-module and therefore the sheaf $\mathcal{L}(D)$ is well defined.

Proposition 9.8. *[32, Ch.2, prop. 6.13, 6.15] Let X be a scheme. Then*

1. *for any Cartier divisor D the sheaf $\mathcal{L}(D)$ is an invertible sheaf on X; the map $D \mapsto \mathcal{L}(D)$ is a one to one correspondence between Cartier divisors on X and invertible subsheaves of the sheaf $\mathcal{K}$;*

2. *$\mathcal{L}(D_1 - D_2) \simeq \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$;*

3. *$D_1 \sim D_2$ iff $\mathcal{L}(D_1) \simeq \mathcal{L}(D_2)$ as abstract invertible sheaves (i.e. not as embedded sheaves);*

4. *on any scheme X the map $D \mapsto \mathcal{L}(D)$ gives an injective homomorphism of the group $CaCl(X)$ of the Cartier divisor class group modulo linear equivalence to the group $\operatorname{Pic} X$;*

5. *for an integral scheme X the homomorphism $CaCl(X) \to \operatorname{Pic} X$ is an isomorphism.*

Remark 9.7. Often geometers deal with the group of Weil divisors $\operatorname{Div} X$ (which are of more geometric nature). Any element of this group has the form $D = \sum n_i Y_i$, where $n_i \in \mathbb{Z}$ and Y_i are integral subschemes of codimension one in X. If X is an integral separated (over $\operatorname{Spec} \mathbb{Z}$) scheme such that all local rings are factorial (a locally factorial scheme), then the groups $\operatorname{Div} X$ and $\Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ are isomorphic, see [32, Ch.2, prop. 6.11]. Weil divisors modulo *principal* Weil divisors form a *divisor class group* $Cl(X)$. Of course, $Cl(X) \simeq \operatorname{Pic}(X) \simeq CaCl(X)$ if X is an integral noetherian locally factorial scheme.

Torsion free sheaves on algebraic curves can be understood as a (natural) generalisation of divisors.

Another useful notion is a tensor, symmetric or antisymmetric product of locally free sheaves.

For any given A-module M denote by $T^n(M) := M \otimes_A \ldots \otimes_A M$ the tensor product of n copies of the module M and by $T(M) = \oplus_{n \geq 0} T^n(M)$ the non-commutative A-algebra (called the tensor algebra; here $T^0(M) := A$). Denote by $S(M)$ and by $\bigwedge(M)$ the symmetric and the exterior algebra correspondingly.

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Define the tensor, symmetric and exterior algebras of the sheaf $\mathcal{F}$ as the sheaves associated to the presheaves $U \mapsto T(\mathcal{F}(U))$ ($S(\mathcal{F}(U))$, $\bigwedge(\mathcal{F}(U))$ correspondingly). These are the sheaves of $\mathcal{O}_X$-algebras, and their homogeneous components are sheaves of $\mathcal{O}_X$-modules. The following exercises are standard, see [32, Ch. 2, §5]:

Exercise 9.8.
1. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank r. Show that the sheaves $T^p(\mathcal{F})$, $S^p(\mathcal{F})$, $\bigwedge^p(\mathcal{F})$ are also locally free of ranks rp, C_{r+p-1}^{r-1}, C_r^p correspondingly.

2. Show that the map $\bigwedge^p(\mathcal{F}) \otimes \bigwedge^{r-p}(\mathcal{F}) \mapsto \bigwedge^r(\mathcal{F})$ is a perfect pairing, i.e. it induces an isomorphism $\bigwedge^p(\mathcal{F}) \simeq (\bigwedge^{r-p}(\mathcal{F}))^\vee \otimes \mathcal{F}^r$.

3. Let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact system of locally free sheaves. Show that for any p there exists a finite filtration of the $\mathcal{O}_X$-module $S^p(\mathcal{F})$
$$S^p(\mathcal{F}) = F^0 \supset F^1 \supset \ldots \supset F^{r+1} = 0$$
with $F^i/F^{i+1} \simeq S^i(\mathcal{F}') \otimes S^{p-i}(\mathcal{F}'')$. The same is true for the antisymmetric products. In particular, if $r' = \operatorname{rk} \mathcal{F}'$, $r'' = \operatorname{rk} \mathcal{F}''$, then $\bigwedge^r(\mathcal{F}) \simeq \bigwedge^{r'}(\mathcal{F}') \otimes \bigwedge^{r''}(\mathcal{F}'')$.

The sheaf $\bigwedge^r(\mathcal{F})$ for a locally free sheaf $\mathcal{F}$ of rank r is called *the determinantal sheaf*, and the corresponding line bundle is called the determinantal bundle.

9.5 A brief introduction to cohomology of sheaves

In this subsection we collect all necessary fact about cohomology groups with coefficients in sheaves. We recommend to learn more about them e.g. in [32]. For our aims it is enough to introduce only Čech cohomology because of their practical usefulness for us.

Let X be a topological space and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. Fix an ordering of I. For any indices $i_0, \ldots, i_p \in I$ denote by $U_{i_0,\ldots,i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$. Let $\mathcal{F}$ be a sheaf of abelian groups on X. For any $p \geq 0$ put $C^p(\mathcal{U}, \mathcal{F}) = \Pi_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0,\ldots,i_p}$. Define the coboundary map $d : C^p \to C^{p+1}$ by
$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$
It can be easily checked that $d^2 = 0$ and therefore we have a complex of abelian groups $\ldots \to C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}) \to C^{p+2}(\mathcal{U}, \mathcal{F}) \to \ldots$ (it depends on the covering $\mathcal{U}$).

Definition 9.18. We define the p-th Čech cohomology group of a sheaf $\mathcal{F}$ with respect to a covering $\mathcal{U}$ as
$$\check{H}^p(\mathcal{U}, \mathcal{F}) := H^p(C^{\cdot}(\mathcal{U}, \mathcal{F})).$$

The first main result about Čech cohomology groups claims that in good cases they are independent of the covering.

Theorem 9.3. *[32, Ch. 3, Th. 4.5] Let X be a noetherian separated scheme, $\mathcal{U}$ be an open affine covering of X and $\mathcal{F}$ be any quasi-coherent sheaf on X. Then for any $p \geq 0$ the Čech cohomology groups don't depend on the covering, and we'll denote them by $H^p(X, \mathcal{F})$.*

Remark 9.8. In fact, these cohomology groups are naturally isomorphic to other cohomology groups defined via the right derived functor of global sections by Grothendieck, see [32]. Moreover, it is known that on a paracompact Hausdorff space Čech cohomology groups of a constant sheaf are isomorphic to other well known in topology *singular cohomology groups*.

The second main result about Čech cohomology groups is standard for all cohomology theories.

Theorem 9.4. *Let*
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
be a short exact sequence of sheaves. Then there is a long exact sequence of cohomology groups
$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \\ \to H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2) \to H^1(X, \mathcal{F}_3) \to \ldots.$$

Exercise 9.9. 1) Let $X=\mathbb{P}^1$. Calculate cohomology groups of $\mathcal{O}_X$.

2) Calculate cohomology groups of a *skyscraper sheaf* T on a scheme X with support $q\in X$, where T is defined as $T(U):=K$ if $q\in U$, and $T(U):=0$ if $q\notin U$ (check that it is really a sheaf). Show that this sheaf is isomorphic to the factor sheaf $\mathcal{O}_X/\mathcal{I}_q$, where $\mathcal{I}_q$ denotes the ideal sheaf of the point q. By [32, Ch. 2, Prop. 5.7] this factor sheaf is a quasi-coherent sheaf on X (try to prove it by yourself).

10 Projective spectral data and classification theorem

In this section we explain how to extend the affine spectral data to a *projective spectral data* consisting of objects from projective geometry. The classification theorem, one of our main aims, states that there is a one to one correspondence between classes of equivalent elliptic commutative subrings in D of rank r and isomorphism classes of projective spectral data of rank r. There are several versions of projective spectral data and of classification theorem. The first version is due to Krichever [38], [39] and has a more analytical nature. The other versions (have more algebraic nature) are due to Mumford [71], Drinfeld, Verdier [114] and Mulase [66] (cf. also an important paper by Segal and Wilson [101]). We'll explain first the most general algebraic form of this theorem and then explain the analytic point of view. The algebraic form which is presented here has one advantage: it can be generalised to commuting *partial differential operators*, cf. [120].

The structure of the classification theorem can be explained by the following diagram:

$$\begin{tikzcd}
{[B\subset D]} \arrow[rr, "\text{direct map}"'] \arrow[dr, bend left, "\text{Schur theory}"] & & {[\text{Projective spectral data}]} \arrow[ll, bend right, "\text{inverse map}"'] \arrow[dl, bend right, "\text{Krichever map}"'] \\
& {[\text{Schur pairs}]} \arrow[ul, bend left, "\text{Sato theory}"] \arrow[ur, bend right, "\text{direct map}"'] &
\end{tikzcd} \tag{24}$$

where $[B]$ denotes an equivalence class of an elliptic commutative subring, and other data will be defined below. Two elliptic subrings B_1,B_2 are *equivalent*, if there is a function $f\in K[[x]]^*$ such that $B_1=f^{-1}B_2f$. As it follows from remark 4.4 and exercise 4.3, each equivalence class contains a unique normalised subring of special type.

The direct maps are just applications of general algebro-geometric Grothendieck constructions, as it will be explained in subsection 10.2. All other arrows will be explained in corresponding subsections 10.1, 11.1, 11.2 as well, except the inverse map, which can be obtained via the analytic theory from section 12. All arrows in these diagram are one-to-one correspondences.

10.1 Schur pairs

In this subsection we explain the map "Schur theory" from diagram (24).

Let $B\subset D$ be an elliptic subring. Then by Schur theory 5.1, 5.1 there exists an invertible operator $S=s_0+s_1\partial^{-1}+\ldots\in E$ such that $A:=S^{-1}BS\subset K((\partial^{-1}))$. Consider the homomorphism

$$E\to E/xE\simeq K((\partial^{-1})) \tag{25}$$

(sometimes it is called *the Sato homomorphism*). It defines a structure of an E-module on the space $K((\partial^{-1}))$: for any $P\in K((\partial^{-1}))$, $Q\in E$ $p\cdot Q=PQ \pmod{xE}$.

Now define the space $W:=F\cdot S\subset K((\partial^{-1}))$. Note that W is an A-module, where the module structure is defined via the multiplication in the *field* $K((\partial^{-1}))$ and this module structure is induced by the E-module structure on $K((\partial^{-1}))$, because $K((\partial^{-1}))\subset E$ and $W\cdot A=(F\cdot S)\cdot(S^{-1}BS)=F\cdot(BS)=(F\cdot B)\cdot S=F\cdot S=W$. Note also that the modules W and F are isomorphic (W is an A-module, F as a B-module, and clearly $A \simeq B$). For convenience of notation, we will replace ∂^{-1} by z in the field $K((\partial^{-1}))$, i.e. $A, W \subset K((z)) \simeq K((\partial^{-1}))$.

For subrings in $K((z))$ we can introduce the same notion of rank as for subrings in D:

Definition 10.1. Let A be a K-subalgebra of $K((z))$, and $r \in \mathbb{N}$. A is said to be an algebra of rank r if $r = \gcd(\operatorname{ord}(a)| \quad a \in A)$, where the order is the same as in D.

Exercise 10.1. $A \subset K((z))$ is a K-subalgebra of rank r if and only if there is a monic element $y \in K[[z]]$ of order $-r$ such that

- $A \subset K((y))$,
- $K((y))/(A + K[[y]])$ is a finitely generated K-module.

Definition 10.2. Let W be a K-subspace in $K((z))$. The *support* of an element $w \in W$ is its highest symbol, i.e. $\sup(w) := HT(w)z^{-\operatorname{ord}(w)}$. The *support* of the space is $\operatorname{Supp} W := \langle \sup(w)| \quad w \in W \rangle$.

Definition 10.3. An embedded Schur pair of rank r is a pair (A, W) consisting of

- $A \subset K((z))$ a K-subalgebra of rank r satisfying $A \cap K[[z]] = K$;
- $W \subset K((z))$ a K-subspace with $\operatorname{Supp} W = K[z^{-1}]$

such that $W \cdot A \subseteq W$.

So, to any elliptic subring $B \subset D$ we can associate an embedded Schur pair.

Definition 10.4. Two embedded Schur pairs (A_i, W_i), $i = 1, 2$ of rank r are *equivalent* if there exists an admissible operator T such that $A_1 = T^{-1}A_2T$, $W_1 = W_2 \cdot T$.

Remark 10.1. It is possible to extend the notion of a Schur pair and define a *category* $\mathcal{S}_r$ of Schur pairs, whose objects are Schur pairs of rank r, see [66]. In loc. cit. the embedded Schur pairs were called as just Schur pairs; our terminology comes from [88], where the relative version of the classification theory was given. In [88] another version of the Schur pair was used.

Namely, by a Schur pair of rank r there the author meant a pair (A, W) consisting of elements $A \in K((z))$, and $W \in K((z))^{\oplus r}$ such that

- A is a K-subalgebra of $K((z))$ and $A \cap K[[z]] = K$,
- the natural action of $K((z))$ on $K((z))^{\oplus r}$ induces an action of A on W s.t. $A \cdot W \subseteq W$.

The relation between Schur pairs is more or less clear (cf. section 10.2 below) and is given in loc. cit.

Remark 10.2. Note that the assignment $B \mapsto (A, W)$ is not uniquely defined. But we claim that the map $[B] \mapsto [(A, W)]$ is well defined.

First, recall that the Schur operator S is defined up to multiplication on an operator with constant coefficients: e.g. we can take another operator $S' = SS_0$, $S_0 \in K((\partial^{-1}))$. In this case, however, $(S')^{-1}BS' = S^{-1}BS = A$.

Second, any Schur operator depends on the choice of a monic operator. If we choose another monic operator from B, then we obtain another Schur operator S' and another ring $A' = (S')^{-1}BS'$. Let's compare them: we have $S'A'(S')^{-1} = SAS^{-1}$, whence $T := S^{-1}S'$ and $T^{-1} = (S')^{-1}S$ are admissible (cf. exercise 5.3). Therefore, $S' = ST$ and $A' = T^{-1}AT$.

Note in both cases $W' = W \cdot T$ for admissible T. At last, note that taking another representative in the equivalence class of B changes the corresponding Schur operators by multiplication on a function: if $B' = f^{-1}Bf$, then we can take $S' = f^{-1}S$, where S is a Schur operator for B. Therefore, $A' = A$ and $W' = F \cdot S' = F \cdot S$ since $F \cdot f = F$ for any invertible function (check it!).

Thus, the map described at the beginning of this section is a well defined map from $[B]$ to $[(A, W)]$.

Exercise 10.2. Here we give an example of all possible embedded Schur pairs for a given ring A.

Consider the ring $A = K[z^{-2}, z^{-3}]$. This is a ring of regular functions of the affine cuspidal curve $y^2 = x^3$. What are the spaces $W \subset K((z))$ such that $A \cdot W \subseteq W$ and $\mathrm{Supp}\, W = K[z^{-1}]$?

Show that all such spaces can be described up to multiplication on $T \in K[[z]]^*$ as follows

$$W_\alpha = \langle z^{-k}, k \geq 1, 1 + \alpha z \rangle, \quad \alpha \in K.$$

So, in this case we obtain a description of all embedded Schur pairs of rank one.

If we apply an isomorphism $z \mapsto cz$, $c \in K$ (a so called "scaling transform", cf. [101, 4.13]) to a given Schur pair (A, W_α), we obtain the Schur pair $(A, W_{c\alpha})$. Note, however, $(A, W_\alpha) \not\cong (A, W_{c\alpha})$, because there are no admissible operators that transform one pair to another.

10.2 Projective spectral data

In this subsection we explain what is the projective spectral data and what are the direct maps in (24).

Given a commutative ring of ODOs $B \subset D$ of rank r, we consider the Rees ring $\tilde{B} = \oplus_{i \geq 0} B_i s^i \subset B[s]$, where $B_i := \{P \in B | \quad \mathrm{ord}(P) \leq ir\}$.

Exercise 10.3. 1) Show that $gr(B) = \oplus_{i \geq 0} B_{i+1}/B_i$ is a subring of a polynomial ring $K[z]$, therefore it is finitely generated.

2) Deduce from 1) that $\tilde{B}$ is also finitely generated over K.

Remark 10.3. Note that a subring in a ring of polynomials in several variables can be not finitely generated. For example, the subring $K[x^i y^j, i \geq 0, j > 0] \subset K[x, y]$ is not finitely generated (check it!).

We define the *projective spectral curve* (a completion of the affine spectral curve C_0) as $C := \mathrm{Proj}\, \tilde{B}$. Notably, our completion of the affine spectral curve can be *not* the usual one.

Exercise 10.4. 1) Consider the affine curve $C_0 = \{y^2 = x^3 - \alpha\}$, $\alpha \neq 0$. Check that the usual closure $\bar{C}_0 = \{zy^2 = x^3 - \alpha z^3\} \subset \mathbb{P}^2$ (this is the so called *elliptic curve*) is a smooth curve.

2) Consider the curve $C_0 = \{y^2 = x^5 - \alpha\}$, $\alpha \neq 0$. Check that the usual closure $\bar{C}_0 = \{z^3 y^2 = x^5 - \alpha z^5\} \subset \mathbb{P}^2$ is a singular curve (at "infinity"). On the other hand, in differential geometry there is a standard way to define a *smooth* completion of C_0 by glueing sheets (spheres) with several cuts along curves connecting different pairs of roots of the polynomial $x^5 - \alpha$ (this is the so called *hyperelliptic curve*). We'll see below that our completion is smooth at "infinity". Thus, this is an alternative algebraic way of constructing smooth completions of curves.

Exercise 10.5. The discrete valuation $-ord$ on D induces a discrete valuation on B and $\mathrm{Quot}(B)$. Now consider the ideal (s) in $\tilde{B}$. Prove that it is prime. Show that $\tilde{B}_{((s))}$ is a DVR with respect to $(-\mathrm{ord})$. So, the ideal (s) defines a smooth point p on C.

Note that $U_s = \mathrm{Spec}\, \tilde{B}_{(s)}$ is just the affine curve C_0 and therefore $C = C_0 \cup p$. The point p is called the *"divisor at infinity"*. It is easy to see (after solving the last exercise) that $\tilde{B}_{((s))}/(s) \simeq K$, i.e. p is a K-point. Then from the Cohen theorem 14.16 it follows that $\widehat{\tilde{B}_{((s))}} \simeq K[[T]]$.

From previous subsection it follows that $\tilde{A} \simeq \tilde{B}$ as graded rings, where a filtration on A and the graded ring $\tilde{A}$ is defined in the same way as for B. Clearly, $\tilde{B}_{((s))} \simeq \tilde{A}_{((s))}$ and by construction $\tilde{A}_{((s))} \subset K[[\partial^{-1}]]$. It is easy to see (cf. exercise 10.1) that any generator of the maximal ideal of $\tilde{A}_{((s))} \subset K[[\partial^{-1}]]$ has order $-r$ and, if we denote this element by y, then $\widehat{\tilde{B}_{((s))}} \simeq \widehat{\tilde{A}_{((s))}} \simeq K[[y]]$. Remembering that $\widehat{\mathcal{O}}_{C,p} \simeq \widehat{\tilde{B}_{((s))}}$, we get an embedding $\pi : \widehat{\mathcal{O}}_{C,p} \hookrightarrow K[[\partial^{-1}]] \simeq K[[z]]$ which is a local K-algebra homomorphism and for any generator f of the maximal ideal p we have $\pi(f)K[[z]] = z^r K[[z]]$. This embedding is a part of *trivialisation associated with the ring* B. The parameter z (or y) plays a role of a local coordinate at p. Note also that $K[[z]]$ is endowed with a $\widehat{\mathcal{O}}_{C,p}$-module structure via π and that it is a *free* module of rank r (e.g. we can take $1, z, \dots, z^{r_1}$ as free generators). The same is true for the subspace $K[[z]] \cdot z$ as this space is preserved by $\pi(\widehat{\mathcal{O}}_{C,p})$.

The embedding π is not uniquely defined: it depends on the choice of an operator S. As we have seen in subsection 10.1, another choice of an operator S leads to a conjugation with the help of an admissible operator T, i.e. for another choice of an operator S the embedding π' will differ from the embedding π by conjugation $z \mapsto T^{-1}zT = z + a_2 z^2 + \dots$ (note that the highest coefficient is preserved by such a conjugation).

Remark 10.4. For a fixed π we can choose such an element $y \in K[[z]]$ that $\pi(\widehat{\mathcal{O}}_{C,p}) = K[[y^r]] \subset K[[y]] = K[[z]]$. Indeed, if we take a generator f of the ideal $\mathfrak{m}_p \subset \widehat{\mathcal{O}}_{C,p}$ such that $\pi(f) = z^r + a_{r+1}z^{r+1} + \dots$, then its root y s.t. $y^r = \pi(f)$ is the element we wanted.

Now we define the (projective) spectral sheaf as follows. The spectral module F is endowed with a natural filtration given by the order function: $F_i := \{f \in F | \quad \mathrm{ord}(f) \le i\}$. With this filtration we can associate a series of graded modules and corresponding associated sheaves:

$$\forall k \in \mathbb{Z} \quad {}^{(k)}\tilde{F} := \oplus_{i\ge 0} F_{ir+k} s^i$$

are graded $\tilde{B}$-modules, and $\mathcal{F}_k := \mathrm{Proj}\, {}^{(k-1)}\tilde{F}$ are associated sheaves of $\mathcal{O}_C$-modules (clearly, they are torsion free as the modules are torsion free). Note that we have embeddings $\ldots \subset {}^{(k)}\tilde{F} \subset {}^{(k+1)}\tilde{F} \subset {}^{(k+2)}\tilde{F} \subset \ldots$ which induce embeddings of sheaves $\ldots \subset \mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \ldots$.

Of course, we have also a natural isomorphism of graded modules $\tilde{W} \simeq \tilde{F}$ and therefore we can construct the sheaves $\mathcal{F}_k$ starting with a Schur pair (A, W).

Exercise 10.6. Show that the modules ${}^{(k)}\tilde{F}$ are finitely generated $\tilde{B}$-modules, i.e. the sheaves $\mathcal{F}_k$ are coherent.

We define the *(projective) spectral sheaf* as $\mathcal{F} := \mathcal{F}_0$.

Remark 10.5. There are several alternative definitions of the spectral sheaf. E.g. in the Krichever classification theorem the spectral sheaf is a shifted version of our sheaf: $\mathcal{F}(p) = \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_C(1)$. In fact, we can take any of the sheaves $\mathcal{F}_k$ as a spectral sheaf. The choice of $\mathcal{F}_0$ is made because this sheaf has zero cohomology and this makes the definition of a projective spectral data easier (see below). For another choice we should add more information about elements of the cohomology groups, cf. [10].

Let's note that in our case the sheaf $\mathcal{O}_C(1)$ is the invertible sheaf corresponding to the Cartier divisor p, although the graded ring $\tilde{B}$ can be not finitely generated by its first graded component (cf. 9.4). The reason is that the variety is a curve in our case: the sheaf is locally free at p because it is torsion free and p is regular (see proposition 8.5 and exercises above) and $(\mathcal{O}_C(1))(C \backslash p) \simeq B \simeq \mathcal{O}_C(C \backslash p)$.

Below we'll use the notation $\mathcal{O}_C(np) := \mathcal{O}_C(n)$ (of course, these are invertible sheaves corresponding to the Cartier divisors np).

Exercise 10.7. Show that $\mathcal{O}_C(-1) \simeq \mathcal{O}_C(-p)$ – the ideal sheaf of the point p.

Now let's construct the last part of the projective spectral data — a *trivialisation of the spectral sheaf* associated with B. First, note that there is an embedding of $\mathcal{O}_{C,p}$-modules

$$\phi : \mathcal{F}_p = ({}^{(-1)}\tilde{F})_{((s))} \simeq ({}^{(-1)}\tilde{W})_{((s))} \subset K[[z]] \cdot z, \quad \frac{f}{b} \mapsto (f \cdot S)(S^{-1}bS)^{-1},$$

which, however, depends on the choice of a Schur operator S. The module $K[[z]] \cdot z$ is complete (cf. section 14.7), and therefore by theorem 14.13 and lemma 14.6 ϕ induces an embedding $\hat{\phi} : \hat{\mathcal{F}}_p \to K[[z]] \cdot z$.

Exercise 10.8. Show that $\hat{\phi}$ is an isomorphism.

The isomorphism $\hat{\phi}$ (a *trivialisation* of $\hat{\mathcal{F}}_p$) is the last part of the projective spectral data. Clearly, we can construct the same isomorphism $\hat{\phi}$ if we'll start with a Schur pair (A, W). The difference is that in this case the homomorphisms $\pi, \hat{\phi}$ will not depend on a choice of the Schur operator (as these subspaces are already in the space $K((z))$).

Remark 10.6. Sometimes it is more convenient to fix another form of trivialisation $\hat{\mathcal{F}}_p \simeq \hat{\mathcal{O}}_{C,p}^{\oplus r}$ (e.g. to obtain a Schur pair in a "matrix form" as in remark 10.1. To have a relation between different trivialisations it is convenient to *fix* an isomorphism of $\hat{\mathcal{O}}_{C,p}$-modules

$$\psi : \hat{\mathcal{O}}_{C,p}^{\oplus r} \simeq K[[z]] \cdot z \quad (\alpha_1, \ldots, \alpha_n) \mapsto z\pi(\alpha_1) + \ldots + z^r \pi(\alpha_r).$$

Then we can take the composition $\hat{\mathcal{F}}_p \overset{\psi^{-1} \circ \hat{\phi}}{\simeq} \hat{\mathcal{O}}_{C,p}^{\oplus r}$.

Now we are going to explain what are the cohomology groups of the sheaves $\mathcal{F}_k$. We can calculate them by using the Čech complex.

Lemma 10.1. *We have*

$$H^0(C, \mathcal{F}_k) \simeq F_{k-1} \simeq W_{k-1}, \quad H^0(C, \mathcal{O}_C(np)) \simeq B_{nr} \simeq A_{nr},$$

$$H^1(C, \mathcal{F}_k) \simeq \frac{K((z))}{W + K[[z]] \cdot z^{-k+1}} \simeq \frac{K((z))}{z^{k-1} \cdot W + K[[z]]}, \quad H^1(C, \mathcal{O}_C(np)) \simeq \frac{K((y^r))}{A \cdot y^{nr} + K[[y^r]]},$$

where y *is taken as in remark 10.4. In particular,* $H^0(C, \mathrm{Proj}(\tilde{W}(n))) \simeq W_{nr}$.

Proof. By definition, $F_{k-1} \simeq W_{k-1} \subset H^0(C, \mathcal{F}_k)$, because all elements of F_k are also sections from $\mathcal{F}_k(C \backslash p)$ and $\mathcal{F}_k(U)$, where U is any open affine set containing p (e.g. we can take $U = D_+(\tilde{f})$, where $\tilde{f} \in A_n \cdot s^n = \tilde{A}_n$ with $\mathrm{ord}\, \tilde{f} = nr$ (i.e. $\deg \tilde{f} = n$)), whose restrictions on the intersection are equal. Let's prove that there are no any other global sections.

Assume there exists $a_1 \in \mathcal{F}_k(C \backslash p)$, $a_2 \in \mathcal{F}_k(U) \simeq ({}^{k-1}\tilde{W})_{(f)}$ such that $a_1 = a_2$ in $\mathcal{F}_k(U \backslash p) \simeq ({}^{k-1}\tilde{W})_{(sf)}$, i.e. the pair (a_1, a_2) defines a global section $a \in \mathcal{F}_k(C)$. Let $a_i = \frac{\tilde{a}_i}{x_i^{k_i}}$, where $x_1 = s$, $x_2 = f$, $\tilde{a}_i \in ({}^{k-1}\tilde{W})_{k_i \deg(x_i)} = W_{rk_i \deg(x_i) + k - 1}$. We can assume that $k_1 > 0$, as if $k_1 = 0$, then $\tilde{a}_1 = a_1 \in W_{k-1} = ({}^{k-1}\tilde{W})_0$ and therefore $a - a_1 = 0$ on $C \backslash p$, whence $a = a_1$ as global section, because $\tilde{W}$ is torsion free. Then $\tilde{a}_1 \in W_{rk_1 + k - 1} \cdot s^{k_1} \backslash W_{rk_1 + k - 1 - k_1} \cdot s^{k_1}$ or, equivalently, $\mathrm{ord}\, \tilde{a}_1 > rk_1 + k - 1 - k_1$. But then $\mathrm{ord}(\tilde{a}_1 f^{k_2}) > r(k_1 + nk_2) + k - 1 - k_1$ and $\tilde{a}_1 f^{k_2} \in W_{r(k_1 + k_2 n + 1) + k - 1 - k_1} \cdot s^{k_1 + nk_2 + 1}$.

On the other hand, $\tilde{a}_1 f^{k_2} = \tilde{a}_2 s^{k_1}$ (since $a_1 = a_2$ in $\mathcal{F}_k(U \backslash p)$ and $\mathrm{ord}(\tilde{a}_2 s^{k_1}) = rnk_2 + k - 1$, a contradiction, as $\mathrm{ord}(\tilde{a}_1 f^{k_2}) > \mathrm{ord}(\tilde{a}_2 s^{k_1})$. Therefore, $k_1 = 0$ and $\mathcal{F}_k(C) = H^0(C, \mathcal{F}_k) \simeq F_{k-1}$.

The same arguments work for the group of global sections of the sheaves $\mathcal{O}_X(np)$, $\mathrm{Proj}(\tilde{W}(n))$.

To calculate the groups $H^1(C,\mathcal{F}_k)$ we'll use the same Čech complex for the covering $C=(C\backslash p)\cup U$. Recall that we have an embedding $\mathcal{F}_k\simeq({}^{(k-1)}W)_{(st)}\hookrightarrow K((z))$ and this subspace in $K((z))$ satisfies the following property: for any $n\in\mathbb{Z}$ there exists $w\in\mathcal{F}_k(U\backslash p)$ such that $-\operatorname{ord}(w)=n$. Analogously, the subspace $\mathcal{F}_k(U)\subset K((z))$ satisfies the following property: for any $n\geq -k+1$ there exists $w\in\mathcal{F}_k(U)$ such that $-\operatorname{ord}(w)=n$, and the subspace $\mathcal{F}_k(C\backslash p)\simeq W$ with $\operatorname{Supp}W=K[z^1]$.

The Čech complex looks like

$$\mathcal{F}_k(C\backslash p)\times\mathcal{F}_k(U)\xrightarrow{d}\mathcal{F}_k(U\backslash p),\quad (w_1,w_2)\mapsto w_1-w_2$$

Note that $\operatorname{Im}(d)=W+\mathcal{F}_k(U)$ is dense in the space $W+K[[z]]\cdot z^{-k+1}$ and $\mathcal{F}_k(U\backslash p)$ is dense in the space $K((z))$ due to the properties of the corresponding spaces, i.e. for any $a\in K((z))$ and $N\gg 0$ $a=w+b$ for some w and b, where $w\in\mathcal{F}_k(U\backslash p)$ and $b\in K[[z]]\cdot z^N$, and for any $a\in W+K[[z]]\cdot z^{-k+1}$ $a=w_1+w_2+b$ for some w_i,b, where $w_1\in W$, $w_2\in\mathcal{F}_k(U)$, $b\in K[[z]]\cdot z^N$. Therefore,

$$\operatorname{Coker}d=H^1(C,\mathcal{F}_k)\simeq\frac{K((z))}{W+K[[z]]\cdot z^{-k+1}}\simeq\frac{K((z))}{z^k\cdot W+K[[z]]}$$

Analogous calculations imply (cf. remark 10.4)

$$H^1(C,\mathcal{O}_C)\simeq\frac{K((y^r))}{A+K[[y^r]]},\quad H^1(C,\mathcal{O}_C(np))\simeq\frac{K((y^r))}{A\cdot y^{nr}+K[[y^r]]}.$$

□

Corollary 10.1. *For a spectral sheaf $\mathcal{F}$ we have $H^0(C,\mathcal{F})=H^1(C,\mathcal{F})=0$.*

Exercise 10.9. Prove that $\mathcal{F}_{k+rq}\simeq\mathcal{F}_k\otimes_{\mathcal{O}_C}\mathcal{O}_C(qp)$. Hint: use the results from section 9.2. Note also that for our subsets $U_1=C\backslash p$, $U_2=U$ we have natural module homomorphisms

$$\mathcal{F}_k(U_i)\otimes_{\mathcal{O}_C(U_i)}\mathcal{O}_C(qp)(U_i)\to\mathcal{F}_{k+rq}(U_i),\quad (f,a)\mapsto fa$$

which are in fact isomorphisms.

As an application of our calculations we can derive the asymptotic Riemann-Roch theorem.

Theorem 10.1. *We have for any $n\in\mathbb{Z}$*

$$\chi(\mathcal{F}_k\otimes_{\mathcal{O}_C}\mathcal{O}_C(np)):=h^0(C,\mathcal{F}_k\otimes_{\mathcal{O}_C}\mathcal{O}_C(np))-h^1(C,\mathcal{F}_k\otimes_{\mathcal{O}_C}\mathcal{O}_C(np))=nr+c,$$

where c is some integer constant.

Proof. The proof is based on the following combinatorial fact (cf. [32, Ch. 1, §7, Prop. 7.3])

Lemma 10.2. *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a function. Assume that $\triangle f:=f(n+1)-f(n)=Q(n)$ for all n, where Q is a polynomial such that $Q(n)\in\mathbb{Z}$ for all $n\gg 0$, $Q\in\mathbb{Q}[T]$.*

Then there exists a polynomial $P(T)\in\mathbb{Q}[T]$ such that $P(n)\in\mathbb{Z}$ for all $n\gg 0$ and $f(n)=P(n)$ for all n. More precisely,

$$P(T)=c_0\cdot C_T^r+c_1\cdot C_T^{r-1}+\ldots+c_r,$$

where $c_i\in\mathbb{Z}$, $C_T^k=T(T-1)\ldots(T-k+1)/k!$.

Proof. By [32, Ch. 1, §7, Prop. 7.3], a polynomial Q has a form $Q(T)=c_0\cdot C_T^{r-1}+c_1\cdot C_T^{r-2}+\ldots+c_{r-1}$ for some integer c_i,r. Put $P(T)=c_0C_T^r+\ldots+c_{r-1}C_T^1$. Then $\triangle P=Q$ so that $\triangle(f-P)(n)=0$ for all n. Therefore, $(f-P)(n)$ is a constant c_r for all n and $f(n)=P(n)+c_r$ for all n. □

In our case $f(n)=\chi(\mathcal{F}_k\otimes\mathcal{O}_C(np))$. From lemma 10.1 we obtain $\triangle f(n)=r$ for all n. So, $f(n)=rn+c$.

In particular, if we take just the sheaf $\mathcal{O}_C(np)$, then we get $\chi(\mathcal{O}_C(np))=n+c$ and thus $\chi(\mathcal{O}_C)=c=1-g$, where $g=h^1(C,\mathcal{O}_C)$ is the *arithmetical genus* of C. $\square$

Remark 10.7. As we'll see later in remark 11.6 the asymptotic Riemann-Roch theorem holds also for arbitrary torsion free sheaves. The number $c+(g-1)r$ (where r is a rank of the sheaf) is called *the degree* of the sheaf.

Now we are ready to give a formal definition of the projective spectral data.

Definition 10.5. The projective spectral data of rank r consists of

1. a projective irreducible curve C over K;
2. a regular point $p\in C$;
3. a torsion free coherent sheaf $\mathcal{F}$ of rank r such that $H^0(C,\mathcal{F})=H^1(C,\mathcal{F})=0$;
4. an embedding of local rings $\pi:\hat{\mathcal{O}}_{C,p}\hookrightarrow K[[z]]$ such that $\pi(f)\cdot K[[z]]=z^r\cdot K[[z]]$;
5. an isomorphism of $\hat{\mathcal{O}}_{C,p}$-modules $\hat{\phi}:\hat{\mathcal{F}}_p\simeq z\cdot K[[z]](\simeq\hat{\mathcal{O}}_{C,p}^{\oplus r})$

Definition 10.6. Two projective spectral data $(C_1,p_1,\mathcal{F}_1,\pi_1,\hat{\phi}_1)$, $(C_2,p_2,\mathcal{F}_2,\pi_2,\hat{\phi}_2)$ are isomorphic if there exists an isomorphism of curves $\beta:C_1\to C_2$ and an isomorphism of sheaves $\psi:\mathcal{F}_2\to\beta_*\mathcal{F}_1$ such that $\beta(p_1)=p_2$ and

- there is an automorphism $\bar{h}:K[[z]]\to K[[z]]$ of rings such that
$$\bar{h}=z+a_2z^2+\dots$$
and the following diagram of ring homomorphisms is commutative:
$$\begin{tikzcd} \hat{\mathcal{O}}_{C_1,p_1} \arrow[d, "\pi_1"] & \hat{\mathcal{O}}_{C_2,p_2} \arrow[l, "\beta^\sharp_{p_2}"'] \arrow[d, "\pi_2"] \\ K[[z]] & K[[z]] \arrow[l, "\bar{h}"'] \end{tikzcd} \tag{26}$$

- there is a $K[[z]]$-module isomorphism $\xi:z\cdot K[[z]]\to z\cdot K[[z]]$, where $z\cdot K[[z]]$ on the right hand side is a $h_*K[[z]]$-module, i.e. $a\cdot v=\bar{h}(a)\cdot v$ (and therefore, ξ is just given by the rule $a\mapsto\bar{h}(a)\xi(1)$, $\xi(1)\in K[[z]]^*$), such that the following diagram of $\hat{\mathcal{O}}_{C_2,p_2}$-modules isomorphisms is commutative:
$$\begin{tikzcd} \hat{\mathcal{F}}_{2,p_2} \arrow[r, "\hat{\psi}"] \arrow[d, "\hat{\phi}_2"] & \widehat{\beta_*\mathcal{F}_1}_{p_2}\simeq\hat{\mathcal{F}}_{1,p_1} \arrow[d, "\hat{\beta}_*(\hat{\phi}_1)"] \\ z\cdot K[[z]] \arrow[r, "(\pi_2)_*\xi"] & z\cdot K[[z]] \end{tikzcd} \tag{27}$$
(more precisely, the isomorphisms in this diagram look as follows: for any $a\in\hat{\mathcal{O}}_{C_2,p_2}$ and $f\in\hat{\mathcal{F}}_{2,p_2}$ we have
$$\hat{\psi}(a\cdot f)=\hat{\beta}^\sharp_{p_2}(a)\hat{\psi}(f)(=a\cdot\hat{\psi}(f)),\quad \hat{\beta}_*(\hat{\phi}_1)(a\cdot\hat{\psi}(f))=\pi_1(\hat{\beta}^\sharp_{p_2}(a))\hat{\phi}_1(\hat{\psi}(f)),$$
$$\hat{\phi}_2(a\cdot f)=\pi_2(a)\hat{\phi}_2(f),\quad ((\pi_2)_*\xi)(\pi_2(a)\hat{\phi}_2(f))=\bar{h}(\pi_2(a))(\xi(\hat{\phi}_2(f)))=\bar{h}(\pi_2(a)\hat{\phi}_2(f))\xi(1);$$
the isomorphism $\widehat{\beta_*\mathcal{F}_1}_{p_2}\simeq\hat{\mathcal{F}}_{1,p_1}$ is also an isomorphism of $\hat{\mathcal{O}}_{C_2,p_2}$-modules, where $\hat{\mathcal{F}}_{1,p_1}$ has a $\hat{\mathcal{O}}_{C_2,p_2}$-module structure via the homomorphism of local rings $\hat{\beta}^\sharp_{p_2}$.)

Remark 10.8. It is possible to extend the notion of a projective spectral datum and define a *category* $\mathcal{G}_r$ of data, whose objects are projective spectral data of rank r, see [66].

Now we are ready to check whether the direct maps $[B] \mapsto$ [Proj. spectral data], $[(A,W)] \mapsto$ [Proj. spectral data] are well defined. Note that the first map is in fact a composition of the Schur map $[B] \mapsto [(A,W)]$ and the second map. As we already noted in remark 10.2, the map $[B] \mapsto [(A,W)]$ is well defined. So, it suffices to check that the second direct map is well defined.

If we have two equivalent Schur pairs (A,W) and (A',W') with $A' = T^{-1}AT$, $W' = W\cdot T$, then the corresponding projective spectral data will be isomorphic with the following isomorphisms: β is an isomorphism induced by the isomorphism of graded rings $\beta^\sharp : \tilde{A}' \to \tilde{A}$, $a' \mapsto a := T\cdot a'\cdot T^{-1}$, and ψ is an isomorphism induced by the isomorphism of graded modules $\psi^\sharp : \tilde{W}' \to \tilde{W}$, $w' \mapsto w := w'\cdot T^{-1}$. An automorphism $\bar{h}$ is given by conjugation $a \mapsto T\cdot a\cdot T^{-1}$, and the isomorphism ξ is just the identity.

Conclusion. We have constructed the direct maps $[B] \mapsto$ [Proj. spectral data] and $[(A,W)] \mapsto$ [Proj. spectral data]. The geometric part of the projective spectral data is canonically defined for both maps, and trivialisations are defined non-canonically, i.e. only up to an isomorphism of data. Besides, the first map is a composition of the Schur map $[B] \mapsto [(A,W)]$ and the second map. The second map is well defined already on the level of sets: $(A,W) \mapsto$ Proj. spectral data.

Remark 10.9. The construction of the spectral data was rewritten several times by different authors. We used in our lectures an approach offered by Mumford [71] developed further my Mulase [66] (see also recent review in [10, Section 1]).

First classification of commutative rings of ODOs of any rank was proposed by Krichever [38], [39] as an algebro-geometric tool in the theory of integrating non-linear soliton systems and the spectral theory of periodic finite-zone operators (see e.g. review [25]). It used an *analytic* version of spectral data (see section 12.1) and worked for rings B in "generic position", i.e. for rings whose spectral curve is smooth. The advantage of this approach is the existence of an explicit formula for the common eigenfunction (the so called Baker-Akhieser function) of a given *rank one* subring B. This formula leads, in particular, to explicit formulae of commuting operators.

Moreover, as we'll see below in remark 11.9, the *rank one subrings* are classified essentially (i.e. up to automorphisms $x \mapsto cx$, $\partial \mapsto c^{-1}\partial$) only by the *geometric part* $(C,p,\mathcal{F})$ of the data (the trivialisations are not important in this case). The higher rank case is much more difficult, see section 13.5. We would like to note that, due to one theorem of Makar-Limanov (see 6.2), the rank of a commutative elliptic subring of the *first Weyl algebra* must be greater that one. By the Schur theory, the maximal subring in D containing such a ring will belong again to A_1. Thus, we see that the most "easy" coefficient ring is the most difficult to study.

11 The Krichever map and the Sato theory

11.1 The Krichever map

In this section we give an overview of the construction of the Krichever map from diagram (24). This map can be defined not only for projective data from previous section, but also for a slightly general data over *any ground field* (not necessarily of characteristic zero).

Let C be a projective irreducible curve over K, let $\mathcal{F}$ be a coherent torsion free sheaf of rank r, let $p \in C$ be a regular point, and let $L := K(p) = \mathcal{O}_{C,p}/\mathfrak{m}_p$ be its residue field.

Let

$$\pi : \widehat{\mathcal{O}}_{C,P} \longrightarrow L[[z]] \tag{28}$$

be a local L-algebra homomorphism[2] satisfying the following property. If f is a local equation of the point p, then $\pi(f)L[[z]] = z^r L[[z]]$. (The definition of π does not depend on the choice of appropriate f. Besides, from this definition it follows that π is an embedding, $L[[z]]$ is a free $\widehat{\mathcal{O}}_{C,p}$-module of rank r with respect to π, as well as $L[[z]]\cdot z$.) We'll consider $L[[z]]$ endowed with the discrete rank one L-valuation ν_z.

Comment 11.1. Alternatively, on the language of schemes, this homomorphism is determined by a L-dominant morphism $j: T \to C$, where $T = \operatorname{Spec} L[[z]] \supset O = \operatorname{Spec} L$, which satisfies the following conditions:

1. $j_*(O) = P \subset C$ (we use here the following notation: for a morphism of noetherian schemes $f: X \to Y$ and a closed subscheme $Z \subset X$, $f_*Z \subset Y$ is the closed subscheme defined by the ideal sheaf $\ker(\mathcal{O}_Y \xrightarrow{f^*} f_*\mathcal{O}_X \to f_*\mathcal{O}_Z)$);
2. $T \times_C P = rO$ (the fiber product is a subscheme of T and rO is an effective Cartier divisor on T).

Now we define a subspace
$$A \subset V_z = L((z)),$$
where A is a filtered subring of $L((z))$ (we mean the filtration defined by the discrete valuation ν_z) as follows:

Let f be a local generator of the ideal $\mathfrak{m}_p = \mathcal{O}_C(-p)_p$. For any $n \geq 0$ we have canonical isomorphisms $\mathcal{O}_C(np)_p \simeq f^{-n}(\mathcal{O}_{C,p})$, where we consider the sheaf $\mathcal{O}_C(np)$ as a subsheaf of the constant sheaf $\mathcal{K}$ corresponding to the Cartier divisor np (check it, see proposition 9.8; the Cartier divisor np is defined by a system $(C\backslash p, 1)$, (U_p, f^n), where f is a local equation of p in a neighbourhood $U_p \ni p$, invertible at all points in $U_p\backslash p$, e.g. we can take f to be a generator of the maximal ideal in the ring $\mathcal{O}_{C,p}$ and take U_p to be a neighbourhood of p where f is regular and vanishes only at p). Then we have natural embeddings for any $n \geq 0$

$$\alpha_n : H^0(C, \mathcal{O}_C(np)) \hookrightarrow \mathcal{O}_C(np)_p \simeq f^{-n}(\mathcal{O}_{C,p}) \hookrightarrow L((z)),$$

where the last embedding is the embedding $f^{-n}\mathcal{O}_{C,p} \xhookrightarrow{\pi} \pi(f)^{-n}L[[z]] \hookrightarrow L((z))$. Hence we have the embedding (a part of the Krichever map)

$$\chi_0 \;:\; H^0(C\backslash p, \mathcal{O}_C) \simeq \varinjlim_{n\geq 0} H^0(C, \mathcal{O}_C(np)) \hookrightarrow L((z)). \tag{29}$$

Remark 11.1. Let's recall a notion of a *direct limit* used in this formula.

First recall that a partially ordered set I is called a *directed set* if it has the additional property that every pair of elements has an upper bound, i.e. for any $i, j \in I$ there must exist $k \in I$ with $i \leq k$ and $j \leq k$.

Now let A be a ring (in our case $A = K$), I be a directed set, and $(M_i)_{i\in I}$ be a family of A-modules indexed by I. Assume that for any pair $i, j \in I$ with $i \leq j$ there is a homomorphism of A-modules $\mu_{ij}: M_i \to M_j$. Assume that these homomorphisms satisfy the following properties: $\mu_{ii} = id$ for any $i \in I$, $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ for any $i \leq j \leq k$. In this case the family (M_i, μ_{ij}), $i, j \in I$ is called an *inductive system*.

Put $C := \oplus_{i\in I} M_i$ and identify each module M_i with its canonical image in C. Denote by D the submodule in C generated by all elements of the form $x_i - \mu_{ij}(x_i)$, $i \leq j$, $x_i \in M_i$. Put $M := C/D$, denote by μ the projection $C \to M$, and by μ_i the restrictions of μ on M_i.

The data consisting of the module M together with the family μ_i is called *a direct limit* of the system (M_i, μ_{ij}) and is denoted by $\varinjlim M_i$. From the construction it follows that $\mu_i = \mu_j \circ \mu_{ij}$ for any $i \leq j$.

[2] Recall that $\widehat{\mathcal{O}}_{C,P} \simeq L[[f]]$ by the Cohen structure theorem 14.16

Note that any element from M can be represented as $\mu_k(x_k)$ for some $k \in I$, $x_k \in M_k$. Indeed, any element has a representative in the module C. This representative is a finite sum of elements from modules M_i, and for any finite set of indices there exists an index k which is an upper bound of them. Then our representative is equal to the sum of elements of the form $\mu_{ik}(x_i)$ from the module M_k, and our original element is the image of this sum under the map μ_k.

The following two standard exercises from [4] show how the map χ_0 is defined and why $H^0(C\backslash p, \mathcal{O}_C) \simeq \varinjlim_{n\geq 0} H^0(C, \mathcal{O}_C(np))$.

Exercise 11.1. Show that the inductive limit is defined (up to an isomorphism) by the following property. Let N be an A-module, and for each $i \in I$ an A-homomorphism $\alpha_i : M_i \to N$ is given such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for all $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Exercise 11.2. Let $(M_i)_{i\in I}$ be a family of submodules of an A-module, and for any pair $i, j \in I$ there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define the order $i \leq j$ iff $M_i \subseteq M_j$, and denote by $\mu_{ij} : M_i \to M_j$ the embedding of M_i into M_j. Show that

$$\varinjlim M_i = \sum M_i = \cup M_i.$$

In particular, any A-module is an inductive limit of its finitely generated submodules.

In our case the K-spaces $H^0(C, \mathcal{O}_C(np))$ are embedded into $H^0(C\backslash p, \mathcal{O}_C)$ via the restriction maps $\rho_{C,C\backslash p}$, because by definition of the sheaf $\mathcal{O}_C(np)$ it is a subsheaf of the constant sheaf $\mathcal{K}$ such that $\mathcal{O}_C(np)(C\backslash p)$ is a $\mathcal{O}_C(C\backslash p)$-submodule generated by 1, and therefore the restriction of its global section on $C\backslash p$ is a section of the structure sheaf. Besides, for any n $\mathcal{O}_C(np) \subset \mathcal{O}_C((n+1)p)$ and therefore $H^0(C, \mathcal{O}_C(np)) \subset H^0(C, \mathcal{O}_C((n+1)p))$.

On the other hand, if $a \in H^0(C\backslash p, \mathcal{O}_C)$ and $n = \nu_p(a)$, where ν_p is a discrete valuation corresponding to the point p (recall that the local ring $\mathcal{O}_{C,p}$ is a DVR since p is a regular point), then $b := af^{-n} \in \mathcal{O}_{C,p} \subset K(C)$ and therefore a can be represented as a global section of the sheaf $\mathcal{O}_C(-np)$, namely as a pair of sections $(C\backslash p, a)$ and (U_p, bf^n) which coincide on the intersection $U_p\backslash p$. So, $H^0(C\backslash p, \mathcal{O}_C)$ is a union of the images of $H^0(C, \mathcal{O}_C(np))$, or the direct limit by the last exercise.

We define $A \stackrel{\text{def}}{=} \chi_0(H^0(C\backslash p, \mathcal{O}_C))$. As it follows from this construction,

$$A \subset L((z')) \subset L((z)), \tag{30}$$

where $z' = \pi(f)$. Thus, on A there is a filtration A_n induced by the filtration $z'^{-n}L[[z']]$ on the space $L((z'))$:

$$A_n := A \cap z'^{-n}L[[z']] = A \cap z^{-nr}L[[z]]. \tag{31}$$

The map χ_0 has the following properties.

Proposition 11.1. *Let (C, p, π) be as in the beginning of this section, and K be an arbitrary field. Let $A = \chi_0(C, p, \mathcal{O}_C, \pi)$ be a filtered subring constructed with the help of chosen homomorphism π.*

Then $H^0(C, \mathcal{O}_C(nP)) \simeq A_n$ for all $n \in \mathbb{Z}$. In particular, $C \simeq C' := \operatorname{Proj} \tilde{A}$, where $\tilde{A} = \sum_{i=0}^{\infty} A_i s^i$, and $\mathcal{O}_{C'}(1) \simeq \mathcal{O}_{C'}(p')$, where p' is a point on C' corresponding to the point p under this isomorphism.

Proof. First note that for $n < 0$ $A_n = 0$ by construction of the map χ_0, and also $H^0(C, \mathcal{O}_C(nP)) = 0$, since in this case $\mathcal{O}_C(np) \subset \mathcal{O}_C$ is an ideal sheaf without global sections (as the only global sections of the sheaf $\mathcal{O}_C$ are constants, cf. lemma 10.1 and remember that $C = \operatorname{Proj} S$, where S is a graded ring finitely generated by S_1 over $S_0 = K$, because C is a projective curve over K).

By construction of the maps α_n, $n \ge 0$, the image $\alpha_n(H^0(C, \mathcal{O}_C(np)))$ belongs to the space A_n for any $n \ge 0$. On the other hand, it is easy to see that $\nu_p(a) = \nu_z/r(\chi_0(a))$ for any $a \in H^0(C\backslash p, \mathcal{O}_C)$, where ν_p is the valuation defined by the point p, and, as we have seen in the last remark, any element a with $\nu_p(a) \ge -n$ belongs to the group $H^0(C, \mathcal{O}_C(-\nu_p(a)p)$. Hence $A_n \simeq H^0(C, \mathcal{O}_C(np))$.

Now consider the scheme $\operatorname{Proj} \tilde{A}$. Note that the basic algebraic results about the ring $\tilde{B}$ are valid also for the ring $\tilde{A}$, because A is a finitely generated integral domain over K, and $\operatorname{gr} A \subset K[z^{-1}]$ (cf. exercise 10.3), i.e. $\tilde{A}$ is a finitely generated graded integral ring over $K = A_0$. The same arguments as in section 10.2 show that $\tilde{A}_{(s)} \simeq A$ and therefore the open set $U_s = \operatorname{Spec} \tilde{A}_{(s)}$ is isomorphic to the affine scheme $C\backslash p$, and that topologically $\operatorname{Proj} \tilde{A} = U_s \cup p'$, where p' is a K-point determined by the discrete valuation ν_z/r. So, there is a natural map of topological spaces $\underline{\operatorname{Proj} \tilde{A}} \to \underline{C}$, which is a homeomorphism, and this map induces a natural map of schemes $\operatorname{Proj} \tilde{A} \to C$, which is an isomorphism of schemes.

Exercise 11.3. Check that this is an isomorphism of schemes (note that we have checked this for affine schemes $U_s \simeq C\backslash p$.

At last, the same arguments as in remark 10.5 show that $\mathcal{O}_{C'}(1) \simeq \mathcal{O}_{C'}(p')$. □

Remark 11.2. This proof illustrates that the sheaf $\mathcal{O}_C(p)$ is an *ample* sheaf (see [32, Ch. 2, §7]). By [32, Ch. 2, Th. 7.6] an invertible sheaf $\mathcal{L}$ on a scheme of finite type over a noetherian ring is ample iff some $\mathcal{L}^d$ is very ample over this ring for some d. In our case d can be found from proposition 9.2.

The map χ_0 can be extended also to sections of coherent torsion free sheaves of $\mathcal{O}_C$-modules, and we'll call this extension as the *Krichever map* and denote by the same letter.

Since p is a regular point, $\mathcal{F}$ is locally free at p. Let $\hat{\phi} : \hat{\mathcal{F}}_p \simeq L[[z]] \cdot z$ be a a trivialisation (a $\hat{\mathcal{O}}_{C,p}$-module isomorphism).

There is a natural extension of the embedding χ_0:

$$\chi_0 \ : \ H^0(C\backslash p, \mathcal{F}) \simeq \varinjlim_{n\ge 0} H^0(C, \mathcal{F}(np)) \hookrightarrow L((z)),$$

which is defined through the natural maps

$$\alpha_n : H^0(C, \mathcal{F}(np)) \hookrightarrow \mathcal{F}(np)_p \simeq f^{-n}(\mathcal{F}_p) \hookrightarrow L((z))$$

for any $n \ge 0$, where the last embedding is the embedding $f^{-n}\mathcal{F}_p \overset{\hat{\phi}}{\hookrightarrow} \pi(f)^{-n} L[[z]] \cdot z \hookrightarrow V_z$.

Remark 11.3. The isomorphism $H^0(C\backslash p, \mathcal{F}) \simeq \varinjlim_{n\ge 0} H^0(C, \mathcal{F}(np))$ can be seen in the following way. First, note that for any n we have embeddings of sheaves $\mathcal{F}(np) \subset \mathcal{F}((n+1)p)$, because there are embeddings $\mathcal{O}_C(np) \subset \mathcal{O}_C((n+1)p)$ and $\mathcal{F}$ is a torsion free sheaf (check it locally). In particular, there are natural embeddings $H^0(C, \mathcal{F}(np)) \subset H^0(C, \mathcal{F}((n+1)p))$ for all n.

Denote by $\mathcal{F}'$ the direct image of the sheaf $\mathcal{F}$ under the isomorphism $i : C \to C'$ from proposition 11.1; then $i_*(\mathcal{F}(np)) \simeq \mathcal{F}'(n)$ for any n. Denote by $j_d : C' \to \operatorname{Proj} \tilde{A}^{(d)}$ the isomorphism from theorem 9.1, where d is chosen as in proposition 9.2. Then by proposition 9.5 $\mathcal{L} \simeq \operatorname{Proj}(\Gamma_*(\mathcal{L}))$, $\Gamma_*(\mathcal{L}) = \oplus_{n\in\mathbb{Z}} \Gamma(C'', \mathcal{L}(n))s^n$ for any quasi-coherent sheaf $\mathcal{L}$ on $C'' = \operatorname{Proj} \tilde{A}^{(d)}$, and by exercise 9.4 combined with remark 9.5, $j_d^*(\operatorname{Proj}(\Gamma_*(\mathcal{L}))) \simeq \operatorname{Proj}(\oplus_{n\in\mathbb{Z}} \Gamma(C', j_d^*(\mathcal{L})(n))s^n)$. So, for any quasi-coherent sheaf $\mathcal{G}$ on C' we also have isomorphisms $\mathcal{G} \simeq \operatorname{Proj}(\Gamma_*(\mathcal{G}))$. In particular, $\mathcal{F}' \simeq \operatorname{Proj}(\Gamma_*(\mathcal{F}'))$ and therefore $\mathcal{F} \simeq \operatorname{Proj}(\oplus_{n\in sdz} \Gamma(C, \mathcal{F}(np))s^n)$.

Then by proposition 9.3 $H^0(C\backslash p,\mathcal{F}) \simeq (\oplus_{n\in sdz}\Gamma(C,\mathcal{F}(np))s^n)_{(s)}$, i.e. $H^0(C\backslash p,\mathcal{F}) = \cup_n H^0(C,\mathcal{F}(np))$. Now again exercise 11.2 shows that $H^0(C\backslash p,\mathcal{F}) \simeq \varinjlim_{n\geq 0} H^0(C,\mathcal{F}(np))$.

We define $W \stackrel{\text{def}}{=} \chi_0(H^0(C\backslash P,\mathcal{F}))$. There is a similar filtration on W which is defined as

$$W_n = W \cap z^{-nr}(L[[z]]\cdot z). \tag{32}$$

Note that this filtration differs from the filtration from section 10.2! (New W_n is the old W_{nr}).

Definition 11.1. The map

$$\chi_0 : (C,p,\mathcal{F},\pi,\hat{\phi}) \to (A,W)$$

is called the *Krichever map.*

Remark 11.4. The Krichever map can be extended also to the relative situation, see [88]. We use the most general definition of this map (not even assuming that p is a K-point), as in [81], [78]. Further generalisation of this construction to higher dimensions see in [78], [43]-[45].

The Krichever map has the following properties.

Proposition 11.2. *Let $(C,p,\mathcal{F},\pi,\hat{\phi})$ be a datum from definition 11.1, and K be an arbitrary field. Let $W=\chi_0(C,P,\mathcal{F},\pi,\hat{\phi})$ be a filtered subspace constructed with the help of chosen trivialisation $\hat{\phi}$.*

Then $H^0(C,\mathcal{F}(nP)) \simeq W_n$ for all $n\in\mathbb{Z}$. In particular, $\mathcal{F}\simeq \operatorname{Proj}\tilde{W}$.

Moreover, we have:

$$H^1(C,\mathcal{O}_C) \simeq \frac{L((z'))}{A+L[[z']]}, \quad H^1(C,\mathcal{F}) \simeq \frac{L((z))}{W+L[[z]]\cdot z} \tag{33}$$

Remark 11.5. If $(A,W)\subset (L((z)),L((z))^{\oplus r})$ is another version of the Schur pair, then

$$H^0(C,\mathcal{F}) \simeq W\cap (L[[z]]\cdot z)^{\oplus r}, \quad H^1(C,\mathcal{F}) \simeq \frac{L((z))^{\oplus r}}{W+(L[[z]]\cdot z)^{\oplus r}},$$

$$H^0(C,\mathcal{O}_C) \simeq A\cap L[[z]], \quad H^1(C,\mathcal{O}_C) \simeq \frac{L((z))}{W+L[[z]]}.$$

Proof. First note that, by definition of the Krichever map,

$$\chi_0(H^0(C,\mathcal{F}(np))) = \alpha_n(H^0(C,\mathcal{F}(np))) \subset W_n$$

for all $n\geq 0$. Conversely, consider an element $w\in W_n$. Then $w\in \chi_0(H^0(C,\mathcal{F}(mp)))$ for some m. Assume that $w\notin \chi_0(H^0(C,\mathcal{F}(np)))$ (thus, $m>n$), $w=\chi_0(b)$, $b\in H^0(C,\mathcal{F}(mp))$, $b\notin H^0(C,\mathcal{F}(np))\subset H^0(C,\mathcal{F}(mp))$.

Note that $w\pi(f)^n \in \hat{\phi}(\mathcal{F}_p)\subset L[[z]]\cdot z$, and therefore (by definition of the sheaf $\mathcal{F}(np) = \mathcal{F}\otimes_{\mathcal{O}_C}\mathcal{O}_C(np)$) there exists a neighbourhood $U\ni p$ such that $b|_U\in\Gamma(U,\mathcal{F}(np))$. Indeed, clearly, $w\pi(f)^n\in L[[z]]\cdot z$ as $\nu_z(w\pi(f)^n)>0$. On the other hand, we know that $w=\hat{\phi}(w')\pi(f)^{-m}$, where $w'\in\mathcal{F}_p\subset\hat{\mathcal{F}}_p$. Since $\mathcal{F}_p$ is a free $\mathcal{O}_{C,p}$-module, $w'=a_1e_1+\ldots a_re_r$ for $a_i\in\mathcal{O}_{C,p}$, where $e_1,\ldots,e_r$ is a basis of this free module (and simultaneously is a basis of the free $\hat{\mathcal{O}}_{C,p}$-module $\hat{\mathcal{F}}_p$). Without loss of generality we can assume $\nu_z(\hat{\phi}(e_i))=i$ and therefore $\nu_z(w\pi(f)^n)>0$ iff $\nu_z(\pi(a_if^{n-m}))\geq 0$ for all i (as $\pi(\mathcal{O}_{C,p})\subset L[[z']]$ and all valuations are therefore divisible by r). But then $a_if^{n-m}\in\mathcal{O}_{C,p}$ for all i (as $\mathcal{O}_{C,p}$ is a DVR) and therefore $w\pi(f)^n=\hat{\phi}(w'')$, where $w''=a_1f^{n-m}e_1+\ldots+a_rf^{n-m}e_r\in\mathcal{F}_p$.

Now consider the following commutative diagram

$$\begin{array}{ccccc} & & b & \longrightarrow & \Gamma(C,\mathcal{F}(mp)/\mathcal{F}(np)) \\ & & \downarrow & & \downarrow \\ \Gamma(U,\mathcal{F}(np)) & \hookrightarrow & \Gamma(U,\mathcal{F}(mp)) & \xrightarrow{\alpha} & \Gamma(U,\mathcal{F}(mp)/\mathcal{F}(np)) \end{array}$$

Note that the first vertical arrow is an embedding, and the second vertical arrow is just an equality, since the sheaf $\mathcal{F}(mp)/\mathcal{F}(np)$ is a skyscraper sheaf with support at p (cf. exercise 9.9). Since $b \in \Gamma(U,\mathcal{F}(np))$, we must have $\alpha(b)=0$, a contradiction. So, $w \in H^0(C,\mathcal{F}(np))$ and $H^0(C,\mathcal{F}(nP)) \simeq W_n$.

The isomorphism $\mathcal{F} \simeq \operatorname{Proj}\tilde{W}$ now follows from remark 11.3. The last assertion can be proved by repeating the proof of lemma 10.1 (replacing $K((z))$ by $L((z))$ in our case). □

From this proposition we immediately get the following corollary.

Corollary 11.1. *If $(C,p,\mathcal{F},\pi,\hat{\phi})$ is a projective spectral data of rank r, then its image under the Krichever map $(A,W)=\chi_0(C,p,\mathcal{F},\pi,\hat{\phi})$ is an embedded Schur pair of rank r.*

In the rest of this section let's check that the Krichever map is well defined on the *isomorphism classes* of projective spectral data.

Let $(C,p,\mathcal{F},\pi,\hat{\phi})$, $(C',p',\mathcal{F}',\pi',\hat{\phi}')$ be two isomorphic projective spectral data, and let (A,W), (A',W') be the corresponding embedded Schur pairs. By definition of an isomorphism of spectral data, there exist a isomorphisms $\beta : C \to C'$, $\psi : \mathcal{F}' \to \beta_*\mathcal{F}$ and an automorphism $\bar{h} : K[[z]] \to K[[z]]$.

By theorem 5.1, item 6, there exists an invertible operator $T \in E(K[[x]])$ such that $\bar{h}(z) = T^{-1}zT$. Moreover, we can choose an operator with the extra property $T|_{x=0}=1$. Since $\bar{h}(z)$ has constant coefficients, the operator T is admissible. Then from the construction of the Krichever map, we'll have $A=\bar{h}(A')=T^{-1}A'T$ and $W=\xi(W')=\bar{h}(W')\cdot\xi(1)$. Note that $\xi(1)=1\cdot(T\xi(1))$, since $T|_{x=0}=1$, and $\bar{h}(W')=T^{-1}W'T=\xi(1)^{-1}T^{-1}W'T\xi(1)$, since operators with constant coefficients commute. Thus,

$$\xi(W') = (1\cdot(T\xi(1)))(\xi(1)^{-1}T^{-1}W'T\xi(1)) = W'\cdot(T\xi(1)).$$

So, $(A,W)\sim(A',W')$, where the equivalence is given by the admissible operator $T\xi(1)$, i.e. the Krichever map is well defined on the set of isomorphism classes.

Remark 11.6. Using the Krichever map and the propositions above about cohomologies, we can improve the Riemann-Roch theorem 10.1 to the case of arbitrary torsion free sheaf and even non-algebraically closed field k: for any torsion free sheaf $\mathcal{F}$ of rank r on a projective curve C over k with a regular point p whose residue field is L, we have

$$\chi(\mathcal{F}(np)) = nr[L:k] + const.$$

The proof is the same.

11.2 The Sato theory

In this section we explain the Sato theory map from diagram (24). It is based on the following statements due to M. Sato [96], cf. [66, Appendix].

Proposition 11.3. *If $G \subset E(K[[x]])$ is a subring that stabilises $K[\partial]$, i.e. for each operator $P \in G$ holds $K[\partial]\cdot P \subseteq K[\partial]$ (cf. (25)), then $G \subseteq D$.*

Proof. Obviously, every differential operator $P \in D$ preserves $K[\partial]$. In order to prove the converse, we need the valuation topology on the ring $K[[x]]$, i.e. the topology induced by the metric associated with the proper discrete K-valuation v such that $v(x) = 1$. Denote by $E := E(K[[x]])$ and let $P \in E$. Let

$$P_- = \sum_{n=1}^{\infty} f_n(x)\partial^{-n} \tag{34}$$

be the $E^{\le -1}$-part of P (see theorem 5.1). The condition $K[\partial] \cdot P \subset K[\partial]$ implies that $D \cdot P \mod xE \subset D \mod xE$, i.e.

$$(QP)_- \in xE \tag{35}$$

for every $Q \in D$. Therefore, $P_- \in xE$ because $1 \cdot P \mod xE \in D \mod xE$. Thus $v(f_n) \ge 1$ for all $n \ge 1$. So let f_m be the coefficient of (34) with the lowest valuation and let $v(f_m) = l \ge 1$. Consider the operator $(\partial^l P)_-$. Then we have

$$(\partial^l P)_- = (\partial^l P_-)_- = (\sum_{n=1}^{\infty} \partial^l f_n \partial^{-n})_- = (\sum_{n=1}^{\infty}\sum_{i=0}^{l} C_l^i f_n^{(i)} \partial^{-n+l-i})_- =$$
$$(\sum_{j=1}^{\infty}\sum_{i=0}^{l} C_l^i f_{j-i}^{(i)} \partial^{l-j})_- = \sum_{j=l+1}^{\infty}\sum_{i=0}^{l} C_l^i f_{j-i}^{(i)} \partial^{l-j}. \tag{36}$$

Since $f_n^{(i)}(0) = 0$ for $0 \le i < l$, we have

$$\partial^l \cdot P_- = \sum_{j=l+1}^{\infty} f_{j-l}^{(l)}(0)\partial^{l-j} = \sum_{j=1}^{\infty} f_j^{(l)}(0)\partial^{-j}.$$

But $(\partial^l P)_- \in xE$ by (35). Thus $f_n^{(l)}(0) = 0$ for all $n \ge 1$. This means that $v(f_m) > l$, a contradiction with our assumption. Therefore, none of the coefficient f_n can have the lowest valuation. Namely, $f_n(x) = 0$ for all $n \ge 1$, i.e. P is a differential operator. □

Theorem 11.1. *Let W be a subspace in the space $K((z)) \simeq K((\partial^{-1}))$ with* $\mathrm{Supp}\, W = K[z^{-1}]$. *Then there exists a unique Sato operator, i.e. a zero-th order invertible operator $S = 1 + s_1\partial^{-1} + \dots$, such that $W = K[\partial] \cdot S$.*

Proof. Since $\mathrm{Supp}\, W = K[z^{-1}]$, we can choose a basis $\{w_n\}_{n\ge 0}$ of the space W in the following form (we identify here ∂^{-1} and z): for every $n \ge 0$

$$w_n = z^{-n} + \sum_{l=1}^{\infty} a_{nl} z^l.$$

Then the equation

$$w_0 = 1 \cdot S = 1 + \sum_{l=1}^{\infty} s_l(0) z^l$$

determines all the constant terms of the coefficients as $s_l(0) = a_{0l}$, $l \ge 1$. Now let's assume that we know $s_l^{(i)}(0)$ for all $l\geq 1$ and $0\leq i<n$. Note that we have

$$z^{-n}\cdot S=\sum_{m=0}^{\infty}\sum_{i=0}^{n}C_n^is_m^{(i)}(0)\partial^{n-m-i}=\partial^n+\sum_{l=1}^{\infty}\sum_{i=0}^{n}C_n^is_{l-i}^{(i)}(0)\partial^{n-l}=$$
$$\partial^n+\sum_{l=1}^{n-1}\sum_{i=0}^{l}C_n^is_{l-i}^{(i)}(0)\partial^{n-l}+\sum_{i=0}^{n-1}C_n^is_{n-i}^{(i)}(0)+\sum_{l=n+1}^{\infty}\sum_{i=0}^{n}C_n^is_{l-i}^{(i)}(0)\partial^{n-l}=$$
$$z^{-n}+\sum_{l=1}^{n-1}\sum_{i=0}^{l}C_n^is_{l-i}^{(i)}(0)z^{-n+l}+\sum_{i=0}^{n-1}C_n^is_{n-i}^{(i)}(0)+\sum_{l=1}^{\infty}\sum_{i=0}^{n}C_n^is_{n+l-i}^{(i)}(0)z^l. \quad (37)$$

The non-negative order terms of the above expression exactly coincides with

$$w_n+\sum_{l=1}^{n-1}\sum_{i=0}^{l}C_n^is_{l-i}^{(i)}(0)w_{n-l}+\sum_{i=0}^{n-1}C_n^is_{n-i}^{(i)}(0)w_0,$$

which contains only known quantities. Therefore, the equation

$$z^{-n}\cdot S=z^{-n}+\sum_{l=1}^{n-1}\sum_{i=0}^{l}C_n^is_{l-i}^{(i)}(0)z^{-n+l}+\sum_{i=0}^{n-1}C_n^is_{n-i}^{(i)}(0)+\sum_{l=1}^{\infty}\sum_{i=0}^{n}C_n^is_{n+l-i}^{(i)}(0)z^l=$$
$$w_n+\sum_{l=1}^{n-1}\sum_{i=0}^{l}C_n^is_{l-i}^{(i)}(0)w_{n-l}+\sum_{i=0}^{n-1}C_n^is_{n-i}^{(i)}w_0 \quad (38)$$

determines $s_l^{(n)}(0)$ for all $l\geq 1$. Thus we have obtained $s_l(x)=\sum_{n=0}^{\infty}\frac{1}{n!}s_l^{(n)}(0)x^n$. Now the operator $S=1+\sum_{l=1}^{\infty}s_l(x)\partial^{-l}$ satisfies $K[\partial]\cdot S=W$ as required. $\square$

Now we can describe the map $[(A,W)]\to[B]$ from (24). Given an embedded Schur pair of rank r we apply theorem 11.1 to get a Sato operator $S\colon\ W=K[z^{-1}]\cdot S$. Then by proposition 11.3 the ring $B:=SAS^{-1}$ is a commutative subring of D, obviously of rank r. If (A',W') is an equivalent Schur pair with $W'=W\cdot T$, $A'=T^{-1}AT$ and $W'=K[z^{-1}]\cdot S'$, we get $S'=ST$ by the uniqueness of the Sato operator and therefore $B'=S'A'(S')^{-1}=S'T^{-1}AT(S')^{-1}=SAS^{-1}=B$. So, the Sato map is well defined.

Remark 11.7. The subspaces W of embedded Schur pairs and even a little bit more general subspaces can be interpreted as closed points of an infinite-dimensional projective variety − *the Sato Grassmanian* (sometimes it is also called a universal Sato grassmanian). There are several ways to define this variety. One way to define its points is to consider *Fredholm* subspaces $W\subset K((z))$, i.e. subspaces W with $\dim_K W\cap K[[z]]\cdot z<\infty$ and $\dim_K\frac{K((z))}{W+K[[z]]\cdot z}<\infty$. Then $Gr=\sqcup_n Gr(n)$ is a union of countable number of irreducible components, where

$$Gr(n)=\{W\subset K((z))|\quad \dim_K(W\cap K[[z]]\cdot z)-\dim_K\frac{K((z))}{W+K[[z]]\cdot z}=n\}.$$

Subspaces with $\operatorname{Supp}W=K[z^{-1}]$ form a *big cell* $Gr^+(0)\subset Gr(0)$. This way is described in [97], [68], [69].

In [101] or in [86] this grassmanian is defined as an infinite-dimensional manifold modeled on Banach spaces. Its points are Fredholm subspaces in a Banach space of square integrable functions on a circle. Alternative descriptions can be found in [108], [117]. We also recommend lecture note [19] for students, where different descriptions of the Sato grassmanian are given.

Exercise 11.4. Calculate the Sato operators for the spaces W_α from exercise 10.2. Calculate the corresponding rings $B_\alpha \subset D$.

On the level of projective spectral data, the sheaves $\mathrm{Proj}\, \tilde{W}_\alpha$ are different torsion free sheaves on the spectral curve C. Due to one-to-one correspondence between spectral data and embedded Schur pairs (see theorem 11.4 below) we have obtained in this exercise a description of all torsion free sheaves of rank one with zero cohomologies on the singular cuspidal projective curve.

Comment 11.2. There is an analogue of the Sato theorem for subspaces $W \subset K((z))^{\oplus r}$:

Theorem 11.2. *([50]) Let $W \subset K((z))^{\oplus r}$ be a subspace with* $\mathrm{Supp}\, W = K[z^{-1}]^{\oplus r}$. *Then there exists a uniquely defined operator $S = E + s_1(x)\partial^{-1} + \dots$ with matrix coefficients $s_i(x) \in M_r(K[[x]])$ such that $W = K[z^{-1}]^{\oplus r} \cdot S$.*

If $A \subset M_r(K[[x]])$ is a stabiliser of W: $A \cdot W \subseteq W$, then $S^{-1}AS \in M_r(K[[x]])[\partial]$ is a matrix differential operator.

This theorem is connected with the theory of commuting *matrix differential operators* and flows on Prym varieties of curves, see [50].

The classification of generic pairs of commuting matrix *ordinary* differential operators (a generalisation of the analytic classification due to Krichever, see the next chapter) was given by Grinevich in [29], and rank one case was developed earlier in works [37], [24]. The approach of [50] leads also to a construction of commuting matrix *partial* differential operators. There are many works with examples of commuting matrix partial differential operators constructed with the help of variety of methods, but there are still no complete theory of them, cf. the problem 13.5 below.

11.3 The classification theorem for commutative rings of ODOs (algebraic version)

Finally we can prove the (algebraic) version of the classification theorem. It can be divided in two steps.

Theorem 11.3. *There is a one-to-one correspondence $[B] \longleftrightarrow [(A, W)]$ given by the Schur map and the Sato map from* (24).

Proof. Since the Sato operator is defined uniquely, the composition of the Schur map combined with the Sato map is the identity. Conversely, if we take the Schur operator to be equal to the Sato operator, then the converse composition will be identity. Since all maps are well defined, we are done. □

Theorem 11.4. *There is a one-to-one correspondence $[(A, W)] \longleftrightarrow [(C, p, \mathcal{F}, \pi, \hat{\phi})]$ given by the direct map and the Krichever map from* (24).

Proof. First note that the composition of maps $(A, W) \mapsto (C, p, \mathcal{F}, \pi, \hat{\phi})$, $(C, p, \mathcal{F}, \pi, \hat{\phi}) \mapsto (A', W') = \chi_0((C, p, \mathcal{F}, \pi, \hat{\phi}))$ from sections 10.2, 11.1 is just the identity map by lemma 10.1 and by construction of the Krichever map.

The composition

$$(C, p, \mathcal{F}, \pi, \hat{\phi}) \mapsto (A, W) \mapsto (C', p', \mathcal{F}', \pi', \hat{\phi}')$$

gives a datum $(C', p', \mathcal{F}', \pi', \hat{\phi}')$ naturally isomorphic to the original datum $(C, p, \mathcal{F}, \pi, \hat{\phi})$ by propositions 11.1 and 11.2. Namely, the isomorphisms (β, ψ) are given by the isomorphisms $C \simeq C' = \mathrm{Proj}(\oplus_n H^0(C, \mathcal{O}_C(np)))$, $\mathcal{F} \simeq \mathrm{Proj}(\oplus_n H^0(C, \mathcal{F}(np)))$, and the trivialisations π' and $\hat{\phi}'$ are compatible with π, $\hat{\phi}$, i.e. $\bar{h} = id$, $\xi = id$. □

Remark 11.8. The one-to-one correspondence from theorem 11.4 can be extended to an equivalence of categories, see [66].

Remark 11.9. In case of *rank one* subrings $B \subset D$ the classification can be simplified. First, examining the proof, one can see that the choice of a trivialisation $\hat{\phi}$ does not matter, and any two projective spectral data of rank one $(C,p,\mathcal{F},\pi,\hat{\phi})$, $(C,p,\mathcal{F},\pi,\hat{\phi}')$ are isomorphic (check it).

Second, we can extend the equivalence of subring by setting $B_1 \sim B_2$ iff $B_1 = \alpha(f^{-1}B_2 f)$, where as before f is an invertible function, and $\alpha : D \to D$ is a scale automorphism, i.e. a linear change of variables $x \mapsto c^{-1}x$, $\partial \mapsto c\partial$. Then, repeating the proofs, we obtain a one-to-one correspondence with extended isomorphism classes of embedded Schur pairs of rank one, where we extend isomorphisms by isomorphisms given by the scaling automorphism $z \mapsto cz$. And the extended isomorphism classes of embedded Schur pairs are in one-to-one correspondence with extended isomorphism classes of projective spectral data of rank one, where we extend isomorphisms of data by letting $\bar{h}(z)$ from definition 10.6 be of the form

$$\bar{h}(z) = a_1 z + a_2 z^2 + \dots, \quad a_1 \neq 0.$$

Exercise 11.5. Show that any two projective spectral data of rank one $(C,p,\mathcal{F},\pi,\hat{\phi})$, $(C,p,\mathcal{F},\pi',\hat{\phi}')$ are isomorphic with respect to the extended notion of isomorphism.

In particular, all such data are defined up to an isomorphism only by a "geometric part" $(C,p,\mathcal{F})$, and an isomorphism of triples is just given by the pair (β,ψ) from definition 10.6.

Combining all together, we obtain the following theorem.

Theorem 11.5. *There is a one-to-one correspondence*

$$[B \subset D \quad \textit{of rank } r] \longleftrightarrow [(C,p,\mathcal{F},\pi,\hat{\phi}) \quad \textit{of rank } r]/\simeq$$

$$[B \subset D \quad \textit{of rank } 1]/\sim \longleftrightarrow [(C,p,\mathcal{F}) \quad \textit{of rank } 1]/\simeq$$

where

- $[B]$ *means a class of equivalent commutative elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in D^*$.
- $\sim$ *means "up to linear changes of variables".*

Remark 11.10. Singular curves and torsion free sheaves which are not locally free were included into the picture by Mumford [71, Section 2] and Verdier [114, Proposition 4]. Mumford's approach was further developed by Mulase [66, Theorem 5.6] and Quandt [88].

Exercise 11.6. Let $(A,W) \subset K((z))$ be an embedded Schur pair. Let $A' \supset A$, $A' \subset K((z))$ be a maximal ring such that $A' \cdot W \subseteq W$ (so, (A',W) is also an embedded Schur pair). Show that the natural embedding $A \subset A'$ induces the embedding of graded rings $\tilde{A} \subset \tilde{A}'$ and the morphism $f : C' := \mathrm{Proj}\, \tilde{A}' \to C := \mathrm{Proj}\, \tilde{A}$ of curves. Show that $\mathcal{F} \simeq f_*\mathcal{F}'$, where $\mathcal{F}' \simeq \mathrm{Proj}(\tilde{W})$ is a sheaf of $\mathcal{O}_{C'}$-modules.

This exercise is closely connected to the notion of a *true (or fake) rank* of the commutative subring $B \subset D$. The rank r of B is *true*, if it is equal to the rank of the *maximal* commutative subring containing B, and is *fake* otherwise.

From this exercise it follows in particular that, if the spectral sheaf is maximal (see the next exercise), then the ring has true rank.

Exercise 11.7. If $\mathcal{F}$ is a sheaf on a curve C such that $\mathcal{F} \not\simeq f_*\mathcal{F}'$ for some morphism $f : C' \to C$ such that f is not an isomorphism and for some sheaf $\mathcal{F}'$ on C', then $\mathcal{F}$ is called maximal.

Let $C = \mathrm{Proj}\, \tilde{A}$, where $A = K[z^{-3}, z^{-4}, z^{-5}] \subset K((z))$. Let $\mathcal{F} = \mathrm{Proj}\, \tilde{W}$, where $W = \langle 1, z^{-1}, z^{-3}, z^{-4}, \dots \rangle$. Show that $\mathcal{F}$ is maximal and that $\mathcal{F}$ is not locally free.

Hint: Show that $A' = A$ for W, then use the Krichever map to show that if $\mathcal{F} \simeq f_*\mathcal{F}'$, then there must be a pair (A',W) with $A' \supset A$.

12 The analytic theory of commuting ODOs

In this section we discuss the analytic counterpart of the theory of commuting differential operators and the meaning of the inverse map from diagram (24). In this approach offered by Krichever in [38], [39] the ground field $K = \mathbb{C}$ and the projective spectral data is replaced by a different data, geometric part of which is the same, but trivialisations are replaced by some (formal) functions. The most important ingredient of this approach is a *Baker-Akhieser function* — a common eigenfunction of commuting differential operators, which can be constructed, in some cases explicitly, by the ring B or by the spectral data.

12.1 The classification of commuting ODOs (analytical version)

Let $B \subset D$ be a *generic* elliptic ring such that its spectral curve C is smooth and the coefficients of operators are germs of *analytic functions*. Since C is smooth, it is a compact Riemann surface, i.e. it is homeomorphic to a sphere with g handles. Moreover, the spectral sheaf is locally free and therefore corresponds to a vector bundle of rank r, see section 9.4.

Remark 12.1. 1) From the Schur theory (section 5) it follows that, if one operator of positive order has analytic coefficients, then all commuting with it operators have analytic coefficients. We'll denote the ring of germs of analytical functions (in a neighbourhood of zero) by $\mathbb{C}\{\{x\}\}$ (it consists of power series convergent in some neighbourhood of zero). Obviously, $\mathbb{C}\{\{x\}\} \subset \mathbb{C}[[x]]$.

2) Let's explain the word "generic". In [38], [39] the ring B assumed to be generated by *two* operators, i.e. the spectral curve is plane, cf. lemma 6.1. It is not difficult to see that generic plane curves are smooth (plane curves can be parametrised by the coefficients of their equations), as the following exercise from [32, Ch. 1, Ex. 5.15] shows.

Exercise 12.1. A homogeneous polynomial f of degree d in 3 variables x, y, z has C_{d+2}^2 coefficients. Consider these coefficients as coordinates of points in $\mathbb{P}^N$, $N = C_{d+2}^2 - 1$.

a) Show that this correspondence is a one to one correspondence between algebraic subsets in $\mathbb{P}^2$ defined by the equations of degree d and closed points in $\mathbb{P}^N$, except the case when f has multiple irreducible factors.

b) Show that irreducible smooth curves of degree d in a) are in one to one correspondence with points of a non-empty Zariski open subset in $\mathbb{P}^N$.

In particular, from this exercise it follows that any plane curve can be included into a *family* of plane curves, with almost all plane curves from this family being smooth (of course, this fact in non-trivial only for singular curves). Notably, for general singular (non-plane) curves this is wrong, see [72]!

However, the restriction to plane curves is not too restrictive: as we know from the Schur theory, the maximal commutative subring of differential operators is completely defined by the Schur operator and by the affine ring of the spectral curve. The Schur operator is defined just by one operator from the ring. As we'll see below, the Schur-Sato operators are closely related with the Baker-Akhieser functions, and can be reconstructed by a spectral data of a smaller subrings. Recall that by exercise 6.1 for any ring $B \subset D$ there are two operators P, Q such that $B/K[P, Q]$ is a finite-dimensional vector space over K, i.e. "almost all" operators of arbitrary ring belong to a subring generated just by two elements (note that *one* element is not enough in general for this claim). On the other hand, as we have already mentioned above and as we'll see below, in some cases the Baker-Akhieser functions can be explicitly reconstructed by such simplified spectral data thus giving information about commuting operators from bigger rings.

3) There is a powerful theory comparing analytic and algebraic properties of complex varieties or complex analytic spaces, see [102] or [32, Appendix B]. For any scheme X of finite type over

$\mathbb{C}$ one can consider the *associated complex analytic space* X_h, which has the same set of complex points, but endowed with the complex topology (which is stronger than the Zariski topology) and with the sheaf of locally holomorphic functions.

This construction can be extended to coherent sheaves on X: any coherent sheaf can be locally represented as a cokernel of a morphism of free sheaves

$$\mathcal{O}_U^{\oplus m} \xrightarrow{\varphi} \mathcal{O}_U^{\oplus n} \to \mathcal{F} \to 0$$

(as it follows from definition 8.15 and basic properties of coherent sheaves [32, Ch.2, §5, Prop. 5.7]). If U is open in the Zariski topology, then U_h is open in the complex topology, and we can define the sheaf $\mathcal{F}_h$ as the cokernel of the free $\mathcal{O}_{X_h}$-sheaves (as φ is defined by the matrix of local sections of the sheaf $\mathcal{O}_U$, which are also local sections of the sheaf $\mathcal{O}_{U_h}$).

The continuous map $\phi : X_h \to X$ induces the natural map of the structure sheaves $\phi^{-1}\mathcal{O}_X \to \mathcal{O}_{X_h}$ which makes the map ϕ into the morphism of locally ringed spaces. In particular, for any coherent sheaf $\mathcal{F}$ on X there is an isomorphism $\mathcal{F}_h \simeq \phi^*\mathcal{F}$, and the natural maps of cohomology groups are defined:

$$\alpha_i : H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h).$$

The following theorem is due to J.P. Serre [102]:

Theorem 12.1. *Let X be a projective scheme over $\mathbb{C}$. Then the functor h induces an equivalence of categories of coherent sheaves on X and coherent analytic sheaves on X_h. Moreover, for any coherent sheaf $\mathcal{F}$ on X the natural maps*

$$\alpha_i : H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h)$$

are isomorphisms for all i.

Let $\mathcal{F}$ be the spectral sheaf of the ring B. By lemma 10.1 and exercise 10.9 we have $H^0(C, \mathcal{F}(p)) \simeq F_{r-1}$, where p is the divisor at "infinity" of the curve C, as before. Let's take the basis $\{\eta_1, \dots, \eta_r\} = \{1, \dots, \partial^{r-1}\}$ in this K-space. Clearly, η_i are free over the field $\mathrm{Quot}(B)$ (and generate the module $\mathrm{Quot}(B) \cdot F$, cf. theorem 6.6). Therefore, we have an embedding of torsion free sheaves

$$\mathcal{O}_C^{\oplus r} \hookrightarrow \mathcal{F}(p), \quad (a_1, \dots, a_r) \mapsto (a_1\eta_1 + \dots + a_r\eta_r)$$

which is an isomorphism at the generic point of C. Denote by T the cokernel of this map. Then T has a finite support, i.e. there are only finitely many closed points on the curve where the stalks of this sheaf are not equal to zero (thus, it behaves like a skyscraper sheaf, cf. exercise 9.9) – these points are exactly the points where the restrictions of $\eta_1, \dots, \eta_r$ are *linearly dependent* over $\mathbb{C}$.

Remark 12.2. In fact, we can say more about the sheaf $\mathcal{F}(p)$ and its global sections. First, note that $H^1(C, T) = 0$, as the support of T is finite (check it). Then from the exact sequence

$$0 \to \mathcal{O}_C^{\oplus r} \to \mathcal{F}(p) \to T \to 0$$

and from its induced exact cohomological sequence (see theorem 9.4)

$$0 \to \mathbb{C}^r \to \mathbb{C}^r = H^0(C, \mathcal{F}(p)) \to H^0(C, T) \to (H^1(C, \mathcal{O}_C))^r \to 0$$

we get $\deg \mathcal{F}(p) = rg$, where $g = h^1(C, \mathcal{O}_C)$ is the genus of the curve C (cf. remark 10.7 about the notion of degree). Now note that the sections $\eta_1, \dots, \eta_r$ induce a global section $\eta_1 \wedge \dots \wedge \eta_r \in H^0(C, \bigwedge^r \mathcal{F}(p))$ of the determinant sheaf (cf. the end of section 9.4), whose restriction vanishes exactly at the same points where the sections $\eta_1, \dots, \eta_r$ are linearly dependent. More precisely, we have an embedding $\mathcal{O}_C \hookrightarrow \bigwedge^r \mathcal{F}(p)$, $1 \mapsto \eta_1 \wedge \dots \wedge \eta_r$, whose cokernel is the sheaf T. Then analogous long exact cohomological sequence gives again $\deg \bigwedge^r \mathcal{F}(p) = r$.

Denote by γ_i, $1 \le i \le rg$ the points from the support of T. At each point γ_i the set of coefficients $\{\alpha_{i,j}\}$ of linear dependence (up to multiplication by a constant) of sections η_j is defined. The set (γ_i, α_{ij}) is called *Tyurin parameters* (or matrix divisor) of the sheaf (vector bundle) $\mathcal{F}(p)$.

Comment 12.1. Tyurin parameters can be thought of as local coordinates of the *moduli space of vector bundles of fixed rank and degree with an enhancement* (or *framed* vector bundles; under the enhancement we mean the fixed basis of global sections of the vector bundle, which generate the fibres of the bundle for almost all points, in our case this is $\eta_1, \dots, \eta_r$), see [111], [112]. Roughly speaking, the moduli space parametrizes isomorphism classes of objects (in this case vector bundles with an enhancement).

There is also the moduli space of *semistable* vector bundles of fixed rank and degree on the curve, which has fine geometric properties: it is a normal (i.e. all its local rings are integrally closed) and irreducible projective variety of dimension $r^2(g-1)+1$, if the curve is smooth of genus $g > 0$, see e.g. the book [74, Rem.5.9]. If the curve is not smooth, it also exists, it is a projective scheme over K, and its points parametrizing the locally free sheaves form an open subset of some irreducible component, see loc. cit., Chapter 5.

By definition, the sheaf $\mathcal{G}$ is called *semistable*, if for any subsheaf $\mathcal{L} \subset \mathcal{G}$ the inequality

$$\frac{\deg(\mathcal{L})}{\mathrm{rk}(\mathcal{L})} \le \frac{\deg(\mathcal{G})}{\mathrm{rk}(\mathcal{G})}$$

holds. The sheaf $\mathcal{G}$ is *stable*, if the strict inequality holds. Stable sheaves are parametrized by points of an open subset in the moduli space of semistable bundles, and this open subset is irreducible and smooth, see [74].

Notably, the spectral sheaves of commutative subrings of ODOs are semistable. Indeed, since $H^0(C, \mathcal{F}) = H^1(C, \mathcal{F}) = 0$, we have for any subsheaf $\mathcal{L} \subset \mathcal{F}$

$$\frac{\deg(\mathcal{L})}{\mathrm{rk}(\mathcal{L})} = \frac{h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) + (g-1)\,\mathrm{rk}\,\mathcal{L}}{\mathrm{rk}\,\mathcal{L}} = g - 1 - \frac{h^1(C,\mathcal{L})}{\mathrm{rk}\,\mathcal{L}} \le g - 1 = \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}.$$

For a *generic* vector bundle of a given rank and degree (i.e. for an open subset of the corresponding moduli space) the set $\{\gamma_i\}$ consists of different points, and the set $\{\alpha_{i,j}\}$ reduces to the following:

$$\eta_r(\gamma_i) = \sum_{j=1}^{r-1} \alpha_{ij}\eta_j(\gamma_i).$$

As it was shown in [39], for any generic Tyurin parameters there exists a commutative ring $B \subset D$ whose spectral sheaf has these parameters. The word "generic" in the beginning of this section will mean also that the spectral sheaf of the ring B is generic.

Now the classification result from [39] for generic subrings $B \subset \mathbb{C}\{\{x\}\}$ can be formulated as follows.

Theorem 12.2. *There is a one to one correspondence between the equivalence classes of generic elliptic subrings $B \subset \mathbb{C}\{\{x\}\}$ of rank r and data*

$$\{C, p, z, \gamma_1, \dots, \gamma_{rg}, (\alpha_{ij}, 1 \le i \le rg, 1 \le j \le r-1), \omega_1(x), \dots, \omega_{r-1}(x)\}$$

where z is a local coordinate at p, $z(p) = 0$, $\omega_i(x) \in \mathbb{C}\{\{x\}\}$ are some functions, $\gamma_i, 1 \le i \le rg$ are different points on the spectral curve C, $\alpha_{ij} \in \mathbb{C}$ are some constants.

We'll give a sketch of proof here (see [38], [39] for details). The correspondence is established via the vector Baker-Akhieser function:

Definition 12.1. A *vector Baker–Akhiezer function* is a function

$$\psi(x,P;x_0) = (\psi_0(x,P;x_0),\dots,\psi_{r-1}(x,P;x_0)), \quad P\in C,$$

on the curve C depending on a formal parameter x (which can be thought of as a local coordinate in some neighbouhood of $0\in\mathbb{C}$) which satisfies the following conditions:

1. In a small (complex) neighbourhood of p (the point "at infinity")
$$\psi(x,P;x_0) = \left(\sum_{s=0}^{\infty}\xi_s(x)z^s\right)\Psi_0(x,z^{-1};x_0),$$
where $z=z(p)$ is a local coordinate near p, $\xi_s(x)$ are vector functions with $\xi_0(x_0) = (1,0,\dots,0)$, $\xi_s(x_0)=(0,\dots,0)$ for $s\ge 1$, and $\Psi_0(x,z^{-1};x_0)$ is a solution of the equation $\frac{d}{dx}\Psi_0 = A\Psi_0$, where
$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1\\ z^{-1}+\omega_1(x) & \omega_2(x) & \omega_3(x) & \dots & \omega_{r-1}(x) & 0\end{pmatrix},$$
with $\Psi_0(x_0,z^{-1};x_0)=Id$ – the identity matrix;
2. on $(C\backslash p)$ the function ψ is meromorphic with simple poles at $\gamma_1,\dots,\gamma_{rg}\in C$;
3. $\mathrm{Res}_{\gamma_i}\psi_j = \alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{r-1}$ for $1\le i\le rg$, $1\le j\le r-2$.

For a given generic elliptic ring $B\subset\mathbb{C}\{\{x\}\}$ the vector BA-function (and the corresponding data) can be constructed as follows. Choose operators $L_1,L_2\in B$ such that $GCD(\mathrm{ord}\,L_1,\mathrm{ord}\,L_2)=r$ (see exercise 6.1). For each point $P=(w,E)\in C_0$ of the affine spectral curve of the operators L_1,L_2 choose a basis $\psi_j(x,P;x_0)$, $j=0,\dots,r-1$ of the subspace of common eigenfunctions of L_1,L_2: $L_1\psi_j=E\psi_j$; $L_2\psi_j=w\psi_j$, with the normalisation $\psi_j^{(i)}(x,P;x_0)|_{x=x_0}=\delta_{i,j}$, $0\le i,j\le r-1$ (cf. the proof of the Burchnall-Chaundy lemma 6.1).

Remark 12.3. The initial value x_0 can be taken to be zero for our aims. However we keep it in our exposition for the convenience of the reader. This parameter is used for the *method of variation of Tyurin parameters*, see [39], [40], where important explicit examples of commuting operators of high rank were constructed. See also section 13.5 below.

Note that for $x_0=0$ the *dual* basis ψ_j^* corresponds via the map η_{χ_P} from theorem 8.1 to the basis $\eta_1,\dots,\eta_r$ above. In particular, the points γ_i from the determinantal spectral sheaf and the coefficients of the linear dependence α_{ij} are defined. In terms of *functions* ψ_j these coefficients can be interpreted as in definition 12.1, item 3. Note also that the points γ_i *depend* on the value of x_0.

Consider the Wronskian matrix $\Psi(x,P;x_0):=W(\psi_0,\dots,\psi_{r-1})$. Then $(\frac{d}{dx}\Psi)\Psi^{-1}=A+O(z)$, where A looks like in definition 12.1 for some functions $\omega_1,\dots,\omega_{r-1}$ (actually, ω_i are differential polynomials in coefficients of L_1; in particular, $\omega_{r-1}=u_{n-2}$ if $L_1=\partial^n+u_{n-2}\partial^{n-2}+\dots$), and the functions ψ_j form a vector BA-function (see [39, §1]).

Conversely, given spectral data from theorem 12.2, one can construct a vector BA-function by solving a system of *singular integral equations* on the Riemann surface C, see [39].

If we know a vector BA-function, then we can find first a *matrix operator* $\bar{L}$ for any meromorphic function $E(P)$ on the curve C with a pole at p of order n:

$$\bar{L}=\sum_{\alpha=0}^{n}w_\alpha(x)\partial^{\alpha n}, \quad \bar{L}\psi=E(P)\psi,$$

here $w_\alpha = (w_\alpha^{ij}$ are *matrices*. Then the corresponding *scalar* operator from B is $L = \sum_{\alpha=0}^{n}\sum_{j=1}^{r} w_\alpha^{ij}(x)\partial^{\alpha r+j-1}$ and again $L\psi = E(P)\psi$.

Remark 12.4. The inverse map can be understood as a way of constructing the ring B via the BA-function.

If $r=1$, then the analytic spectral data from theorem 12.2 has no functional parameters, and can be easily compared with the algebraic projective spectral data. In case $r=1$ the Tyurin parameters determine the spectral bundle (hence, the spectral sheaf $\mathcal{F}$ from the projective spectral data, see section 9.4), and the parameter z determines the trivialisation $\pi : \mathcal{O}_{C,p} \simeq \mathbb{C}[[z]]$. As it was already mentioned in remark 11.9, the choice of a trivialisation $\hat{\phi}$ is not important, i.e. we can choose it arbitrarily.

Moreover, we can compare the BA-function with the Schur-Sato operators as follows.

Consider the (formal) function $\varepsilon = e^{xz^{-1}}$ and the K-vector space $(K[[x]]((z)))\cdot\varepsilon$. This space can be endowed with a $E(K[[x]])$-module structure and with a $K[[x]]((z))$-module structure (a *bimodule* structure) as follows: $K[[x]]((z))$ acts just by multiplication, i.e. for $a\in K[[x]]((z))$, $w = b\cdot\varepsilon$, $a\cdot w := (ab)\cdot\varepsilon$, and $E(K[[x]])$ acts by the rule $Q\cdot w := (Q\cdot b(x,\partial^{-1}))(\varepsilon)$, where $b(x,\partial^{-1})$ means the series b with z replaced by ∂^{-1} (i.e. we obtain an element from $E(K[[x]])$), $(Q\cdot b(x,\partial^{-1}))$ is a product in the ring $E(K[[x]])$, and $E(K[[x]])$ acts on ε by the rule $Q(\varepsilon) := Q|_{\partial^{-1}\mapsto z}\cdot\varepsilon = Q(x,z)\cdot\varepsilon$.

Now let's take an elliptic operator $Q\in B$. Let S be the Schur operator for P such that $S|_{x=0} = 1$ (any Schur operator can be normalized in such a way, because Schur operators are defined up to multiplication on operators with constant coefficients, see section 5) and $a(Q)\in A\subset K((z))$ be the corresponding meromorphic function, $a(Q) = S^{-1}PS$ (see section 10.1). Then $S(\varepsilon)$ will be the BA-function for the ring B. To see this just note that $\psi := S(\varepsilon)$ is a common eigenfunction of operators L_1, L_2 as above with $GCD(\operatorname{ord}L_1, \operatorname{ord}L_2) = 1$. Indeed, for any z (i.e. for almost all points $P\in C_0$) we have

$$L_i(\psi) = (L_i\cdot S)(\varepsilon) = (S\cdot a(L_i)|_{z=\partial^{-1}})(\varepsilon) = a(L_i)S|_{\partial^{-1}\mapsto z}\varepsilon = a(L_i)\psi.$$

Besides, $\psi|_{x=0} = 1$ and therefore ψ is the BA-function. Note that the function ψ (as well as all BA-functions) can be thought of as a function of some differential extension of the ring $K[[x]]$ (see remark 6.5). For *singular* curves the function $\psi := S(\varepsilon)$ can be thought of as a definition of BA-function.

Problem 12.1. Find an explicit dictionary between the spectral data (and between BA-functions and Schur-Sato operators) in general rank $r>1$ case.

Remark 12.5. The BA-function for rank one commuting operators was first constructed by Baker in [5]. Later on Akhiezer used such functions to the investigation of the spectral theory of ordinary differential operators in [1]. For commuting operators of arbitrary rank they were introduced by Krichever in [38], [39].

The algebraic counterpart of *BA-bimodules* was introduced by Drinfeld in [22]. BA-functions were used since that works many times in different context and motivation, see e.g. lecture notes for students [58] and survey [107]. In particular, they were studied also for some singular curves, see loc. cit. Other useful surveys are [25], [101], [83].

To explain the explicit Krichever formula for the BA-function we need first to establish a connection of the thoery of commuting ordinary differential operators with the theory of integrable systems (more precisely, with the KP-theory, where "KP" means a shortening of the names Kadomtsev and Petviashvili, who invented another famous non-linear partial differential equation of mathematical physics).

12.2 Connection to the KP theory

The explicit Krichever formula (see below) gives not only explicit examples of commuting differential operators, but also *explicit solutions* of some famous non-linear partial differential equations (PDEs for short), a big part of which are contained in the *KP hierarchy* — an infinite system of no-linear PDEs. In this section we'll give an overview of the theory appeared around this hierarchy. We'll first describe a purely formal algebraic version of this system (suitable for various generalisations), and then consider an important partial case of analytic algebro-geometric solutions of this system appeared in the Krichever approach.

First we extend the ring $K[[x]]$ by adding infinite number of additional variables ("times"). Namely, consider the completion of the ring $K[x, t_1, t_2, \ldots]$ with respect to the K-valuation ν defined as $\nu(t_n) := n$, $\nu(x) = 1$. Denote this completion by $K[[x, t]]$.

We set $\mathcal{D} := K[[x, t]][\partial]$, $\mathcal{E} := E(K[[x, t]]) = K[[x, t]]((\partial^{-1}))$. Recall (see theorem 5.1) that $\mathcal{E} = \mathcal{E}_- \oplus \mathcal{D}$. Define the *KP hierarchy* as the system

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n \geq 1, \qquad KPH$$

where $L = \partial + u_1\partial^{-1} + u_2\partial^{-1} + \ldots \in \mathcal{E}$.

The KP-hierarchy is equivalent to the system of consistency conditions of *isospectral deformations* of a given differential operator $P \in D$ (in other words, it is a master equation of all isospectral deformations). We'll briefly explain this equivalence following the works [67], [70], [20], [124], [19].

Consider an analytic family $\{P(t) \in D | t \in M\}$ of operators, where the parameter space M is an open domain of $\mathbb{C}^N$ and $P(t)$ is an ordinary differential operator of the form

$$P(t) = \partial^n + a_1(x, t)\partial^{n-1} + \ldots + a_n(x, t)$$

depending on both $x \in \mathbb{C}$ and $t = (t_1, \ldots, t_N) \in M$ analytically (actually, the analyticity condition can be omitted).

Definition 12.2. We say $\{P(t) \in D | t \in M\}$ is a family of isospectral deformations if there exist ordinary differential operators $Q_1(t), Q_2(t), \ldots, Q_N(t)$ depending on the parameter $t \in M$ analytically such that the following system of equations has a nontrivial solution $\psi(x, t; \lambda)$ for every $\lambda \in \mathbb{C}$:

$$\begin{cases} P(t)\psi(x, t; \lambda) = \lambda\psi(x, t; \lambda) \\ \frac{\partial}{\partial t_1}\psi(x, t; \lambda) = Q_1(t)\psi(x, t; \lambda) \\ \ldots \\ \frac{\partial}{\partial t_N}\psi(x, t; \lambda) = Q_N(t)\psi(x, t; \lambda) \end{cases} \qquad (39)$$

The compatibility conditions of this system

$$\frac{\partial}{\partial t_i}(P(t)\psi - \lambda\psi), \quad \frac{\partial^2}{\partial t_i \partial t_j} = \frac{\partial^2}{\partial t_j \partial t_i}$$

are equivalent (check it!) to the equations

$$\frac{\partial}{\partial t_i}P(t) = [Q_i(t), P], \qquad (40)$$

$$\frac{\partial}{\partial t_i}Q_j - \frac{\partial}{\partial t_j}Q_i = [Q_i, Q_j] \qquad (41)$$

Therefore, finding a family $P(t)$ of isospectral deformations of a given operator $P = P(0)$ is equivalent to finding a solution of the *Lax equation* (40) for differential operators $Q_i(t)$ satisfying (41) together with the initial condition $P(t)|_{t=0} = P(0)$.

The simplest example of an isospectral deformation is the *spatial translation* $P(x,t_1) = P(x+t_1)$. Since

$$\frac{\partial}{\partial t_1} P(x+t_1) = [\partial, P(x+t_1)],$$

we have $Q_1(t) = \partial$ in this case.

If $P(t)$ is normalised (as we know from theorem 4.2 this can always be done), then by theorem 5.1 $P(t) = L(t)^n$, $n = \operatorname{ord} P$, where $L(t)$ has the form $L(t) = \partial + u_1(x,t)\partial^{-1} + u_2(x,t)\partial^{-2} + \dots$. Since partial derivations $\partial/\partial t_i$ and commutators $[Q_i, \cdot]$ are *derivations*, the equation (40) is equivalent to the equation

$$\frac{\partial}{\partial t_i} L(t) = [Q_i(t), L(t)] \tag{42}$$

(check it!). Here the left hand side of (42) is a pseudo-differential operator of order at most -1. Therefore, the differential operator $Q_i(t)$ must satisfy

$$[Q_i(t), L(t)] \in \mathcal{E}_-.$$

Lemma 12.1. *Let $L = \partial + u_1\partial^{-1} + u_2\partial^{-1} + \dots$ be an arbitrary normalized pseudo-differential operator of order 1. Then the space*

$$V_L = \{Q \in \mathcal{D} | \quad [Q, L] \in \mathcal{E}_-\}$$

coincides with the $\mathbb{C}$-linear space generated by the operators $(L^m)_+$, $m \geq 0$.

Proof. Since $[L^m, L] = [L^m_+ + L^m_-, L] = 0$, we have

$$[L^m_+, L] = -[L^m_-, L] \in \mathcal{E}_-.$$

Conversely, let $Q \in V_L$ be an element of order m. The condition $[Q, L] \in \mathcal{E}_-$ implies that the leading coefficient of Q is a constant, say $c \in \mathbb{C}$. Since L^m_+ is monic, the linear combination $Q - cL^m_+$ has order less than m. Since $[Q - cL^m_+, L] \in \mathcal{E}_-$, the lemma follows by induction on m. □

So, to find all possible isospectral deformations (with $N \to \infty$) it is necessary to solve the KP system with the initial condition $L(0)^n = P(0) = P$ (then $P(t) = L(t)^n$, see below). Below we'll show that it is also sufficient, and we'll show that this system is uniquely solvable. Note that the first equation of the KP system gives exactly the spatial translations.

Let $\mathbb{C}^\infty = \varinjlim_n \mathbb{C}^n$ be an ∞-dimensional space. Then the algebra $\mathbb{C}[[t]] = \mathbb{C}[[t_1, t_2, \dots]]$ can be thought of as an algebra of formal functions in a small neighbourhood of zero in $\mathbb{C}^\infty$. For a given $L \in \mathcal{E}$ consider the following 1-forms on $\mathbb{C}^\infty$ with values in $\mathcal{E}$:

$$Z^\pm_L = \pm \sum_{n \geq 1} (L^n)_\pm dt_n.$$

These forms are called *Zakharov-Shabat* connections, and can be thought of as connections on the trivial bundle $\mathbb{C}^\infty \times E(\mathbb{C}[[x]])$ over $\mathbb{C}^\infty$ on which the Lie algebra $\mathcal{E}$ acts by the commutator.

Lemma 12.2. *The KP hierarchy is equivalent to the equations $dL = [Z^+_L, L] \Longleftrightarrow dL = [Z^-_L, L]$, where $d = \sum dt_n \frac{\partial}{\partial t_n}$.*

The proof is clear.

Proposition 12.1. *The KP hierarchy is equivalent to the system*

$$dZ^\pm_L = Z^\pm_L \wedge Z^\pm_L$$

(which means that the connections $Z^\pm_L$ are flat).

Proof. Consider the form $Z_L := Z_L^+ - Z_L^- = \sum L^n dt_n$. Then, clearly, $Z_L \wedge Z_L = 0$. Now note that the equation $dL = [Z_L^+, L]$ is equivalent to the equation $dZ_L = Z_L^+ \wedge Z_L + Z_L \wedge Z_L^+$. Indeed, from the equations of the KP hierarchy it follows

$$\frac{\partial L^m}{\partial t_n} = [(L^n)_+, L^m] \quad \forall n, m \geq 1,$$

as $\partial/\partial t_n$ and $[(L^n)_+, \cdot]$ are derivations. The coefficient at $dt_m \wedge dt_n$ of dZ_L is equal to

$$\frac{\partial L^n}{\partial t_m} - \frac{\partial L^m}{\partial t_n} = [(L^m)_+, L^n] - [(L^n)_+, L^m]$$

and the corresponding coefficient of the right hand side is equal to the same expression. Now note that

$$0 = dZ_L - Z_L^+ \wedge Z_L - Z_L \wedge Z_L^+ + Z_L \wedge Z_L = (dZ_L^+ - Z_L^+ \wedge Z_L^+) - (dZ_L^- - Z_L^- \wedge Z_L^-).$$

Since the expression $(dZ_L^+ - Z_L^+ \wedge Z_L^+)$ has coefficients in $\mathcal{D}$ and the expression $(dZ_L^- - Z_L^- \wedge Z_L^-)$ has coefficients in $\mathcal{E}$, they must be equal to zero simultaneously.

Conversely, assume that $dZ_L^+ - Z_L^+ \wedge Z_L^+ = 0$. Then all coefficients at $dt_m \wedge dt_n$ are equal to zero:

$$\frac{\partial (L^n)_+}{\partial t_m} - \frac{\partial (L^m)_+}{\partial t_n} - [(L^m)_+, (L^n)_+] = 0,$$

where from

$$\frac{\partial (L^m)_+}{\partial t_n} - [(L^n)_+, L^m] \in \mathcal{E}^{\leq n-2}.$$

We have also

$$\left(\frac{\partial}{\partial t_n} - [(L^n)_+, \cdot]\right) L^m = m \left(\frac{\partial L}{\partial t_n} - [(L^n)_+, L]\right) L^{m-1} + \ldots (l.o.t.),$$

whence

$$\frac{\partial L}{\partial t_n} - [(L^n)_+, L] \in \mathcal{E}^{\leq n-m+1}.$$

In the limit $m \to \infty$ we get the KP hierarchy. □

As a corollary, we get that KP hierarchy implies equation (42), and therefore it is equivalent to (40) + (42).

Exercise 12.2. The equations

$$\frac{\partial (L^n)_+}{\partial t_m} - \frac{\partial (L^m)_+}{\partial t_n} - [(L^m)_+, (L^n)_+] = 0, \qquad ZS_{n,m}$$

from previous proposition are called *Zakharov-Shabat* equations.

The KP hierarchy contains many known non-linear partial differential equations like KdV and KP. Show that

$$(L^2)_+ = \partial^2 + 2u_{-1}, \quad (L^3)_+ = \partial^3 + 3u_{-1}\partial + 3u_{-2} + 3u'_{-1}.$$

Using this show that the equation $ZS_{2,3}$ is equivalent to the *KP equation* (cf. this equation with the KdV equation on the first page):

$$\frac{3}{4}u_{yy} = \left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x\right)_x \qquad KP,$$

if we put $u = 2u_{-1}$, $t_2 = y$, $t_3 = t$.

Now we are ready to explain the unique solvability of the KP hierarchy.

Proposition 12.2. *Let* $L(t) = \partial + u_1(x,t)\partial^{-1} + \dots$ *be an operator of the first order (looking like a solution of the KP hierarchy). Let* $S \in \mathcal{E}$ *be a Schur operator:* $L = S\partial S^{-1}$ *. Then the KP hierarchy is equivalent to the system*

$$dS = -(S\omega_\partial S^{-1})_- S = Z_L^- S, \qquad SW$$

where $\omega_\partial = \sum_{n\geq 1} \partial^n dt_n$ *. This system is called Sato-Wilson hierarchy.*

Proof. If $S(t)$ is a solution of the system (SW) (and S is a Schur operator, i.e. $S = 1 + \dots$), then put $L(t) = S\partial S^{-1}$. Then

$$dL = (dS)\partial S^{-1} - S\partial S^{-1}(dS)S^{-1} = Z_L^- S\partial S^{-1} - S\partial S^{-1} Z^- = [Z_L^-, L].$$

Conversely, let $L(t)$ be a solution of the KP hierarchy and let S_0 be a Schur operator. Set $Z_0^- := S_0^{-1} Z_L^- S_0 - S_0^{-1} dS_0$ (a gauge transform of the connection Z_L^-). Then $dZ_0^- = Z_0^- \wedge Z_0^-$ (this can be checked purely formally). Let's show that the coefficients of Z_0^- are series with *constant coefficients*, i.e. from the ring $K[[t]]((\partial^{-1}))$:

$$\begin{aligned}[Z_0^-, \partial] &= S_0^{-1}[S_0 Z_0^- S_0^{-1}, S_0\partial S_0^{-1}]S_0 = \\ &S_0^{-1}[Z_L^- - (dS_0)S_0^{-1}, L]S_0 = S_0^{-1}([Z_L^-, L] - [(dS_0)S_0^{-1}, L])S_0 = S_0^{-1}(dL - dL)S_0 = 0,\end{aligned}$$

because

$$dL = (dS_0)\partial S_0^{-1} - S_0\partial S_0^{-1}(dS_0)S_0^{-1} = [(dS_0)S_0^{-1}, L].$$

Therefore there exists an operator $C \in K[[t]]((\partial^{-1}))$ such that $C = 1 + l.o.t.$ and $dC = Z_0^- C$ (check it!). Set $S := S_0 C$, then

$$dS - Z_L^- S = (dS_0)C + S_0(dC) - Z_L^- S_0 C = S_0(S_0^{-1}dS_0 + (dC)C^{-1} - S_0^{-1} Z_L^- S_0)C = 0,$$

and, moreover, $S\partial^{-1}S^{-1} = S_0 C\partial C^{-1} S_0^{-1} = S_0^{-1}\partial S_0^{-1} = L$. □

Corollary 12.1. *Denote by* $\mathcal{V}$ *the subgroup of zeroth order monic invertible operators from* $\mathcal{E}$ *(i.e. the group of Schur-Sato operators), by* $\mathcal{V}_{K[[t]]}$ *the same subgroup with coefficients from* $K[[t]]$ *, and by* $\mathcal{V}_K$ *the same subgroup with coefficients from* K *.*

There is a one to one correspondence between the sets
$Sol(KP) := \{L \in \mathcal{E} | \quad L$ *satisfies KP hierarchy*$\}$ *and*
$Sol(SW) = \{S \in \mathcal{V} | \quad S$ *satisfies SW hierarchy*$\}/\mathcal{V}_K$ *.*

Proof. The proof is straightforward. Recall that by Schur theorem 5.1 the Schur operator is defined up to an operator with constant (i.e. not depending on x) coefficients. In our case, if $L = S_1\partial S_1^{-1} = S_2\partial S_2^{-1}$, then $S_2 = S_1 C$ with $C \in \mathcal{V}_{K[[t]]}$. From the SW equations it follows that

$$d(S_1 C) = Z_L^-(S_1 C), \quad (dS_1)C + S_1(dC) = Z_L^- S_1 C,$$

whence $S_1(dC) = 0$, i.e. $dC = 0$ and $C \in \mathcal{V}_K$. □

To solve the Sato-Wilson hierarchy, we need two more assertions.

Consider a completion of the ring $\mathcal{E}$ with respect to the valuation v defined as $v(t_i) := i$, $v(x) = 0$: $\hat{\mathcal{E}} := \{\sum_{i=-\infty}^{\infty} a_i\partial^i | \quad v(a_i) \overset{i\to\infty}{\longrightarrow} \infty\}$. Denote by $\hat{\mathcal{D}}$ the corresponding completion of the ring $\mathcal{D}$, $\hat{\mathcal{D}} = \hat{\mathcal{E}}_+$. Denote by $Pr : K[[x,t]] \to K[[x,t]]/(t_1, t_2, \dots) \simeq K$ the natural projection. Define the subgroups

$$\hat{\mathcal{D}}^* = \{P \in \hat{\mathcal{D}} | \quad Pr(P) = 1 \text{ and } \exists \ P^{-1} \in \hat{\mathcal{D}}\},$$

$$\hat{\mathcal{E}}^* = \{P \in \hat{\mathcal{D}} | \quad Pr(P) \in \mathcal{V} \text{ and } \exists \ P^{-1} \in \hat{\mathcal{E}}\}.$$

Theorem 12.3. *(generalized Birkhoff decomposition)*
For any $\Psi \in \hat{\mathcal{E}}^*$ *there exists a unique operators* $S \in \mathcal{V}$, $Y \in \hat{\mathcal{D}}^*$ *such that* $\Psi = YS$.

Proof. Consider the space $W := K[z^{-1}] \cdot \Psi \subset K[[t]][z^{-1}]$ defined with the help of the Sato action. Note that it has an admissible basis $\{w_i\}$ as in the proof of Sato theorem 11.1: $w_i = z^{-i} + w_{i1}z + \dots$ (where now $w_{ij} \in K[[t]]$), because $Pr(z^{-i} \cdot \Psi) = z^{-i} + \dots$. Then the proof of the Sato theorem can be applied to get a uniquely defined operator $S \in \mathcal{V}$ such that $W = K[z^{-1}] \cdot S$. Therefore, $K[z^{-1}] \cdot (\Psi S^{-1}) \subseteq K[z^{-1}]$ and by proposition 11.3 the operator $Y := \Psi S^{-1}) \subseteq K[z^{-1}] \in \hat{\mathcal{D}}$, and, obviously, $Y \in \hat{\mathcal{D}}^*$.

The decomposition $\Psi = YS$ is unique: if $Y_1 S_1 = Y_2 S_2$, then $Y_2^{-1} Y_1 = S_2 S_1^{-1}$, but $\hat{\mathcal{D}}^* \cap \mathcal{V} = \{1\}$, therefore $Y_1 = Y_2$ and $S_1 = S_2$. □

Remark 12.6. This theorem generalizes the Birkhoff decomposition for loop groups, see e.g. [86]. The original proof of this theorem in [70] is more complicated, but contains formulae for coefficients of the operators S, Y . The analytic counterpart of these formulae was later developed in [89], [90].

Theorem 12.4. *If* S *is a solution of the Sato-Wilson hierarchy, then there exists a uniquely defined operator* $Y \in \hat{\mathcal{D}}^*$ *such that*

$$dY = YZ_L^+, \quad Z_L^+ = (S\omega_\partial S^{-1})_+.$$

The proof of this theorem is technical; its idea – the inductive construction of the operator Y . We refer to the original proof in [70, Th.4.1] (or leave it as an exercise).

Theorem 12.5. *The Sato-Wilson hierarchy is equivalent to the linear differential equation*

$$d\Psi = \Psi\omega_\partial, \quad on \ \Psi \in \hat{\mathcal{E}}^* .$$

Proof. If S is a solution of the SW system, and Y is a solution from theorem 12.4, then put $\Psi := YS$. Then we have

$$d\Psi = (dY)S + Y(dS) = YZ_L^+ S - YZ_L^- S = \Psi S^{-1}(Z_L^+ - Z_L^-)S = \Psi\omega_\partial.$$

Vice versa, if Ψ is a solution of the linear equation, then by the generalized Birkhoff theorem 12.3 $\Psi = YS$ for uniquely defined Y and S . Then

$$\omega_\partial = \Psi^{-1}(d\Psi) = S^{-1}Y^{-1}((dY)S + Y(dS)) = S^{-1}(Y^{-1}dY)S + S^{-1}dS,$$

whence

$$S\omega_\partial S^{-1} = Z_L^+ - Z_L^- = Y^{-1}dY + (dS)S^{-1}.$$

Since the first summand has coefficients in $\hat{\mathcal{D}}$, and the second summand has coefficients in $\mathcal{E}_-$, we have $dS = Z_L^- S$ and $dY = YZ_L^+$. □

Theorem 12.6. *For any operator* $\Psi_0 \in 1 + E(K[[x]])_-$ *the equation* $d\Psi = \Psi\omega_\partial$ *has a unique solution* $\Psi(t) \in \hat{\mathcal{E}}^*$ *with the initial condition* $\Psi(0) = \Psi_0$.
Namely, $\Psi(t) = \Psi_0 E(T)$ *, where* $E(t) := \exp(\sum_{i\geq 1} t_i \partial^i)$.

Proof. Note that $dE(T) = E(T)\omega_\partial$ and therefore $\Psi_0 E(T)$ is a solution of the linear equation $d\Psi = \Psi\omega_\partial$.

Uniqueness: If $\Psi(t)$ is any solution with $\Psi(0) = 0$, then $\frac{\partial}{\partial t_n}\Psi = \Psi\partial^n$ for any $n \geq 1$, hence $\frac{\partial}{\partial t_{n_1}}\frac{\partial}{\partial t_{n_2}} \dots \frac{\partial}{\partial t_{n_k}}\Psi = \Psi\partial^{n_1+\dots+n_k}$ and therefore $(\frac{\partial}{\partial t_{n_1}}\frac{\partial}{\partial t_{n_2}} \dots \frac{\partial}{\partial t_{n_k}}\Psi)(0) = \Psi(0)\partial^{n_1+\dots+n_k}$, whence $\Psi(t) \equiv 0$. □

Corollary 12.2. *For any initial condition $S_0 \in 1 + E(K[[x]])_-$ there exists a unique solution $S(t)$ of the Sato-Wilson hierarchy with $S(0) = S_0$.*

Proof. First solve the linear equation $d\Psi = \Psi\omega_\partial$ with the initial condition $\Psi(0) = S_0$. Then $\Psi = S_0 E(T) = YS(t)$ by theorem 12.3. Then by theorem 12.5 $S(t)$ is a solution of the Sato-Wilson hierarchy. □

Remark 12.7. If S_0 is a Schur operator of some differential operator $P \in D$, then the space $W_0 := K[z^{-1}] \cdot S_0$ has a non-trivial stabiliser in the field $K((z))$ (namely, the pair (A, W_0) consisting of the maximal stabiliser A and the space W_0 is a Schur pair corresponding to the maximal commutative subring $B_P = \{Q \in D| \quad [Q, P] = 0\}$. Then A will be also a stabiliser of the space $W := W_0 \cdot E(t) = K[[t]][z^{-1}] \cdot S(t)$ by theorem 12.3. Then the arguments of the proof of proposition 11.3 imply that $S(t) \cdot A \cdot S(t)^{-1} \in \mathcal{D}$ (cf. the proof of theorem 12.3), i.e. by solving the KP or SW hierarchy with the initial condition S_0 we indeed obtain the family of isospectral deformations of the whole ring of differential operators B_P.

Comment 12.2. There is the following geometric interpretation of the Sato-Wilson hierarchy (see e.g. [101], [67]). It can be interpreted as a flow on the Sato Grassmanian defined by the Sato action as follows. Consider the SW hierarchy with the initial condition S_0. Then take the space $W(0) := K[z^{-1}] \cdot S_0 \in Gr^+(0)$ as an initial point of the flow. Then the flow is defined by the rule $W(t) := W(0) \cdot E(t) \in Gr(0)$ (strictly speaking, the space $W(t)$ belong to a bigger space $K[t]((z))$, i.e. this space defines a $K[t]$-point of the Sato Grassminian; we'll see below in remark 13.3 that in the case when the initial condition S_0 of the Sato-Wilson hierarchy comes from a differential operator $P \in D$ with $\mathrm{rk}\, B_P = 1$, then the flow can be defined on the Grassmanian $Gr(0)$ over K). By Sato theorem 11.1 $W(t) = K[z^{-1}] \cdot S(t)$, and, as we have seen above, $S(t)$ is the solution of the Sato-Wilson hierarchy. The following theorem can be found in the loc. cit. papers:

Theorem 12.7. *If the initial condition S_0 of the Sato-Wilson hierarchy comes from a differential operator $P \in D$ (i.e. S_0 is a Schur operator of P) with $\mathrm{rk}\, B_P = 1$, then the closure of the flow $W(t)$ in $Gr(0)$ is isomorphic to the moduli space of coherent torsion free sheaves of rank one.*

In particular, if the spectral curve is smooth, this moduli space is the *Jacobian*, which will be described in more details in the next section.

13 Jacobians of curves and explicit formulae of BA-functions

A great part of this section consists of classical results about compact Riemann surfaces (smooth projective algebraic curves over $\mathbb{C}$). The most usable book for the first read about complex algebraic varieties is [27]. We'll need mostly the second chapter of this book. At the end of this section we'll explain the explicit Krichever formulae of BA-functions.

13.1 Jacobians of curves

Let C be a smooth curve over $\mathbb{C}$ (recall that C is a projective spectral curve, i.e. an irreducible projective curve over a field). First let's recall terminology about differential 1-forms on C. Locally any such a form ω can be written as $f(z)dz$, where z is local coordinate (a coordinate in a local chart on C).

If all functions $f(z)$ in all charts are *holomorphic*, then ω is called a *holomorphic differential form*, i.e. $\omega \in \Omega^1_C$, where Ω^1_C denotes the sheaf of holomorphic differential 1-forms on C. It is known that Ω^1_C is a locally free sheaf of rank one, see e.g. [32, Ch. 2, Th.8.15].

If the functions $f(z)$ are meromorphic of the form

$$f(z) = \frac{a_{-n}}{z^n} + \frac{a_{-n+1}}{z^{n-1}} + \ldots + \frac{a_{-2}}{z^2} + a_0 + a_1 z + \ldots$$

(i.e. $\operatorname{res}_p \omega = 0$ for any $p \in C$), then ω is called a *differential of the second kind.*

If the functions $f(z)$ are meromorphic,

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + \ldots$$

and $\sum_{p \in C} \operatorname{res}_p \omega = 0$, then ω is called a *differential of the third kind.*

By the Serre duality theorem (see e.g. [32, Ch. 3, Cor. 7.7.]), $h^0(C, \Omega^1_C) = g = h^1(C, \mathcal{O}_C)$. Choose a basis $\{\omega_1, \ldots, \omega_g\}$ of the space $H^0(C, \Omega^1_C)$. Fix a point $p_0 \in C$. Now let's associate to any point $p \in C$ the vector

$$j_\gamma(p) = (\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g),$$

where γ is a (real) curve on C connecting the points p_0 and p. If $\gamma' = \gamma + \lambda$, where λ represents a class in $H_1(C, \mathbb{Z})$, then $j_{\gamma'}(p) = j_\gamma(p) + j_\lambda(p_0)$. Vectors $j_\lambda(p_0)$ generate a lattice $\Lambda \subset \mathbb{C}^g$ of rank $2g$ with a basis $\{j_{a_i}, j_{b_k}\}$, where $\{a_i, b_j\}$ is a basis in $H_1(C, \mathbb{Z})$ such that $a_i \cdot b_j = \delta_{ij}$, $a_i \cdot a_j = b_i \cdot b_j = 0$ (recall that any compact Riemann surface of genus g, or, equivalently, the sphere with g handles, can be glued from a $4g$-polygon by identifying pairs of edges indexed by letters a_i, b_j).

Therefore, the map

$$j : C \to \mathbb{C}^g/\Lambda, \quad p \mapsto j_\gamma(p) + \Lambda$$

does not depend on γ and p_0, and is called the *Abel map.* The lattice Λ is called the *period lattice*, and the torus $JacC := \mathbb{C}^g/\Lambda$ is called the *Jacobian* of the curve C.

Theorem 13.1 (Riemann)**.** *For given forms $\omega, \eta \in H^0(C, \Omega^1_C)$ there are the following relations (Riemann relations):*

1. $$\sum_{i=1}^{g} (\int_{a_i} \omega \cdot \int_{b_i} \eta - \int_{b_i} \omega \cdot \int_{a_i} \eta) = 0$$

2. $$Im(\sum_{i=1}^{g} \overline{\int_{a_i} \omega} \cdot \int_{b_i} \omega) > 0$$

Corollary 13.1. *There is a basis $\{\omega_1, \ldots, \omega_g\} \subset H^0(C, \Omega^1_C)$ such that $\int_{a_i} \omega_j = \delta_{ij}$.*

If $B_{ij} = \int_{b_i} \omega_j$, then $B_{ij} = B_{ji}$ and $Im(B)$ is a positively defined matrix. In particular, in this basis $\Lambda \simeq \mathbb{Z}^g + B\mathbb{Z}^g$.

Expanding remark 9.7, consider the group $Div(C)$ of Weil divisors on the curve C. Recall that it is a free abelian group generated by points of the curve C, i.e. finite sums $\sum n_i p_i$, $n_i \in \mathbb{Z}$, $p_i \in C$. For any $D \in Div(C)$ define the *degree* as $\deg D = \sum n_i$. For any function $f \in K(C)$ define the *principal* divisor $(f) = \sum_{p \in C} v_p(f) p$, where v_p is the discrete valuation associated to the point p. Note that $\deg(f) = 0$.

Introduce the *linear equivalence* on the group $Div(C)$: $D, D' \in Div(C)$ are equivalent, if $D - D' = (f)$ for some $f \in K(C)$. Denote by Div^0 the subgroup of degree zero divisors. Now extend the Abel map to the group $Div(C)$:

$$j(D) := \sum n_i j(p_i).$$

Theorem 13.2 (Abel). *For $f \in K(C)$ we have $j((f)) = 0$. Vice versa, if $j(D) = 0$ with $D \in Div(C)$, $\deg D = 0$, then $D = (f)$ for some $f \in K(C)$.*

Corollary 13.2. *$Jac(C) \simeq Div^0(C)/\sim$, where $\sim$ is the linear equivalence of divisors.*

Remark 13.1. Analogues of the Jacobian variety exist also for non-smooth curves and for curves over non-algebraically closed fields (and of course, there are algebraic constructions of these varieties, cf. comments 12.1, 12.2), see e.g. the book [103]. Unlike the case of smooth curves, Jacobians of non-smooth curves are not compact, and the moduli spaces of torsion free rank one sheaves serve as natural compactifications. In higher dimensions these varieties are known under the name of the *Picard schemes*. The extensive theory of the Picard schemes and their compactifications see in [2], [3].

13.2 Theta-functions

The explicit formulae of Krichever use special functions, namely the *theta-functions*. In this subsection we give a short review of them. The main reference is the book of Mumford [73].

Let U be a complex symmetric matrix of order g with $Im(U)$ positive defined (such matrices are called *Siegel matrices*). Set

$$\theta(z, U) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t U n + 2\pi i n^t z), \quad z = (z_1, \ldots, z_g)^t \in \mathbb{C}^g$$

Then (see [73]) θ is a holomorphic function on $\mathbb{C}^g$ and is U-quasi-periodic, i.e. $\forall m \in \mathbb{Z}^g$

$$\theta(z+m, U) = \theta(z, U)$$
$$\theta(z+Um, U) = \exp(-\pi i m^t U m - 2\pi i m^t z)\theta(z, U).$$

Definition 13.1. Let B be a Siegel matrix and $l \in \mathbb{N}$ be an integer number. An integral function $f(z)$ on $\mathbb{C}^g$ is called B-quasi-periodic of weight l, if for any $m \in \mathbb{Z}^g$

$$f(z+m) = f(z), \quad f(z+Bm) = \exp(-\pi i l m^t B m - 2\pi i l m^t z) f(z).$$

Denote by θ_B^l the space of such functions.

There are several useful applications of theta-functions. One of them is an embedding of a complex torus $X_B = \mathbb{C}^g/\mathbb{Z}^g + B\mathbb{Z}^g$ to the projective space. If $f_0, \ldots, f_N$ is a basis of the space θ_B^l and for any $z \in \mathbb{C}^g$ there exists $f_i(z) \neq 0$, then the map

$$z \mapsto (f_0(z) : \ldots : f_N(z)),$$

is an embedding $X_B \hookrightarrow \mathbb{P}^N$ (for l big enough).

Another application is a construction of functions with given zeros. Consider the function $f(z, P) := \theta(z + j(p), B)$, where $p \in C_0$ runs over the $4g$-polygon, $z \in \mathbb{C}^g$ and B is defined as in corollary 13.1.

Theorem 13.3. *[73, Ch. 2, §3] There is a vector $\triangle \in \mathbb{C}^g$ such that for all $z \in \mathbb{C}^g$ the function $f(z, P) = \theta(z + j(p), B)$ is either identically zero, or has g zeros $Q_1, \ldots, Q_g$ such that*

$$\sum_{i=1}^{g} j(Q_i) = -z + \triangle.$$

Corollary 13.3. *For any divisor $D = P_1 + \ldots + P_g$ there exists a vector $\zeta = \zeta(D) \in \mathbb{C}^g$ such that the divisor of zeros of the function $\theta(j(P) + \zeta, B)$ is equal to D.*

Remark 13.2. The set $\{z \in X_B | \quad \theta(z, B) = 0\}$ is called the θ-divisor. It is important for the geometric characterisation of the abelian variety.

The Weierstrass functions mentioned above can be also represented with the help of theta-functions. There are formulae for solutions of polynomial equations of degree greater than four that use theta-functions, see e.g. [73].

13.3 Formal BA-functions

Let's extend the notion of formal BA-function from remark 12.4. Consider the formal function $\varepsilon(t) := \exp(xz^{-1} + \sum_{i=1}^{\infty} t_i z^{-i})$ and extend the action from 12.4 as follows:

$$\partial(\varepsilon(t)) = z^{-1}\varepsilon(t), \quad \frac{\partial}{\partial t_n}\varepsilon(t) = z^{-n}\varepsilon(t).$$

So, we obtain a $\mathcal{E}$-module BA of formal BA-functions with the generator $\varepsilon(t)$.

Definition 13.2. The formal BA-function is an element of the module BA

$$\psi(x,t,z) = \hat{\psi}(x,t,z)\varepsilon(t) = (1 + \sum_{i=1}^{\infty} w_{-i}(x,t)z^i)\varepsilon(t).$$

Proposition 13.1. *Let ψ be a formal BA-function, and $P = \sum_{n\geq 1} dt_n \cdot P_n \in \Omega^1 \otimes \mathcal{D}$. If $d\psi = P\psi$, then the operator L defined by ψ as $L = S\partial S^{-1}$, where $\psi = S(\varepsilon(t))$, satisfies KP hierarchy. If, moreover, $P_n\psi = z^{-n}\psi$, then $L^n \in \mathcal{D}$.*

Proof. From the equation $d\psi = P\psi$ it follows $d(S(\varepsilon(t)) = P(S(\varepsilon(t)$, whence

$$(dS)\varepsilon(t) + S\omega_\partial\varepsilon(t) = PS\varepsilon(t)$$

and $dS = PS - S\omega_\partial$ or $P = (dS)S^{-1} + S\omega_\partial S^{-1}$. But

$$P_n = (P_n)_+ = (S\partial^n S^{-1})_+ = (L^n)_+,$$

whence $dS = Z_L^- S$, $Z_L^- = -(S\omega_\partial S^{-1})_- = -\sum dt_n (L^n)_-$ – the Sato-Wilson hierarchy. Therefore, L is a solution of the KP hierarchy.

If $P_n\psi = z^{-n}\psi$, where $\psi = \hat{\psi}\varepsilon(t)$, then

$$\frac{\partial\hat{\psi}}{\partial t_n}\varepsilon(t) + z^{-n}\hat{\psi}\varepsilon(t) = \frac{\partial\psi}{\partial t_n} = P_n\psi = P_n\psi = z^{-n}\hat{\psi}\varepsilon(t)$$

and therefore $\frac{\partial\hat{\psi}}{\partial t_n} = 0$, whence

$$0 = \frac{\partial S}{\partial t_n} = P_n S - S\partial^n$$

and $L^n = S\partial^n \S^{-1} = P_n \in \mathcal{D}$. □

13.4 Krichever explicit formulae

Now we can explain the Krichever explicit formulae.

Theorem 13.4 (Krichever)**.** *Let C be a smooth curve, $p \in C$, $z = z(p)$, and $D = P_1 + \ldots + P_g$ be a non-special divisor, i.e. $h^0(C, \mathcal{O}_C(D)) = 1$ (here $\mathcal{O}_C(D)$ denotes the invertible sheaf corresponding to the divisor D, cf. remark 9.7; according to comment 12.1 $h^0(C, \mathcal{O}_C(D)) = 1$ for almost all such divisors). Then there exists a uniquely defined function $\psi = \psi(x,t,p)$ on C such that*

1. *ψ is meromorphic on $C \backslash p$ with poles at P_i (i.e. $D + (\psi) \geq 0$).*

2. *in a neighbourhood of p*

$$\psi = \psi(x,t,z) = (1 + \sum_{s\geq 1} w_s(x,t)z^s)\varepsilon(t).$$

Proof. Fix differentials of the secon kind with given singular parts at p:

$$\Omega_0 = d(z^{-1}) + \dots, \quad \Omega_k = d(z^{-k}) + \dots, \quad k = 1, 2, \dots$$

such that $\int_{a_i} \Omega_k = 0$. Denote as $2\pi i A_k := (\int_{b_1} \Omega_k, \dots, \int_{b_g} \Omega_k)$. Consider the function

$$\hat{\psi}(x, t, p) = e^{\int_{p_0}^{p} (x\Omega_0 + \sum_{k=1}^{\infty} t_k \Omega_k)} \frac{\theta(j(p) + A_0 x + \sum_{k=1}^{\infty} A_k t_k + \zeta)}{\theta(j(p) + \zeta)}, \tag{43}$$

where $\theta = \theta(z, B)$ is a theta-function of the jacobian $Jac(C)$ (i.e. B is defined as in corollary 13.1) and $\zeta = \zeta(D)$ is the Riemann vector from corollary 13.3. Then $\hat{\psi}$ is well defined because of quasi-periodicity of θ and by definition of ω_k (check it).

By Riemann theorem 13.3 $\hat{\psi}$ is meromorphic with poles in D, i.e. the first item holds. In a neighbourhood of p the fraction of theta functions does not vanish. Therefore we can normalize our function setting

$$\psi := \hat{\psi} \frac{\theta(\zeta)}{\theta(A_0 x + \sum A_k t_k + \zeta)}.$$

This function satisfies also item two.

The uniqueness of ψ follows from the Riemann-Roch theorem, cf. 10.7. Namely, let ψ_1, ψ_2 be two such functions and let D' be the divisor of zeros of the function ψ_2. Consider the function ψ_1/ψ_2. Then $\psi_1/\psi_2 \in H^0(C, \mathcal{O}_C(D'))$ (as the essential singularity at p is cancelled), and by the Riemann-Roch theorem (and since D' is non-special) $h^0(C, \psi_1/\psi_2) = 1$. Therefore, $\psi_1 = c\psi_2$, and $c = 1$ because of the normalisation condition at p. □

Proposition 13.2. *Let ψ be the formal BA-function from theorem 13.4. Then for any $n \in \mathbb{N}$ there exist a uniquely defined differential operator $P_n \in \mathcal{D}$ of order* $\operatorname{ord} P_n = n$ *such that* $\frac{\partial \psi}{\partial t_n} = P_n \psi$.

If z^{-m} (where z is a local parameter at p from theorem 13.4) is a meromorphic function on C with a unique pole at p,[3] *then $P_m \psi = z^{-m} \psi$ (and, as a corollary, ψ is a solution of the KP hierarchy and isospectral deformations of the ring B_{P_m} are determined).*

Proof. In a neighbourhood of p we have

$$\frac{\partial \psi}{\partial t_n} = z^{-n} \psi + O(z)\varepsilon(t), \quad \partial^m \psi = (z^{-m} + O(z^{-m+1}))\varepsilon(t).$$

Therefore, $\frac{\partial \psi}{\partial t_n} = P_n \psi (\mod O(z)\varepsilon(t))$. But $\frac{\partial \psi}{\partial t_n}$, $P_n \psi$ are formal BA-functions and therefore $\frac{\partial \psi}{\partial t_n} - P_n \psi$ is also a BA-function. But it is equal to zero at p and therefore it is identically zero. Now

$$P_n \psi - z^{-n} \psi = \frac{\partial \psi}{\partial t_n} - z^{-n} \psi = O(z)\varepsilon(t),$$

and on the left hand side the function is global holomorphic. Therefore, it is identically zero. □

Remark 13.3. The Krichever formulae (43) permit to interpret the part $\hat{\psi}$ of the BA-function (recall that $\hat{\psi} = S(t)|_{\partial \mapsto z}$) in a broader context: namely, as a meromorphic function on the product of varieties $Jac(C) \times C$, i.e. due to Serre theorem 12.1 as an *algebraic* function. An interpretation of the BA-function via the *Fourier-Mukai transform* is given in [92], a further development of these ideas see in [93], [94].

[3] We can always choose the local parameter z with such a property. E.g. we can take the $-m$-th root of a meromorphic function regular outside p of order m. Recall that the ring of functions can be embedded into the space $K((z))$ via the Krichever map.

The solutions of the KP hierarchy from proposition 13.2 (so called *algebro-geometric solutions*) played an important role in the solution of the Schottky problem by T. Shiota, see [105] and e.g. [101], [19] and references therein.

Besides, now we can explain why the flow $W(t)$ from comment 12.2 is well defined as a subspace in $\mathbb{C}((z))$ (not just as a space in $\mathbb{C}[t]((z))$).

For complex varieties there is the exact exponential sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{C} \stackrel{\exp}{\to} \mathbb{C}^* \to 0$$

and the induced exact sequence of sheaves on X with respect to the complex topology (obtained by considering holomorphic functions with values in these groups)

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

where $\mathbb{Z}$ denotes the constant sheaf and $\mathcal{O}_X^*$ denotes the sheaf of invertible functions (with respect to multiplication). If $X = C$ is a smooth curve, in the induced long exact sequence of cohomologies

$$0 \to H^1(C_h, \mathbb{Z}) \to H^1(C_h, \mathcal{O}_{C_h}) \stackrel{\exp}{\to} H^1(C_h, \mathcal{O}_{C_h}^*) \stackrel{\deg}{\to} H^2(C_h, \mathbb{Z}) \to H^2(C_h, \mathcal{O}_{C_h}) \to \dots$$

we have the following groups: $H^1(C_h, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, $H^2(C_h, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1(C_h, \mathcal{O}_{C_h}) \simeq H^1(C, \mathcal{O}_C) \simeq \mathbb{C}^g$ and $H^2(C_h, \mathcal{O}_{C_h}) \simeq H^2(C, \mathcal{O}_C) = 0$ by theorem of Serre 12.1, $H^1(C_h, \mathcal{O}_{C_h}^*) \simeq \operatorname{Pic} C_h$ by [32, Ch. III, Ex. 4.5] and $\operatorname{Pic} C_h \simeq \operatorname{Pic} C$ by Serre theorem. In particular, $\operatorname{Pic}^0 C \simeq H^1(C, \mathcal{O}_C)/\mathbb{Z}^{2g}$.

Now note that $H^1(C_h, \mathcal{O}_h) \subset \mathcal{O}_{C_h}(U \backslash p)$ for some small neighbourhood of p (see section 9.5), and the ring $\mathcal{O}_C(U \backslash p)$ of algebraic regular functions is dense in the ring $\mathcal{O}_{C_h}(U \backslash p)$ of holomorphic functions. The space $W(t) = W(0)e^{\sum t_i z^{-i}}$ is a well defined subspace of $\mathcal{O}_{C_h}^*(U \backslash p)$ (and determines a point of the analytic version of the Sato Grassmanian, see [101]). The sum $(\sum t_i z^{-i})$ determines an element in the group $H^1(C_h, \mathcal{O}_{C_h}) \simeq H^1(C, \mathcal{O}_C)$, and therefore $e^{\sum t_i z^{-i}} \in H^1(C_h, \mathcal{O}_{C_h}^*) \simeq \operatorname{Pic} C_h \simeq \operatorname{Pic} C$. By the construction of the isomorphism $H^1(C_h, \mathcal{O}_{C_h}^*) \simeq \operatorname{Pic} C_h$ from [32, Ch. III, Ex. 4.5] the space $W(t)$ is isomorphic to the space $\mathcal{F} \otimes \mathcal{F}_\varepsilon(C \backslash p)$, where $\mathcal{F}$ is the original spectral sheaf (determined by the space $W(0)$) and $\mathcal{F}_\varepsilon \in \operatorname{Pic} C$ is the sheaf determined by the element $e^{\sum t_i z^{-i}}$. Since these sheaves are algebraic, the space $W(t)$ can be generated by algebraic functions, i.e. it is determined as a subspace in $\mathbb{C}((z))$ (as the image of $\mathcal{F} \otimes \mathcal{F}_\varepsilon(C \backslash p)$ under the Krichever map). As we have seen in comment 12.2, $W(t) = \mathbb{C}[z^{-1}] \cdot S(t)$, and $S(t)$ has meromorphic coefficients as $S(t)|_{\partial \mapsto z} = \hat{\psi}$ is a meromorphic function.

13.5 Other explicit formulae, explicit examples of commuting ODOs and problems related with their construction

Explicit formulae for the Baker-Akhieser functions exist also in the case when C is a *rational* singular curve (and the rank of the spectral sheaf is one). First formulae of such type were connected with so-called N-soliton solutions of the KP system invented by Hirota and Sato, see [33], [95]. Later Wilson gave universal formulae for all rational curves in [116]. In other cases there are no such formulae.

Comment 13.1. The soliton solutions of the KP hierarchy were extensively studied by many authors. The most convenient way to encode them is a *tau-function* – another way of representing solutions of certain non-linear partial differential equations. Of course, these functions are closely connected with the Sato operators or BA-functions. More on these objects see in [16], [101].

Let's mention some other points of view on the KP hierarchy and other analogous systems of equations. In the work [26] they are interpreted as infinite-dimensional hamiltonian systems, see also review [52] for further references. A way of obtaining soliton solutions to KP and KdV hierarchies via the so called boson-fermion correspondence and affine Lie algebras approach is given in the book [34]. In [23] a series of hierarchies associated with the *Kac-Moody Lie algebras* were defined. Further development of these approaches you can find e.g. in Math. Sci. Net database.

If the rank of the spectral sheaf is *greater than one*, there are no general explicit formulae for any curves. Though there are many explicit examples of higher rank commutative operators (see below), there are still many open questions.

First non-trivial explicit examples of higher rank commutative operators were already mentioned in Introduction: these are the polynomial examples of Dixmier. But a great number of explicit examples were obtained with the help of the *method of deformations of Tyurin parameters* from works [39], [40]. We'll give here only some particular interesting examples, for a complete list we refer e.g. to the Math. Sci. Net database.

The main idea of this method is to study the linear differential operator which vanishes the common eigenfunctions of commuting operators (cf. sections 8.4, 12.1). The common eigenfunctions of commuting differential operators of rank r satisfy the linear differential equation of order r

$$\psi^{(r)}(x,P;x_0)=\chi_0(x,P)\psi(x,P;x_0)+\cdots+\chi_{r-1}(x,P)\psi^{(r-1)}(x,P;x_0).$$

The coefficients χ_i are rational functions on the spectral curve C with the simple poles $\gamma_1(x)$, $\dots$, $\gamma_{rg}(x)\in C$, and with the following expansions in the neighbourhood of p

$$\chi_0(x,P)=z^{-1}+g_0(x)+O(z),\qquad \chi_j(x,P)=g_j(x)+O(z),\quad 0<j<r-1,$$

$$\chi_{r-1}(x,P)=O(z).$$

The divisors $D(x_0)=\sum_i\gamma_i(x_0)$ and $D(x)=\sum_i\gamma_i(x)$ are equivalent (see [39, §3]), and determine the determinantal divisors of the spectral sheaf (from the analytic classification theorem). Let $z^{-1}-\gamma_i(x)$ be a local parameter at $\gamma_i(x)$. Then

$$\chi_j=\frac{c_{i,j}(x)}{z^{-1}-\gamma_i(x)}+d_{i,j}(x)+O(z^{-1}-\gamma_i(x)).$$

Functions $c_{ij}(x),d_{ij}(x)$ satisfy the following equations which determine the dependence of the spectral sheaf on x_0 (see [39]):

$$c_{i,r-1}(x)=-\gamma_i'(x), \tag{44}$$

$$d_{i,0}(x)=\alpha_{i,0}(x)\alpha_{i,r-2}(x)+\alpha_{i,0}(x)d_{i,r-1}(x)-\alpha_{i,0}'(x), \tag{45}$$

$$d_{i,j}(x)=\alpha_{i,j}(x)\alpha_{i,r-2}(x)-\alpha_{i,j-1}(x)+\alpha_{i,j}(x)d_{i,r-1}(x)-\alpha_{i,j}'(x),\, j\geq 1, \tag{46}$$

where $\alpha_{i,j}(x)=\frac{c_{i,j}(x)}{c_{i,l-1}(x)}$, $0\le j\le r-1,\ 1\le i\le rg.$ To find χ_i one should solve the equations (44)–(46). Using χ_i one can find coefficients of the operators.

At $g=1$, $l=2$ Krichever and Novikov [40] solved these equations and found commuting operators of order 4 and 6. The operators of order 4 from their list looks as follows:

$$L_4=\left(\partial_x^2+u\right)^2+2c_x(\wp(\gamma_2)-\wp(\gamma_1))\partial_x+(c_x(\wp(\gamma_2)-\wp(\gamma_1)))_x-\wp(\gamma_2)-\wp(\gamma_1),$$

where $\gamma_1(x)=\gamma_0+c(x),\gamma_2(x)=\gamma_0-c(x)$,

$$u(x)=-\frac{1}{4c_x^2}+\frac{1}{4}\frac{c_{xx}^2}{c_x^2}+2\Phi(\gamma_1,\gamma_2)c_x-\frac{c_{xxx}}{2c_x}+c_x^2(\Phi_c(\gamma_0+c,\gamma_0-c)-\Phi^2(\gamma_1,\gamma_2)),$$

$$\Phi(\gamma_1,\gamma_2)=\zeta(\gamma_2-\gamma_1)+\zeta(\gamma_1)-\zeta(\gamma_2),$$

$\zeta(z), \wp(z)$ are the Weierstrass functions, $c(x)$ is an *arbitrary* smooth function, γ_0 is a constant.

Operators of rank 3 corresponding to elliptic spectral curves were found by Mokhov [61]. Various properties of the Krichever–Novikov operators were studied in [28], [31], [48], [49], [64], [65], [85], [18], [10]. In particular, Grinevich in [28] found degeneracy conditions on coefficients of L_4, L_6 to obtain the Dixmier examples:

- The operator L_4 corresponding to the curve $w^2 = 4z^3 + g_2 z + g_3$ has rational coefficients if and only if
$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{4t^3 + g_2 t + g_3}},$$
where $q(x)$ is a rational function. If $\gamma_0 = 0$ and $q(x) = x$, then L_4 coincides with the Dixmier operator.

Problem 13.1. Extend this answer to the case of genus $g > 1$ curves and rank $r > 2$ operators. Some partial results in this direction were obtained in other works of Grinevich and Mokhov.

Grünbaum in [31], using a supercomputer, completed the list of all possible commuting operators of orders 4 and 6 (not only of true rank two!). Previato and Wilson gave a description of spectral sheaves corresponding to given ring $\mathbb{C}[L_4, L_6]$ for the case of smooth spectral curves. In [10] the authors completed this description for the case of singular curves. However, there is still a problem:

Problem 13.2. Classify all commuting operators of orders 4 and 6 with *polynomial coefficients.* A partial answer on this question was given in [59], [60].

Further examples were invented in connection with the following question of I.M. Gelfand: are there commuting operators with polynomial coefficients such that their spectral curve has any given genus and their rank is a given one?

In [56] commuting operators of rank two of order 4 and $4g+2$ corresponding to hyperelliptic spectral curves were studied

$$L_4\psi = z\psi, \quad L_{4g+2}\psi = w\psi, \quad w^2 = F_g(z) = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0.$$

Common eigenfunctions of L_4 and L_{4g+2} satisfy the second order differential equation

$$\psi'' - \chi_1(x,P)\psi' - \chi_0(x,P)\psi = 0, \; P = (z,w) \in \Gamma,$$

where $\chi_0(x,P), \chi_1(x,P)$ are rational functions on Γ satisfying equations (44)–(46).

Theorem 13.5. *([56]) If L_4 is formally self-adjoint, i.e. $L_4 = (\partial_x^2 + V(x))^2 + W(x)$, then*

$$\chi_0 = -\frac{1}{2}\frac{Q_{xx}}{Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q_x}{Q},$$

where $Q = z^g + a_{g-1}(x)z^{g-1} + \cdots + a_0(x)$, $a_0(x), \ldots, a_{g-1}(x)$ are some functions. The function Q satisfies the equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} + 2Q(2V_xQ_x + 4VQ_{xx} + \partial_x^4 Q). \quad (47)$$

With the help of Theorem 13.5 many examples of rank 2 operators with high genus spectral curves were constructed. For example

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \qquad \alpha_3 \neq 0$$

commutes with an operator $L_{4g+2}^\sharp$ [56]. Mokhov [62] proved that if one apply elements of $Aut(A_1)$ to $L_4^\sharp, L_{4g+2}^\sharp$, then one can obtains operators of rank $r = 2k$ and $r = 3k$, where k

is a positive integer. For example, if we apply the automorphism $\varphi(x)=\partial_x, \varphi(\partial_x)=-x$ to $L_4^\sharp, L_{4g+2}^\sharp$, we obtain rank 3 operators. Herewith

$$\varphi(L_4^\sharp)=(\alpha_3\partial_x^3+\alpha_2\partial_x^2+\alpha_1\partial_x+\alpha_0+x^2)^2+g(g+1)\alpha_3\partial_x.$$

Another important example constructed in [57] is the following. The operator

$$L_4^\natural=(\partial_x^2+\alpha_1\cosh x+\alpha_0)^2+\alpha_1 g(g+1)\cosh x, \quad \alpha_1\neq 0$$

commutes with $L_{4g+2}^\natural$. Using $L_4^\natural, L_{4g+2}^\natural$ Mokhov constructed examples of operators of arbitrary rank $r>1$ [63].

Let us consider commuting operators $L_4^\natural, L_{4g+2}^\natural$ [57]. The polynomial Q for $L_4^\natural, L_{4g+2}^\natural$ has the form (see [57])

$$Q(x,z)=A_g(z)\cosh^g x+\dots+A_1(z)\cosh x+A_0(z),$$

where

$$A_s=\frac{1}{8(2s+1)\alpha_1(g(g+1)-s(s+1))}\left(4A_{s+5}\frac{(s+5)!}{s!}-8A_{s+3}\frac{(s+3)!}{s!}(2\alpha_0+s^2+4s+5)-\right.$$

$$\left.-8A_{s+2}\frac{(s+2)!}{s!}(2s+3)\alpha_1+4A_{s+1}(s+1)((s+1)^2(4\alpha_0+(s+1)^2+4z)\right), \quad 0\leq s<g, \quad (48)$$

we assume that $A_s=0$ at $s<0$ and $s>g$, A_g is a constant.

Lemma 13.1. *([56]) The spectral curve of $L_4^\natural, L_{4g+2}^\natural$ is given by the equation*

$$w^2=F_g(z)=\frac{1}{4}\left(4A_0^2z-4A_0A_1\alpha_1-16A_2(\alpha_0+1)+48A_4)+4\alpha_0A_1^2+4A_2^2-2A_1(6A_3-A_1)\right),$$

where $A_j(z)$ are defined in (48).

Examples:

1) $g=1$

$$F_1(z)=z^3+(\frac{1}{2}-2\alpha_0)z^2+\frac{1}{16}(1-8\alpha_0+16\alpha_0^2-16\alpha_1^2)z+\frac{\alpha_1^2}{4}.$$

2) $g=2$, let's for simplicity of formulae put $\alpha_0=0$:

$$F_2(z)=z^5+\frac{17}{2}z^4+\frac{1}{16}(321-336\alpha_1^2)z^3+\frac{1}{4}(34-531\alpha_1^2)z^2+(1-189\alpha_1^2+108\alpha_1^4)z+24\alpha_1^2+513\alpha_1^4.$$

The spectral curves defined by the above equations are not singular for generic values of parameters.

Mokhov in [63] found a remarkable change of variable

$$x=\ln(y+\sqrt{y^2-1})^r, \quad r=\pm1,\pm2,\dots,$$

which reduces the operators $L_4^\natural, L_{4g+2}^\natural$ to the operators with polynomial coefficients. In particular, $L_4^\natural$ in new variable y gets the form

$$L_4^\natural=((1-y^2)\partial_y^2-3y\partial_y+aT_r(y)+b)^2-ar^2g(g+1)T_r(y), \quad a\neq 0,$$

b is arbitrary constant, $T_r(y)$ is the *Chebyshev polynomial* of degree $|r|$. Recall that

$$T_0(y)=1, \quad T_1(y)=y, \quad T_r(y)=2yT_{r-1}(y)-T_{r-2}(y), \quad T_{-r}(y)=T_r(y).$$

Chebyshev polynomials are commuting polynomials, i.e.

$$T_n(T_m(y)) = T_m(T_n(y)) = T_{n+m}(y).$$

If one applies the automorphism

$$\varphi(y) = -\partial_y, \qquad \varphi(\partial_y) = y, \quad \varphi \in Aut(A_1)$$

to the operators $L_4^{\natural}, L_{4g+2}^{\natural}$ written in y variable, then one gets operators of orders $2r, (2g+1)r$ of rank r (see [63]) and

$$\varphi(L_4^{\natural}) = (aT_r(\partial_y) - y^2\partial_y^2 - 3y\partial_y + y^2 + b)^2 - arg(g+1)T_r(\partial_y).$$

Further interesting examples of high rank operators were obtained in works [17], [75], [76] – by analytic methods, in [82], [84] – via certain invented (different) computer algorithms (in [82] - with the help of the algebraic classification theory, in [84] - with the help of subdifferential resultants).

Problem 13.3. An interesting but difficult problem, connected with the Berest and Dixmier conjectures, is to describe the action of the group $\mathrm{Aut}(A_1)$ on the spectral data of commuting operators with *polynomial coefficients*, at least for hyperelliptic spectral curves.

Problem 13.4. Yet one another interesting problem is to recognise whether the maximal commutative subring B_P containing a given differential operator P has rank one. Such operators or rings are called *algebraically integrable* in [9]. In this paper a criterion of algebraic integrability was given in terms of the *Differential Galois group* (see e.g. the book [87] for necessary definitions). Namely, the ring B_P has rank one if the Differential Galois group of P is commutative.

Thus, the problem of explicit calculation of the Differential Galois group appears.

Remark 13.4. Of course, similar problems of constructing explicit examples of commuting matrix ordinary differential operators exist. For rank one subrings there are explicit formulae (see [37], [24]). For generic rank there are no such formulae, but there are many explicit examples obtained by similar methods, see e.g. the recent work [77] for such examples.

Remark 13.5. We considered in our lectures so far only differential operators with coefficients in subrings of the ring $K[[x]]$. However, many results, including the classification ones, can be extended to operators with coefficients in different rings. First, recall that many purely algebraic results (like the Schur theory) hold in a very generic situation, and therefore can be applied to study a large class of operators.

Second, in the most evident case of operators with *smooth real functions* practically all results from these lecture notes remain valid because of the following fact ([38], [101]:

Theorem 13.6. *Let $P = \partial^n + u_{n-2}\partial^{n-2} + \ldots + u_0$ be a differential operator with coefficients being smooth functions in a neighbourhood I of zero in $\mathbb{R}$. Assume there is a differential operator Q of order m, $GCD(n,m) = 1$, commuting with P. Then the functions u_i can be extended to meromorphic functions on the whole complex plane with poles of order not greater than $n-i$, and all finite singular points of P are regular.*

Problem 13.5. At last, let's mention a big problem: how to extend the whole theory to rings of commuting partial differential operators? Of course, a lot of pieces of such theory is already developed, see works [79], [80], [81], [78], [123], [118], [43], [44], [119], [45], [46], [11], [121], [42], [122], but there are still many open questions. We refer for a list of contemporary problems to [120] and other recent papers listed above.

14 Appendix

In this section we collect basic facts and constructions from Commutative algebra, and also review some results about non-commutative Noetherian rings.

14.1 Localisation of rings

The main reference for this part is the book [55] about non-commutative Noetherian rings.

Definition 14.1. The multiplicatively closed (m.c.) subset $S \subset \mathcal{R}$ is a set such that $1 \in S$, and for any $s_1, s_2 \in S$ we have $s_1 s_2 \in S$. We assume also that $0 \notin S$.

The element $x \in \mathcal{R}$ is called *right regular* if from the equality $xr = 0$ for some $r \in \mathcal{R}$ it follows that $r = 0$. The *left regular* element is defined analogously. The element $x \in \mathcal{R}$ is called *regular* if it is left and right regular.

The m.c. set of all regular elements is denoted by $C_{\mathcal{R}}(0)$.

For a given m.c. set $S \subset \mathcal{R}$ denote by

$$\mathbf{ass}\, S = \{r \in \mathcal{R} | rs = 0 \quad \text{fo some } s \in S\}$$

The following definition of a *localisation* with respect to S is available in both commutative and non-commutative cases.

Definition 14.2. A *right quotient ring* of $\mathcal{R}$ with respect to S is a ring Q together with a homomorphism $\theta : \mathcal{R} \to Q$ such that

1. for any $s \in S$ the element $\theta(s)$ is a unit in Q;
2. for any element $q \in Q$ there are elements $r \in \mathcal{R}$, $s \in S$ such that $q = \theta(r)\theta(s)^{-1}$;
3. $\ker \theta = \mathbf{ass}\, S$.

The *left quotient ring* can be defined analogously. The following properties are standard:

Proposition 14.1. *If there exists a right quotient ring Q of $\mathcal{R}$ with respect to S, then it is unique up to isomorphism.*

If $\mathcal{R}$ also has a left quotient ring Q' with respect to S then $Q \simeq Q'$.

Exercise 14.1. Prove this proposition.

Definition 14.3. A m.c. set $S \subset \mathcal{R}$ is said to satisfy the *right Ore condition* if, for each $r \in \mathcal{R}$ and $s \in S$ there exist $r' \in \mathcal{R}$, $s' \in S$ such that $rs' = sr'$.

Lemma 14.1. *If the right quotient ring $\mathcal{R}_S$ exists then S satisfies the right Ore condition.*

If S satisfies the right Ore condition then $\mathbf{ass}\, S$ *is an ideal of $\mathcal{R}$.*

Proof. Consider the element $\theta(s)^{-1}\theta(r) \in \mathcal{R}_S$. By definition

$$\theta(s)^{-1}\theta(r) = \theta(r_1)\theta(s_1)^{-1} \quad \text{with } r_1 \in \mathcal{R},\ s_1 \in S.$$

Therefore

$$\theta(r)\theta(s_1) = \theta(s)\theta(r_1) \quad \text{and so } rs_1 - sr_1 \in \ker\theta = \mathbf{ass}\, S.$$

Therefore $(rs_1 - sr_1)s_2 = 0$ for some $s_2 \in S$. Setting $s_1 s_2 = s'$ and $r_1 s_2 = r'$ establishes the first result.

Let $a, b \in \mathbf{ass}\, S$ and $r \in \mathcal{R}$. Then $as = bt = 0$ for some $s, t \in S$. The right Ore condition provides t_1, s_1 such that $ss_1 = tt_1$ and r', s' such that $rs' = sr'$, where $s_1, s' \in S$. Hence $(a - b)ss_1 = 0$ and $ars' = 0$; and so $a - b \in \mathbf{ass}\, S$ and $ar \in \mathbf{ass}\, S$. $\square$

Theorem 14.1. *Let S be a m.c. subset of $\mathcal{R}$. Then the right quotient ring $\mathcal{R}_S$ exists if and only if S satisfies the right Ore condition and $\bar{S} = \{$image of S in $\bar{\mathcal{R}} = \mathcal{R}/\,\mathbf{ass}\,S\}$ consists of regular elements.*

Proof. The necessity of these conditions has been shown, see lemma 14.1 (it is clear that $\bar{S}$ consists of regular elements, because they are units in $\mathcal{R}$). It remains to construct the ring $\mathcal{R}_S$. This can be done, as in the commutative case, by imposing appropriate operations on equivalence classes in $\mathcal{R} \times S$, but the verifications are arduous. For this reason, an alternative route is described below. Note first, however, that by passing to homomorphic images, it will suffice to consider the case when $\mathbf{ass}\,S = 0$, and then elements of $\mathcal{R}_S$ will be of the form rs^{-1}.

Construction. First consider the set $\mathcal{F}$ of those right ideals $A \subset \mathcal{R}$ such that $A \cap S \neq 0$. The right Ore condition makes it true, and easily verified, for $A_1, A_2 \in \mathcal{F}$ and $\alpha \in Hom(A_1, \mathcal{R})$ (homomorphisms of right $\mathcal{R}$-modules) that

(i) $A_1 \cap A_2 \in \mathcal{F}$ and

(ii) $\alpha^{-1}A_2 \overset{\text{def}}{=} \{a \in A_1 | \alpha(a) \in A_2\} \subset \mathcal{F}$.

Exercise 14.2. Check these assertions.

Consider the set $\cup\{Hom(A, \mathcal{R}) | A \in \mathcal{F}\}$ together with the equivalence relation given, for $\alpha_i \in Hom(A_i, \mathcal{R})$ by $\alpha_1 \sim \alpha_2$ if α_1 and α_2 coincide on some $A \in \mathcal{F}$, $A \subset A_1 \cap A_2$. Operations on equivalence classes $[\alpha_i]$ are defined by $[\alpha_1] + [\alpha_2] = [\beta]$, where $\beta = \alpha_1|_{A_1 \cap A_2} + \alpha_2|_{A_1 \cap A_2}$; and by $[\alpha_1][\alpha_2] = [\gamma]$, with $\gamma = \alpha_1\alpha_2|_{\alpha_2^{-1}A_1}$.

It can be readily checked:

(i) that these operations are well defined and, under them, the equivalence classes form a ring, $\mathcal{R}_S$ say;

(ii) that, if $r \in \mathcal{R}$ is identified with the equivalence class of the homomorphism $\lambda(r) : \mathcal{R} \to \mathcal{R}$ given by $x \mapsto rx$, this embeds $\mathcal{R}$ in $\mathcal{R}_S$;

(iii) that, under this embedding, each $s \in S$ has an inverse $s^{-1} = [\alpha]$, where $\alpha : s\mathcal{R} \to \mathcal{R}$ is given by $sx \mapsto x$; and

(iv) that, if $\alpha \in Hom(A, \mathcal{R})$ with $A \in \mathcal{F}$, then $[\alpha] = as^{-1}$, where $s \in A \cap S$ and $a = \alpha(s)$.

These facts complete the proof of the theorem.

Exercise 14.3. Check these assertions.

□

Definition 14.4. An integral domain $\mathcal{R}$ is called a *right Ore domain* if $C_{\mathcal{R}}(0)$ is a right Ore set.

Theorem 14.2. *Any right Noetherian integral domain $\mathcal{R}$ is a right Ore domain.*

Proof. It is enough, given nonzero $a, b \in \mathcal{R}$, to show that $ab' = ba' \neq 0$ for some $a', b' \in \mathcal{R}$. But we have either $a\mathcal{R} \cap b\mathcal{R} \neq 0$ (and in this case we are done) or the sum $\sum b^n a\mathcal{R}$ is direct, but this contradicts with the Noetherity of $\mathcal{R}$. □

Remark 14.1. Let R be a commutative ring. Note that the conditions of theorem 14.1 are satisfied for any m.c. set $S \subset R$. So, the right (and left) quotient ring is defined. Usually in Commutative Algebra such quotient rings are called *localizations* $S^{-1}R$ of R with respect to S. In this case it is more convenient to define it as the set of formal fractions $\frac{a}{b}$, with $a \in R$ and $b \in S$, up to the equivalence relation: $\frac{a}{b} = \frac{a_1}{b_1}$ if $(ab_1 - a_1b)s = 0$ for some $s \in S$.

When R has no zero divisors, the natural map $a \to \frac{a}{1}$ is an injective homomorphism of rings, so that we can think of R as a subset of $S^{-1}R$. Then $S^{-1}R$ is simply the fractions with "restricted denominators". The set formal fractions form a ring with a usual addition and multiplication of fractions. Note that if $I \subset R$ is an ideal, then $S^{-1}I$ is an ideal in $S^{-1}R$.

Exercise 14.4. Prove that a localization of a Noetherian ring is Noetherian.

Example 14.1. If $S = R\backslash\{0\}$ and R is an integral domain, then $S^{-1}R$ is just the field of fractions.

Another important example: let $S = \{a^n\}$, $n = 0, 1, 2, \ldots$, $a \neq 0$. The localisation $S^{-1}R$ is usually denoted by R_a, and can be understood as the ring of polynomials $R[a^{-1}] \subset \mathrm{Quot}(R)$ if R has no zero divisors. Such rings play important role in algebraic geometry: they form rings of regular functions on open affine sets (see section about affine morphisms of algebraic varieties below). Besides, if R is finitely generated over a field K, then R_a is also finitely generated.

14.2 Resultants, transcendence basis, factorial rings

In this subsection all rings assumed to be commutative. Main references for this section are the books [4], [47].

Theorem 14.3. *[47, Ch. V §6] The ring $k[T_1, \ldots T_n]$, where k is a field, is a unique factorisation domain (UFD for short), i.e. any polynomial $f \in k[T_1, \ldots T_n]$ has a unique decomposition (up to multiplication on a unit)*

$$f = f_1^{k_1} \ldots f_q^{k_q},$$

where f_i are irreducible polynomials (i.e. they are not divisible by any other non-constant polynomials).

Comment 14.1. Rings with the property to be a UFD are also called *factorial* rings, see more details in [7, Ch.VII, §3].

Definition 14.5. The elements $r_1, \ldots, r_n$ of a ring R over K are *algebraically independent* (over K) if the only *irreducible* polynomial $f(x_1, \ldots, x_n)$ with coefficients in K such that $f(r_1, \ldots, r_n) = 0$, is the zero polynomial (here we assume that f depends on all variables $x_1, \ldots, x_n$).

Remark 14.2. Our definition slightly differs from usual standard definition of algebraically independent elements. Usually it is assumed that all monomials of the form $\prod r_1^{i_1} \cdots r_n^{i_n}$ are linearly independent over K for all n-tuples of non-negative integers. In particular, by this definition, the elements $1, t$ are algebraically dependent over K in the ring $K[t]$. It seems to us to be contra-intuitive, so we decided to change it. If we consider *mutually* algebraically independent elements in the sense of our definition (i.e. that any finite number of elements from a given set are algebraically independent), then we obtain a usual definition, so there will be no conflicts with standard results from algebra.

Definition 14.6. Let $A \subset R$ be an extension of integral domains. An element $x \in R$ is called *integral* over A if there exists a *monic* polynomial $f \in A[T]$ such that $f(x) = 0$.

Let $\tilde{K} \supset k$ be a field extension. An element $x \in \tilde{K}$ is *algebraic* over k if it is integral over k. $\tilde{K} \supset k$ is an algebraic extension if any $x \in \tilde{K}$ is algebraic over k.

Theorem 14.4. *[47, Ch. V, §10] Two polynomials $F, G \in A[T]$, where A is an integral domain, $F = a_0 + \ldots + a_n t^n$, $G = b_0 + \ldots + b_m t^m$ have a common zero (in some extension of $\mathrm{Quot}(A)$) if and only if their resultant*

$$Res(F, G) = \det \begin{pmatrix} a_0 & a_1 & \ldots & a_n & 0 & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & a_{n-1} & a_n & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & a_0 & \ldots & \ldots & \ldots & a_n \\ b_0 & b_1 & \ldots & \ldots & b_m & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ldots & b_0 & \ldots & \ldots & \ldots & \ldots & b_m \end{pmatrix}$$

is equal to zero. Moreover, $Res(F,G)=0$ *if and only if* r_1F *,* r_2G *have a common divisor of positive degree for some non-zero* $r_1,r_2\in A$ *.*

The resultant can be used to prove standard results from algebra about algebraic extension of rings in an effective way. Besides it is useful to play with commuting differential operators. Below we illustrate this.

Lemma 14.2. *Let* R *be an integral domain over* K *. If* $a\in R$ *is algebraic over* K *and* $a,b\in R$ *are algebraically dependent over* K *then* b *is algebraic over* K *.*

Proof. First note that a is algebraic over K if and only if there exists an irreducible *homogeneous* polynomial $F(T_1,T_2)$ such that $F(1,a)=0$ (just take the homogenisation of the corresponding irreducible monic polynomial). Let $G(T_2,T_3)$ be an irreducible polynomial such that $G(a,b)=0$. Let $n=\deg F$, $m=\deg G$, $F=a_0T_1^n+\ldots+a_nT_2^n$, $G=b_0+\ldots+b_mT_2^m$, where $a_i\in K$, $b_i\in K[T_3]$. Then the polynomial $H(T_1,T_3):=Res(F(T_2),G(T_2))\in K[T_1,T_3]$ is not zero. For, if $H=0$, then F and G must have a common divisor of positive degree in $K[T_1,T_2,T_3]$ by theorem 14.4 (since $K[T_1,T_2,T_3]$ is a UFD), a contradiction.

Note that H depend on T_1 non-trivially. For, the determinant of the Sylvester matrix contains the unique monomial of degree $n\cdot m$ with respect to T_1, namely, $(a_0T_1^n)^m\cdot b_m^n$ (all other monomials have smaller degree, and $a_0,b_m\neq 0$ since F,G are irreducible). Also note that $H(1,T_3)\neq 0$. For, if it is zero, then again $F(1,T_2)$ and $G(T_2,T_3)$ must have a common divisor of positive degree in $K[T_2,T_3]$ by theorem 14.4, a contradiction with irreducibility of G.

Note that $H(1,b)=Res(F(1,T_2),G(T_2,b)=0$ in the ring $K(b)\subset R$, since $a\in R$ is a common zero. Hence, any irreducible factor of $H(T_1,T_3)$ that vanishes at $(1,b)$ depend either on T_1,T_3 or on T_3. In both cases we are done: b is algebraic over K. □

Corollary 14.1. *Let* R *be an integral domain over* K *. If* $a,b\in R$ *are algebraic over* K *, then* a *and* b *are algebraically dependent over* K *. Moreover,* $a\cdot b$ *is algebraic over* K *.*

Proof. Let $F(T_1,T_2)$, $G(T_2,T_3)$ be irreducible homogeneous polynomials such that $F(a,1)=G(1,b)=0$. Then $H(T_1,T_3)$ from the proof of previous lemma is a *homogeneous* polynomial. Indeed, the determinant of the Sylvester matrix is a sum of monomials of the form $\pm a_{1,j_1}\cdots a_{m,j_m}\cdot b_{1,k_1}\cdots b_{n,k_n}$, where a_{i,j_i} denotes the element at i-th row and j_i-th column and b_{i,k_i} denotes the element at $(i+m)$-th row and k_i-th column, and $\{j_1,\ldots,j_m,k_1,\ldots,k_n\}=\{1,\ldots n+m\}$. Note that $a_{i,j_i}=c_{i,j_i}T_1^{n-(j_i-i)}$ for some $c_{i,j_i}\in K$ and $b_{i,k_i}=d_{i,k_i}T_1^{m-(k_i-i)}$ for some $d_{i,k_i}\in K$. So,

$$\deg(a_{1,j_1}\cdots a_{m,j_m}\cdot b_{1,k_1}\cdots b_{n,k_n})=2mn-\frac{(m+n)(m+n+1)}{2}+\frac{m(m+1)}{2}+\frac{n(n+1)}{2}=mn.$$

Since any factor of a homogeneous polynomial H is again homogeneous, there exists an irreducible homogeneous polynomial $\tilde{H}$ in two variables which vanishes at (a,b).

Let $\tilde{H}=\sum_{i=0}^n c_{i,n-i}T_1^iT_2^{n-i}$. Consider the polynomial $H_n(T_1,T_2)=\sum_{i=0}^n c_{i,n-i}T_1^iT_2^{2(n-i)}$. Then $H_n(ab,b)=b^n\tilde{H}(a,b)=0$. Note that H_n has no irreducible factors depending on T_2 and vanishing at b, since otherwise $\tilde{H}(T_1,b)=0$, a contradiction. Therefore, there is an irreducible factor vanishing at (ab,b) and depending on T_1. Thus, ab is algebraic over K by lemma 14.2. □

Lemma 14.3. *Let* R *be an integral domain over* K *. If* $a,b\in R$ *are algebraically dependent over* K *and* $b,c\in R$ *are algebraically dependent over* K *, then* $a,c\in R$ *are algebraically dependent over* K *.*

Proof. Let $F(a,b)=0$, $G(b,c)=0$, where $F(T_1,T_2)$, $G(T_2,T_3)$ are irreducible polynomials.

As in proofs above consider F as a polynomial in the variable T_2 with coefficients in $K[T_1]\subset K[T_1,T_3]$ and G as a polynomial in T_2 with coefficients in $K[T_3]\subset K[T_1,T_3]$. Then the polynomial $H(T_1,T_3):=Res(F(T_2),G(T_2))\in K[T_1,T_3]$ is not zero. For, if $H=0$, then F and G have a common divisor in $K[T_1,T_2,T_3]$ by theorem 14.4, a contradiction.

Again we have $H(a,c)=Res(F(a,T_2),G(T_2,c))=0$ in the ring $K[a,c]$, since $b\in R\supset K[a,c]$ is a common zero. Any irreducible factor of H that vanishes at (a,c) depends either on two or one variable. If it depends on two variables, we are done. If it depends on one variable, then a (or c) is algebraic over K. Therefore, b is algebraic over K by lemma 14.2, and therefore c (or a) is algebraic over K by the same reason. Hence, a,c are algebraically dependent by corollary 14.1. □

Corollary 14.2. *If any two non-algebraic over K elements in R are algebraically dependent, then any two non-algebraic over K elements in $\mathrm{Quot}(R)$ are algebraically dependent.*

Proof. Take any element $\frac{1}{a}\in\mathrm{Quot}(R)$, where a is not algebraic. Then $(\frac{1}{a})a=1$, so $\frac{1}{a}$, a are algebraically dependent over K and therefore $\frac{1}{a}$ and b are algebraically dependent for any non-algebraic over K element $b\in R$ by lemma 14.3. In particular, $\frac{1}{a}$ is not algebraic by lemma 14.2.

Now take any element $\frac{b}{a}\in\mathrm{Quot}(R)$ which is not algebraic over K. If a is algebraic over K, then $1/a$ is algebraic over K by lemma 14.2 and therefore b (and $1/b$) can not be algebraic over K by corollary 14.1. Let $F(T)$ be an irreducible polynomial such that $F(1/a)=0$. Then the polynomial $\tilde{F}(T_1,T_2)=F(T_1T_2)$ vanishes at $(b/a,1/b)$. Clearly, $\tilde{F}$ has no irreducible factors depending on one variable and vanishing at $(b/a,1/b)$ (because $b/a,1/b$ are not algebraic over K). Thus, $b/a,1/b$ are algebraically dependent and therefore b/a is algebraically dependent with any non-algebraic element from R by lemma 14.3. The same arguments work if b is algebraic over K.

If a,b are not algebraic over K, let $F(T_1,T_2)=\sum_{i,j\geq 0}c_{ij}T_1^iT_2^j$ be an irreducible polynomial such that $F(1/a,b)=0$. Then there exists some $N\in\mathbb{Z}$ such that $j-i-N\geq 0$ for all j,i from the finite sum, so that

$$0=\sum_{i,j\geq 0}c_{ij}\frac{1}{a^i}b^j=b^N(\sum_{i,j\geq 0}c_{ij}\frac{b^i}{a^i}b^{j-i-N}).$$

Therefore, the polynomial $\tilde{F}(T_1,T_2)=\sum_{i,j\geq 0}c_{i,j}T_1^iT_2^{j-i-N}$, where sum is taken over the same set if indices, (note it is not identically zero) vanishes at $(b/a,b)$. Since $b/a,b$ are not algebraic over K, $\tilde{F}$ has no irreducible factors vanishing at $(b/a,b)$ and depending only on one variable. Thus, $b/a,b$ are algebraically dependent.

Now by lemma 14.3 any two non-algebraic over K elements in $\mathrm{Quot}(R)$ are algebraically dependent. □

Corollary 14.3. *If non-algebraic over K elements $a,b\in R$ are algebraically dependent, then any two non-algebraic over K elements from $K[a,b]$ are algebraically dependent.*

Proof. Obviously, any non-algebraic element in $K[a]$ is algebraically dependent with a (we can write down the polynomial explicitly). Therefore, any two non-algebraic elements are algebraically dependent in $K[a]$. Analogously any two non-algebraic elements are algebraically dependent in $K[b]$. So, by 14.3, 14.2 any two non-algebraic elements from $K(a)$ and $K(b)$ are algebraically dependent.

Now take any non-algebraic over K element $f\in K[a,b]$, say $f=\sum_{i=0}^n f_ib^i$, $f_i\in K(a)$. We claim that f,b are algebraically dependent. Use induction on n: if $n=0$, then we already know that f is algebraically dependent with any element from $K[b]$. In general situation

$$f = f_n(b^n + \frac{f_{n-1}}{f_n}b^{n-1} + \ldots + \frac{f_0}{f_n}) =: f_n(b^n + f').$$

If $(b^n + f')$ is algebraic over K, then by the arguments from the proof of corollary 14.2 f, f_n are algebraically dependent, whence f, b are algebraically dependent by lemma 14.3. So, we can assume $(b^n + f')$ is not algebraic over K. If f' is algebraic over K, then we can use the same trick: let $F(T)$ be an irreducible polynomial such that $F(f') = 0$. Then the polynomial $F(T_2 - T_1^n)$ vanishes at $(b^n + f'), b$, and this leads to algebraic dependence of $(b^n + f'), b$. If f' is not algebraic over K, by induction f', b are algebraically dependent. Therefore, $(b^n + f'), b$ are algebraically dependent: if $F(T_1, T_2)$ is an irreducible polynomial with $F(f', b) = 0$, then $F(T_1 - T_2^n, T_2)$ is a polynomial vanishing at $(b^n + f'), b$, and this leads to algebraic dependence of $(b^n + f'), b$.

If f_n is algebraic, the arguments above say that f, b are algebraically dependent. If not, by lemma 14.3 $(b^n + f'), f_n$ are algebraically dependent, and we can use again the trick from previous corollary to show that f, f_n are algebraically dependent, whence f, b are algebraically dependent and we are done. □

Definition 14.7. Let $\tilde{K} \supset k$ be a field extension. A *transcendence basis* of $\tilde{K} \supset k$ is a set T of mutually algebraically independent elements over k (here we mean that any *finite number* of elements from T are algebraically independent) such that $\tilde{K}$ is algebraic over $k(T)$.

On a set σ of algebraically independent subsets there is a partial order: $T_1 \le T_2$ if $T_1 \subseteq T_2$. For any chain $T_1 \subset T_2 \subset \ldots$ there is an upper bound $T = \cup_i T_i$ (i.e. $T \ge T_i$ for any i). Then by Zorn's lemma[4] there are maximal elements in σ. Therefore, the transcendence basis exists. We'll consider only extensions with *finite* transcendence bases.

Theorem 14.5. *Let $\tilde{K} \supset k$ be an extension of fields. Then any two transcendence bases of $\tilde{K}$ over k have equal cardinality.*

If $\tilde{K} = k(\Gamma)$, where Γ is a set of generators, and $T \subset \Gamma$ is a subset of algebraically independent elements, then there exists a transcendence basis β of $\tilde{K}$ over k such that $T \subset \beta \subset \Gamma$.

Proof. Let $\{x_1, \ldots x_m\}$ be a transcendence basis, and $\{w_1 \ldots, w_n\}$ are algebraically independent elements. It is suffice to prove that $n \le m$, since then by symmetry $m \le n$ and therefore $m = n$.

Let's prove it. There exists a non-zero polynomial $f_1(w_1, x_1, \ldots, x_m) = 0$ (since w_1 is algebraic over $k(x_1, \ldots, x_m)$). Without loss of generality we can assume that f_1 depends on x_1. Then it means that w_1, x_1 are algebraically dependent over $k(x_2, \ldots x_m)$ and x_1 is algebraic over $k(w_1, x_2, \ldots, x_m)$. Therefore, $\tilde{K}$ is algebraic over $k(w_1, x_2, \ldots, x_m)$. For, for any non-algebraic over $k(x_2, \ldots, x_m)$ element $\alpha \in \tilde{K}$ the elements α, x_1 are algebraically dependent over $k(x_2, \ldots, x_m)$, hence by lemma 14.3 α, w_1 are algebraically dependent over $k(x_2, \ldots, x_m)$ and therefore α is algebraic over $k(w_1, x_2, \ldots, x_m)$.

Now we use induction: if $\tilde{K}$ is algebraic over $k(w_1, \ldots, w_r, x_{r+1}, \ldots, x_m)$ for $r < n$ then there exists a non-zero polynomial $f(w_{r+1}, w_1, \ldots, w_r, x_{r+1}, \ldots, x_m) = 0$. Without loss of generality (by renumbering the variables) we can assume that f depends on x_{r+1} (if f contains no x_i this would mean that $w_1, \ldots, w_{r+1}$ are algebraically dependent, a contradiction). Then x_{r+1} is algebraic over $k(w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m)$.

[4] Zorn's lemma says: Let S be a non-empty partially ordered set (i.e. we are given a relation $x \le y$ on S which is reflexive and transitive and such that $x \le y$ and $y \le x$ together imply $x = y$). A subset T of S is a chain if either $x \le y$ or $y \le x$ for every pair of elements x, y in T. If every chain of T has an upper bound in S (i.e. if there exists $x \in S$ such that $t \le x$ for all $t \in T$) then S has at least one maximal element.

Now again for any non-algebraic over $k(w_1,\dots,w_r,x_{r+2},\dots,x_m)$ element $\alpha\in\tilde K$ the elements α, x_{r+1} are algebraically dependent over $k(w_1,\dots,w_r,x_{r+2},\dots,x_m)$ and x_{r+1},w_{r+1} are algebraically dependent over
$k(w_1,\dots,w_r,x_{r+2},\dots,x_m)$. Hence by lemma 14.3 α, w_{r+1} are algebraically dependent over $k(w_1,\dots,w_r,x_{r+2},\dots,x_m)$ and therefore α is algebraic over $k(w_1,\dots,w_{r+1},x_{r+2},\dots,x_m)$. Then, if $n>m$ we deduce that $\tilde K$ is algebraic over $k(w_1,\dots,w_m)$, a contradiction, since $w_1,\dots,w_n$ are algebraically independent. □

Notation 14.1. We denote by $\mathrm{trdeg}(\tilde K/k)$ the cardinality of a transcendence basis.

Exercise 14.5. Show that $\mathrm{trdeg}(K((x))/K)=\infty$. Show also that $K[[x]]$ is not finitely generated over K.

14.3 Commutative Noetherian rings, Hilbert's basis theorem

The main references for this subsection and for subsections 14.4, 14.5 are the books [4], [7] and the course [106].

Let R be a commutative ring. We shall always consider ideals $I\subset R$ different from R itself. Then the quotient ring R/I is also a commutative ring.

Definition 14.8. An ideal $I\subset R$ is called *prime* if the quotient ring R/I has no zero divisors.

An ideal I is *maximal* if it is not contained in another ideal (different from R). Then the ring R/I has no non-zero ideals (otherwise its preimage in R would be an ideal containing I), hence every element $x\in R/I$, $x\neq 0$, is invertible (since the principal ideal (x) must coincide with R/I, and thus contain 1), in other words, R/I is a field. Since a field has no non-trivial ideals, the converse is also true, so that $I\subset R$ is maximal iff R/I is a field. By Zorn's lemma, each ring contains a maximal ideal.

An important finiteness property of rings is encoded in a notion *Noetherian ring*, given in the following proposition-definition.

Proposition 14.2. *A ring R satisfying any of the following equivalent properties is called Noetherian:*

1. *any chain of ideals $I_1\subset I_2\subset I_3\subset\dots$ of R stabilizes (that is, there is an integer m such that $I_m=I_{m+1}=I_{m+2}=\dots$),*
2. *any set of ideals of R contains a maximal element,*
3. *any ideal of R is generated by finitely many elements, that is, is an R-module of finite type.*

Proof. The equivalence of (1) and (2) is completely formal.

(3) $\Rightarrow$ (1): Let $I=\sum I_j$, then I is an ideal which is generated, say, by $x_1,\dots,x_n$ as an R-module. Take m such that I_m contains all the x_i, then the chain stabilizes at I_m.

(2) $\Rightarrow$ (3) is based on a trick called "Noetherian induction" (cf. [32, Ch.2, Exer. 3.16]). Suppose that $I\subset R$ is an ideal which is not of finite type as an R-module. Consider the set of subideals of I which are of finite type as R-modules. This set is not empty: it contains 0. Now it has a maximal element $J\neq I$. Take $x\in I\backslash J$, then the ideal $J+(x)\subset I$ is strictly bigger than J, but is of finite type as an R-module. Contradiction. □

Exercise 14.6. i) Prove equivalence of (1) and (2);

ii) Let R be a Noetherian ring, $I\subset R$ is an ideal. Show that R/I as also a Noetherian ring.

The easiest example of a Noetherian ring is a field. Hilbert's basis theorem produces a lot of examples of Noetherian rings.

Theorem 14.6 (Hilbert's basis theorem)**.** *If R is a Noetherian ring, then so are the polynomial ring $R[z]$ and the formal power series ring $R[[z]]$.*

Proof. Let $I \subset R[z]$ be an ideal. We associate to it a series of ideals in R: $I_0 \subset I_1 \subset I_2 \subset \dots$, where I_j is generated by the leading coefficients of polynomials in I of degree j. Since R is Noetherian, this chain of ideals stabilizes, say, at I_r. Then we have a finite collection of polynomials whose leading coefficients generate $I_0, \dots, I_r$. Then the ideal of $R[z]$ generated by these polynomials is I. The same idea works with $R[[z]]$, if we use the discrete valuation instead of degree function. □

Exercise 14.7. Prove the theorem 14.6 for $R[[z]]$.

14.4 Integral elements, Noether's normalisation lemma, Hilbert's Nullstellensatz

In this subsection all rings assumed to be commutative.

Proposition 14.3. *Let $A \subset B$ be integral domains. The following conditions are equivalent:*

1. *$x \in B$ is integral over A,*
2. *$A[x]$ is an A-module of finite type,*
3. *There exists an A-module M of finite type such that $A \subset M \subset B$ and $xM \subset M$.*

Proof. The proof of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is direct. Suppose we know (3). Let $m_1, \dots, m_n$ be a system of generators of M. Then $xm_i = \sum_{i=1}^n b_{ij} m_j$, where $b_{ij} \in A$.

Recall that in any ring R we can do the following "determinant trick". Let N be a matrix with entries in R. Let $adj(N)$ be the matrix with entries in R given by

$$adj(N)_{ij} = (-1)^{i+j} \det(N(j,i)),$$

where $N(i,j)$ is N with i-th row and j-th column removed. It is an exercise in linear algebra that the product $adj(N) \cdot N$ is the scalar matrix with $\det(N)$ on the diagonal.

We apply this trick to the polynomial ring $R = A[T]$. For N we take the $n \times n$-matrix $Q(T)$ such that $Q(T)_{ij} = T\delta_{ij} - b_{ij}$. Let $f(T) = \det(Q(T)) \in A[T]$ (this is the analogue of the characteristic polynomial of x). We have a matrix identity

$$adj(Q(T)) \cdot Q(T) = diag(f(T)).$$

We consider this as an identity between matrices over the bigger ring $B[T]$. We are free to assign T any value in B. Substitute $T = x \in B$, and apply these matrices to the column vector $(m_1, \dots, m_n)^T$. Then the left hand side is zero. Hence $f(x)m_i = 0$ for any i. Since the m_i generate M, the whole module M is annihilated by $f(x) \in B$. In particular, $f(x) \cdot 1 = 0$, that is, $f(x) = 0$. Now note that $f(T)$ has coefficients in A and leading coefficient 1. □

Definition 14.9. Let $A \subset B$ be integral domains, then B is integral over A if every its element is integral over A. The set of elements of B which are integral over A is called the *integral closure* of A in B.

Let us prove some basic properties of integral elements.

Proposition 14.4. *1. The integral closure is a ring.*

2. *Suppose that B is integral over A, and is of finite type as an A-algebra. Then B is of finite type as an A-module.*

3. *Suppose that C is integral over B, and B is integral over A, then C is integral over A.*

Proof. (1) Let $x, y \in B$ be integral over A. Consider the A-module generated by all the monomials $x^i y^j$, $i, j \geq 0$. All higher powers of x, y can be reduced to finitely many of its powers using monic polynomials whose roots are x, y. So, the module is of finite type, and xy and $x+y$ act on it. Then by proposition 14.3 (3) xy and $x+y$ are integral.

(2) Suppose that B is generated by $b_1, \dots, b_n$ as an A-algebra, then B is generated by monomials $b_1^{i_1} \dots b_n^{i_n}$ as an A-module. As in item 1), all higher powers of each of the b_i's can be reduced to finitely many of its powers using a monic polynomial whose root is b_i. There remain finitely many monomials which generate B as an A-module.

(3) Let $x \in C$. Consider the A-subalgebra $F \subset C$ generated by x and the coefficients b_i of a monic polynomial with coefficients in B, whose root is x. Then F is an A-module of finite type, as only finitely many monomials generate it (the b_i are integral, and the higher powers of x can be reduced to lower powers). Now use item 3) of proposition 14.3. □

Proposition 14.5. *[7, Ch. V, §1, Prop. 13] Let A be a ring, R be a commutative A-algebra, A' be the integral closure of A in R. Then the integral closure of the ring $A[X_1, \dots, X_n]$ in the ring $R[X_1, \dots, X_n]$ is $A'[X_1, \dots X_n]$.*

Proposition 14.6. *[4, Ch. V, T. 5.10] Let $A \subseteq B$ be rings, B is integral over A, $\wp \subset A$ is a prime ideal. Then there exists a prime ideal $q \subset B$ such that $q \cap A = \wp$.*

Theorem 14.7. *([4, Ch. V, T. 5.11], Cohen-Seidenberg) Let $A \subseteq B$ be rings, B is integral over A. Consider a chain of prime ideals $\wp_1 \subseteq \wp_2 \subseteq \dots \subseteq \wp_n \subset A$ and a chain of prime ideals $q_1 \subseteq q_2 \subseteq \dots \subseteq q_m \subset B$ ($m < n$) such that $q_i \cap A = \wp_i$ for any $i \leq m$.*

Then the second chain can be continued to a chain of prime ideals $q_1 \subseteq q_2 \subseteq \dots \subseteq q_n \subset B$ such that $q_i \cap A = \wp_i$ for any $i \leq n$.

Theorem 14.8. *([4, Ch. V, T. 5.16], Cohen-Seidenberg) Let $A \subseteq B$ be integral domains, A is integrally closed, B is integral over A. Let $\wp_1 \supseteq \wp_2 \supseteq \dots \supseteq \wp_n \subset A$ be a chain of prime ideals in A, $q_1 \supseteq q_2 \supseteq \dots \supseteq q_m \subset B$ ($m < n$) be a chain of prime ideals in B such that $q_i \cap A = \wp_i$ for any $i \leq m$.*

Then the second chain can be continued to a chain of prime ideals $q_1 \supseteq q_2 \supseteq \dots \supseteq q_n \subset B$ such that $q_i \cap A = \wp_i$ for any $i \leq n$.

Definition 14.10. A ring is *integrally closed* or *normal* if it is integrally closed in its field of fractions.

Example 14.2. The rings $K[x]$ and $K[x, y]$ are integrally closed, but $K[x, y]/(y^2 - x^2 - x^3)$ is not.

Theorem 14.9 (Noether's normalization lemma). *Let k be any field, and $I \subset k[T_1, \dots, T_n]$ be an ideal, $R = k[T_1, \dots, T_n]/I$. There exist algebraically independent elements $Y_1, \dots, Y_m \in R$ such that R is integral over $k[Y_1, \dots, Y_m]$.*

Proof. If $I = 0$ there is nothing to prove. Suppose we have a non-zero polynomial $f \in I$. Let d be a positive integer greater than $\deg(f)$. Let us choose new variables in the following tricky way:

$$T_2' = T_2 - T_1^d, \quad T_3' = T_3 - T_1^{d^2}, \quad T_4' = T_4 - T_1^{d^3}, \quad \dots, \quad T_n' = T_n - T_1^{d^{n-1}}.$$

Substituting $T_2 = T_2' + T_1^d, \dots$ into f we rewrite it as a linear combination of powers of T_1 and a polynomial, say, g containing no pure powers of T_1. We observe that the pure powers of T_1 are of the form $i_1 + di_2 + d^2 i_3 + \ldots + d^{n-1} i_n$. Since $d > i_s$, all these integers are different, hence there is no cancellation among the pure powers of T_1. At least one such power enters with a nonzero coefficient. On the other hand, any power of T_1 in g is strictly less than the corresponding pure power. Therefore, we get a polynomial in T_1 with coefficients in $k[T_2', \ldots, T_n']$ and leading coefficient in k. Normalizing this polynomial we conclude that T_1 is integral over $R_1 = k[T_2', \ldots, T_n']/(I \cap k[T_2', \ldots, T_n'])$. Hence R is integral over R_1. We now play the same game with R_1 instead of R, and obtain a subring R_2 over which R_1 is integral. By proposition 14.4, item (3) R is also integral over R_2. We continue like that until we get a zero ideal, which means that the variables are algebraically independent. $\square$

Exercise 14.8. Show that for $B \subset D$ from proposition 6.1 the field $\mathrm{Quot}(B)$ is a finite algebraic extension, i.e. a module of finite type, over $K(Q)$, $Q \in B$, $\mathrm{ord}(Q) > 0$.

Theorem 14.10. *Let k be an algebraically closed field. All maximal ideals of $k[X_1, \ldots, X_n]$ are of the form $(X_1 - a_1, \ldots, X_n - a_n)$, $a_i \in k$, that is, consist of polynomials vanishing at a point $(a_1, \ldots, a_n) \in k^n$.*

Proof. Any polynomial has a Taylor expansion at the point $(a_1, \ldots, a_n)$. The canonical map

$$k[X_1, \ldots, X_n] \to k[X_1, \ldots, X_n]/(X_1 - a_1, \ldots, X_n - a_n)$$

sends f to $f(a_1, \ldots a_n)$, hence is surjective onto k. It follows that the ideal $(X_1 - a_1, \ldots, X_n - a_n)$ is maximal.

Let M be a maximal ideal (recall that $M \neq k[X_1, \ldots, X_n]$), then $\tilde{K} := k[X_1, \ldots, X_n]/M$ is a field containing k. By Noetherian normalization $\tilde{K}$ is integral over its subring $A = k[Y_1, \ldots, Y_m]$. But $\tilde{K}$ is a field, and we now show that then A must also be a field, in which case $k[Y_1, \ldots, Y_m] = k$ (no variables at all), and hence $\tilde{K}$ is integral over k. Indeed, let $x \in A$, then it is enough to show that $x^{-1} \in \tilde{K}$ also belongs to A. Since $x^{-1} \in \tilde{K}$ is integral over A it is subject to a polynomial relation $(x^{-1})^n + a_{n-1}(x^{-1})^{n-1} + \ldots + a_1 x^{-1} + a_0 = 0$, for some $a_i \in A$. Multiplying this by x^{n-1} we express x^{-1} as a polynomial in x with coefficients in A, hence $x^{-1} \in A$.

The k-algebra of finite type $\tilde{K}$ is integral over k, hence by proposition 14.3, item 2) $\tilde{K}$ is a k-module (= vector space over k) of finite type (= of finite dimension). Since k is algebraically closed, we must have $k = K$. Now let $a_i \in k$ be the image of X_i under the map $k[X_1, \ldots, X_n] \to k = k[X_1, \ldots, X_n]/M$. Then M contains the maximal ideal $(X_1 - a_1, \ldots, X_n - a_n)$, hence coincides with it. $\square$

Remark 14.3. When k is not supposed to be algebraically closed, this proof shows that the quotient by a maximal ideal of $k[X_1, \ldots, X_n]$ is a finite extension of k.

Corollary 14.4. *Let k be an algebraically closed field. If the polynomials of an ideal $I \subset k[X_1, \ldots, X_n]$ have no common zeros in k^n, then $I = k[X_1, \ldots, X_n]$.*

Proof. Assume $I \neq k[X_1, \ldots, X_n]$. Hilbert's basis theorem says that $k[X_1, \ldots, X_n]$ is Noetherian. Then I is contained in a maximal ideal, since the set of ideals that contain I has a maximal element, by item 2) of proposition 14.2 above. Therefore $I \subset (X_1 - a_1, \ldots, X_n - a_n)$, for some $a_i \in k$, since all the maximal ideals are of this form by the previous result. But then all the polynomials of I vanish at the point $(a_1, \ldots, a_n)$, which is a contradiction. $\square$

Theorem 14.11 (Nullstellensatz)**.** *Let k be an algebraically closed field. If a polynomial f vanishes at all the zeros of an ideal $I \subset k[X_1, \ldots, X_n]$, then $f^m \in I$ for some positive integer m.*

Remark 14.4. Let $I \subset A$ be an ideal in a ring A. The ideal

$$\sqrt{I} = \{f \in A| \quad f^r \in I \text{ for some integer } r > 0\}$$

is called the *radical* of the ideal I.

Proof. We know that I is generated by finitely many polynomials, say, $I = (g_1, \ldots, g_r)$. Let T be a new variable. Consider the ideal $J \subset k[T, X_1, \ldots, X_n]$ generated by $g_1, \ldots, g_r$ and $Tf - 1$. We observe that these polynomials have no common zero. The previous corollary implies that $J = k[T, X_1, \ldots, X_n]$, in particular, J contains 1. Then there exist polynomials $p, p_1, \ldots, p_r$ in variables $T, X_1, \ldots, X_n$ such that

$$1 = p(Tf - 1) + p_1 g_1 + \ldots + p_r g_r.$$

Note that this is an identity in variables $T, X_1, \ldots, X_n$. Thus we can specialize the variables anyway we like. For example, we can set $T = 1/f$. Multiplying both sides by an appropriate power of f we get an identity between polynomials in variables $X_1, \ldots, X_n$, which gives that some power of f belongs to $I = (g_1, \ldots, g_r)$. □

The Nullstellensatz and Hilbert's basis theorem form a foundation of a "dictionary" between commutative algebra and algebraic geometry. Below we give an overview of basic concepts from affine and projective algebraic geometry. We need these concepts to explain the connection between algebraic objects arising from a commutative ring of differential operators and geometric objects from complex algebraic geometry.

14.5 Localisation of modules, local rings, DVR

In this subsection all rings assumed to be commutative.

Definition 14.11. The localisation of a R-module M with respect to the multiplicative subset $S \subset R$ is defined in analogous way to the localisation of commutative rings. Namely, the localization $S^{-1}M$ of M with respect to $S \subset R$ is defined as the set of formal fractions $\frac{a}{b}$, with $a \in M$ and $b \in S$, up to the equivalence relation: $\frac{a}{b} = \frac{a_1}{b_1}$ if $(ab_1 - a_1 b)s = 0$ for some $s \in S$.

Fact: the $S^{-1}R$-module $S^{-1}M$ is canonically isomorphic to the $S^{-1}R$-module $S^{-1}R \otimes_R M$ ([4, Prop.3.5.]).

Definition 14.12. Let $\wp \subset R$ be a prime ideal, then $S = R \backslash \wp$ is a multiplicative system. Then $S^{-1}R$ is denoted $R_\wp$ and is called the *localization* of R at $\wp$. The ring $R_\wp$ has a very important property: $S^{-1}\wp$ is its only maximal ideal (every element not in $S^{-1}\wp$ is by definition invertible, hence $S^{-1}\wp$ contains all other ideals). Such rings have a name:

Rings with just one maximal ideal are called *local rings*. If R is a local ring and $\mathcal{M}$ is its maximal ideal then the field $R/\mathcal{M}$ is called the *residue field*. Note that if R is a localisation of a finitely generated ring over an algebraically closed field K at a maximal ideal, then the residue field is a finite extension over K, i.e. it is K.

Note that the ring $R_\wp$ is not finitely generated even if R is finitely generated over K.

Example 14.3.

1. Rational functions in one variable over a field K such that the denominator does not vanish at 0,

2. Formal power series $K[[z]]$,

3. Rational functions in two variables such that the denominator does not vanish at $(0, 0)$.

In all these examples, except the last one, the maximal ideal is principal. Such rings form the simplest class of local rings.

Definition 14.13. Let $\tilde{K}$ be a field. A subring $R \subset \tilde{K}$ is called a *discrete valuation ring* (DVR for short) if there is a discrete valuation v on $\tilde{K}$ such that $R\backslash\{0\} = \{x \in \tilde{K}^* | v(x) \geq 0\}$. In particular, $\tilde{K} = \mathrm{Quot}(R)$.

Remark 14.5. Recall that in our lectures we work with fields over K and K-valuations.

Proposition 14.7. *The following conditions are equivalent*

1. R *is a DVR.*
2. R *is a local ring which is a Noetherian integral domain and whose maximal ideal is principal.*

Moreover, all ideals of R *are principal.*

Proof. Let R be a DVR. Then it has the ideal $\mathcal{M} = \{x \in R | v(x) > 0\}$. Note that any element of $R\backslash\mathcal{M}$ is invertible, thus $\mathcal{M}$ is a unique maximal ideal. Obviously, $\mathcal{M}$ is generated by any element of valuation one.

To prove the converse statement we need the following lemma.

Lemma 14.4. *Let* R *be a Noetherian integral domain,* $t \in R\backslash R^*$. *Then* $\cap_{i=1}^{\infty}(t^i) = 0$.

Proof. For contradiction let $x \neq 0$ be contained in (t^i), for any $i \geq 1$. We write $x = t^i x_i$, then $(x) \subset (x_1) \subset (x_2) \subset \ldots$ is an ascending chain of ideals. Then $(x_{i+1}) = (x_i) = (tx_{i+1})$ for some i. Hence $x_{i+1} = tax_{i+1} \Leftrightarrow x_{i+1}(1 - ta) = 0$, but this implies $t \in R^*$ since $x_{i+1} \neq 0$ and R is an integral domain, a contradiction. □

Let R be a local ring which is a Noetherian integral domain and whose maximal ideal $\mathcal{M} = (t)$ is principal. By lemma above every $x \in R$, $x \neq 0$, is in $\mathcal{M}^i \backslash \mathcal{M}^{i+1}$ for some $i \geq 0$. Then $x = t^i u$, where $u \in R$ must be a unit. Define $v(x) = i$. If $y = t^j u'$ with $u' \in R^*$, then $xy = t^{i+j}uu'$, hence $v(xy) = v(x) + v(y)$. Suppose that $i \leq j$, then $x + y = t^i(u + t^{j-i}u')$, hence $v(x+y) \geq v(x) = \min\{v(x), v(y)\}$. We now can extend v to the field of fractions $\tilde{K}$ by the formula $v(x/y) = v(x) - v(y)$. The remaining properties are clear.

Now let I be a non-zero ideal of R. Let s be the infinum of v on the ideal $I \subset R$, then there exists $x \in I$ such that $v(x) = s$. Then $\mathcal{M}^s = (x) \subset I$. On the other hand, $v(I\backslash\{0\}) \subset \{s, s+1, \ldots\}$, and $\mathcal{M}^s\backslash\{0\} = \{x \in \tilde{K}^* | v(x) \geq s\}$, hence $I \subset \mathcal{M}^s$. Combining all together we obtain $I = \mathcal{M}^s$.

□

Exercise 14.9. Prove that a DVR is normal (i.e. integrally closed in its field of fractions).

14.6 Tensor product of rings and modules

For basic definitions of the tensor product we refer to [4, Ch.2]. We collect here only the most important for us definitions and theorems about the behaviour of tensor products.

Recall that $K \supset k$ is a *separable algebraic extension* of fields (where k is any field) if it is algebraic and a minimal polynomial of any element $x \in K$ has no multiple roots, i.e. its derivative does not vanish at x.

Definition 14.14. Let k be any field. The field K is *separably generated* over k if there exists a transcendence basis $\{x_\lambda\}_{\lambda \in \Lambda}$ such that K is separable algebraic extension over $k(x_\lambda)$.

K is *separable* over k if any subfield in K finitely generated over k is separably generated over k.

Theorem 14.12. *[8, Ch.4-7], [51] (MacLane) Let $K \supset k$ be an extension of fields. K is separable over k if and only if for any extension $L \supset k$ the ring $L \otimes_k K$ is reduced (i.e. its nilradical $\sqrt{(0)}$ is zero).*

More generally, if R is an integral domain over k, S is a reduced ring over k, then

1. *if $\mathrm{Quot}(R)$ is separable over k, then $R \otimes_k S$ is a reduced ring;*

2. *if $\mathrm{Quot}(R)$ is separable over k and k is algebraically closed in $\mathrm{Quot}(R)$, then $R \otimes_k S$ is integral.*

14.7 Completion

In this section we collect all necessary results about completion of groups, rings and modules. The main references are the books [4], [113].

Let G be an abelian topological group, not necessarily Hausdorff, i.e. G is a topological space and a group, and these two structures are compatible in the following sense: the maps $G \times G \to G : (x, y) \mapsto x + y$ and $G \to G : x \mapsto -x$ are continuous. If the set $\{0\}$ is closed in G, then the diagonal is closed in $G \times G$, hence G is Hausdorff. The topology on G is uniquely defined by a system of *neighbourhoods of zero* (we use here the additive group law, i.e. zero is the unity of the group).

Lemma 14.5. *[4, L. 10.1] Let H be the intersection of all neighbourhoods of zero in G. Then*

1. *H is a subgroup;*

2. *H coincides with the closure of $\{0\}$;*

3. *the factor group G/H is Hausdorff;*

4. *the group G is Hausdorff iff $H = 0$.*

In general case a notion of completion can be complicated (see [113]), but in the case when zero has a countable base of neighbourhoods (this is the case in our lectures), this notion admit a simpler description. Namely, the completion $\hat{G}$ is a group of equivalence classes of Cauchy sequences. Recall that a sequence (x_ν) of elements of the group G is called a Cauchy sequence, if for any neighbourhood of zero U there exists an integer $s(U)$ such that $x_\mu - x_\nu \in U$ for all $\mu, \nu \geq s(U)$. Two Cauchy sequences are *equivalent* if $x_\nu - y_\nu \to 0$ in G. If (x_ν), (y_ν) are Cauchy sequences, then $(x_\nu + y_\nu)$ is also a Cauchy sequence and the class of this sequence depends only on the classes of (x_ν) and (y_ν). We have a natural map $\varphi : G \to \hat{G}$ of abelian groups that sends an element $x \in G$ to a class (x) of a constant sequence. We have $\ker \varphi = \cap U$, where U runs all neighbourhoods of zero in G, i.e. by the lemma above φ is injective iff G is Hausdorff. For any two abelian topological groups H, G and a continuous homomorphism $f : G \to H$ there is a homomorphism of completions $\hat{f} : \hat{G} \to \hat{H}$ that sends a Cauchy sequence to a Cauchy sequence.

If the topology on G is defined by a *sequence of subgroups* (and this is the case in our lectures), then the completion can be defined via a *projective limit*: $\hat{G} \simeq \varprojlim G/G_n$. The following properties of completions of aforementioned groups are widely used in these lectures:

Lemma 14.6. *[4, Cor. 10.3, 10.4, 10.5]*

Let $0 \to G' \to G \xrightarrow{p} G'' \to 0$ be an exact sequence of groups. Assume that the topology on G is given by a sequence of subgroups $\{G_n\}$, and the topologies on G', G'' are induced by subgroups $\{G' \cap G_n\}$, $\{p(G_n)\}$ correspondingly. Then the sequence

$$0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$$

is exact. Moreover, $\hat{G}_n$ is a subgroup in $\hat{G}$ and $\hat{G}/\hat{G}_n \simeq G/G_n$ and $\hat{\hat{G}} \simeq \hat{G}$.

The most important for us examples of topological groups and completions are *topological rings and modules*, where the topology is given with the help of powers of ideals. If $G = A$ is a ring and $\wp$ is an ideal, put $G_n = \wp^n$. The topology defined by these subgroups is called $\wp$-*adic topology*. The ring A is a topological *ring* in this topology, i.e. not only the additive group law is continuous, but all ring operations are continuous. The completion $\hat{A}$ is a gain a topological ring.

If M is an A-module, put $G_n := \wp^n M$. The completion $\hat{M}$ with respect to this topology is a *topological* $\hat{A}$-*module*, i.e. the map $\hat{A} \times \hat{M} \to \hat{M}$ is continuous. The following properties of completed rings and modules are important for us (the proofs are contained in [4, Ch.10].

Theorem 14.13. *1. Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated modules over a Noetherian ring A. Then for any ideal $\wp \subset A$ the sequence of $\wp$-adic completions

$$0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$$

is exact.

2. *For any A module M consider the natural homomorphism of $\hat{A}$-modules $\hat{A} \otimes_A M \to \hat{A} \otimes_A \hat{M} \to \hat{A} \otimes_{\hat{A}} \hat{M} = \hat{M}$. If M is finitely generated, then this homomorphism is surjective. If moreover the ring A is Noetherian, then it is an isomorphism.*

3. *If A is a Noetherian local ring and $\mathfrak{m}$ is its maximal ideal, then the $\mathfrak{m}$-adic completion $\hat{A}$ of the ring A is a local ring with maximal ideal $\hat{\mathfrak{m}}$.*

4. *Let A be a Noetherian ring and let $\wp$ be its ideal. Let M be a finitely generated A-module, $\hat{M}$ be its $\wp$-adic completion. Then the kernel of the homomorphism $M \to \hat{M}$ consists of elements $x \in M$ annihilated by some element from $1 + \wp$. In particular, if A is an integral domain and M is a torsion free module, then the homomorphisms $A \to \hat{A}$, $M \to \hat{M}$ are injective.*

5. *If A is a Noetherian ring and $\wp$ is an ideal, then the $\wp$-adic completion $\hat{A}$ is a Noetherian ring.*

14.8 Krull dimension

It is useful to mention another notion of "measure" on the commutative ring (except the transcendence degree), the *Krull dimension*. Here we list important results on the Krull dimension of rings, which we will not prove. The proofs one can find e.g. in [4, Ch.11].

Definition 14.15. Let R be a ring. The *height* of a prime ideal $\wp$ in the ring R is a supremum of the lengths of all chains of prime ideals $\wp_0 \subset \wp_1 \subset \ldots \wp_n = \wp$, and is denoted by $\mathrm{ht}(\wp)$.

The *Krull dimension* of R is a supremum of the lengths of all chains of prime ideals, and is denoted by $\dim(R)$.

Examples of rings with finite dimension: rings finitely generated over K, local Noetherian rings.

Proposition 14.8. *[4, Ch. 11] Let B be an integral domain finitely generated over K. Then*

1. $\dim(B) = trdeg_K(\mathrm{Quot}(B))$*;*

2. *for any prime ideal $\wp \subset B$ we have*

$$\mathrm{ht}(\wp) + \dim(B/\wp) = \dim B.$$

Theorem 14.14. *(i) Let A be a factorial integral domain. Then any localisation AS^{-1} is factorial.*

(ii) A Noetherian integral domain A is factorial if and only if every prime ideal of height one is principal.

(iii) If a completion of a noetherian local ring is factorial, then the ring itself is factorial.

(iv) Let A be a field or DVR. Then the ring $A[[T_1,\ldots,T_n]]$ is factorial.

The proof is contained in [7, Ch. VII, §3].

Theorem 14.15. *([4, Ch. 11, Cor. 11.17], Krull) Let A be a Noetherian ring, and $x \in A$ be an element which is neither a unit nor a zero divisor. Then the height of any minimal prime ideal associated with (x) is equal to one.*

Exercise 14.10. ([4, Ch.11], Nagata) Let $A = k[x_1,\ldots,x_n,\ldots]$ be a polynomial ring in infinitely many variables. Let $m_1, m_2, \ldots$ be a sequence of natural numbers with the property $m_{i+1} - m_i > m_i - m_{i-1}$ for any $i > 1$. Set $\wp_i := (x_{m_i+1},\ldots,x_{m_{i+1}})$ and $S := A\backslash\{\cup_i \wp_i\}$. Note that $\wp_i$ are prime ideals and S is a m.c. set.

Show that the ring $S^{-1}A$ is Noetherian and $\mathrm{ht}(S^{-1}\wp_i) = m_{i+1} - m_i$. In particular, $\dim S^{-1}A = \infty$.

Hint: Prove first the following statement. If A is a ring such that A_m is a Noetherian ring for any maximal ideal m and the set of maximal ideals containing a given non-zero element $x \in A$ is finite for any such x, then A is Noetherian.

14.9 More facts about regular and factorial rings

Theorem 14.16. *[4, Ch. 11, Rem. to prop. 11.24] or [53, Cor 2, p.206] Let A be a local regular complete ring over a field k (or A contains a field). Then $A \simeq k(\mathfrak{m})[[T_1,\ldots,T_n]]$, where $n = \dim A$ and $k(\mathfrak{m})$ is the residue field of A.*

Remark 14.6. In [4, Ch. 11, Rem. to prop. 11.24] the proof is given for the case when A contains a field isomorphic to $k(\mathfrak{m})$. For general case see [53, Cor 2, p.206].

Theorem 14.17. *Any regular local ring is factorial.*

Remark 14.7. If a regular local ring contains a field, then this theorem follows from theorems 14.14 and 14.16. In general case see [53, Th. 49, p.142].

15 List of Exercises

For convenience of the reader, here we collect the references to all exercises from the text.

Section 3: Exercise 3.1, 3.2, 3.3, 3.4.
Section 4: Exercise 4.1, 4.2, 4.3.
Section 5: Exercise 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, 5.10.
Section 6: Exercise 6.1, 6.2, 6.3, 6.4, 6.5.
Section 7: Exercise 7.1, 7.2, 7.3, 7.4.
Section 8: Exercise 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7, 8.8, 8.9.
Section 9: Exercise 9.1, 9.2, 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, 9.9.
Section 10: Exercise 10.1, 10.2, 10.3, 10.4, 10.5, 10.6, 10.7, 10.8, 10.9.
Section 11: Exercise 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7.
Section 12: Exercise 12.1, 12.2.
Section 14: Exercise 14.1, 14.2, 14.3, 14.4, 14.5, 14.6, 14.7, 14.8, 14.9, 14.10

16 List of Problems

For convenience of the reader, here we collect the references to all problems from the text.

Section 4: Problem 4.1.

Section 5: Problem 5.1, 5.2, 5.3, Exercise* 5.8

Section 6: Problem 6.1, 6.2.

Section 12: Problem 12.1.

Section 13: Problem 13.1, 13.2, 13.3, 13.4, 13.5.

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A. Zheglov, Lomonosov Moscow State University, faculty of mechanics and mathematics, department of differential geometry and applications, Leninskie gory, GSP, Moscow, 119899, Russia
e-mail *azheglov@mech.math.msu.su*