Cheeger–Colding Theory

Teaching Assistant Session (3)

Volume convergence theorem

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Part 1: Colding's volume convergence theorem

An n–splitting map supplies n independent coordinate directions.

Volume convergence theorem

For every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(n, \varepsilon) > 0$ such that if (M^n, g, p) is complete and **assume**

$$\mathrm{Ric}_g \geq -(n-1)\delta g, \qquad d_{\mathrm{GH}}(\overline{B}_2(p), \overline{B}_2(0^n)) < \delta, \qquad 0 < \delta \leq \delta_0.$$

Then

$$\left|\mathrm{Vol}_g(B_1(p)) - \omega_n\right| < \varepsilon, \qquad \omega_n := \mathcal{L}^n(B_1(0^n)).$$

In particular, Vol is continuous with respect to Gromov–Hausdorff distance.

No separate noncollapse hypothesis is required. Closed balls carry their ambient distances.

The proof and its main difficulty

- **The upper bound is trivial by Bishop–Gromov comparison.**

$$\operatorname{Vol} B_1(p) \le V_{-\delta}^n(1) \le \omega_n + C(n)\delta.$$

- **The lower bound needs an actual large image.**

On a prepared ball, build harmonic splitting function u. A good point and mod-2 degree show that $u(B_1(p))$ contains an large inner Euclidean ball.

$$B_{1-\Psi}(0^n) \subset u(B_1(p)), \qquad J_u = \sqrt{\det(\langle \nabla u_i, \nabla u_j \rangle)} \le 1 + \Psi,$$

for some small $\Psi = \Psi(\delta|n) > 0$.

Idea is simple: the image of splitting map u contains an almost full Euclidean unit ball, and since this map cannot significantly enlarge volume, the original ball cannot have too small a volume.

Volume convergence on large comparison balls

We will only indicate the proof of the following weak volume convergence, general version needs a further tenichal fine covering argument.

Weak volume convergence

If

$$\mathrm{Ric}_g \geq -(n-1)\delta g, \qquad d_{\mathrm{GH}}(\overline{B}_{\delta^{-1}}(p), \overline{B}_{\delta^{-1}}(0^n)) \leq \delta,$$

then

$$\mathrm{Vol}\, B_1(p) \geq (1 - \Psi(\delta \mid n))\omega_n, \qquad \Psi(\delta \mid n) \longrightarrow 0.$$

- **Use the n–splitting function constructed from Session 2.**

By Cheeger-Colding's almost splitting theorem, there exists (n, κ)-splitting map u on $B_{10}(p)$, where $\kappa = \Psi(\delta \mid n) \to 0$.

$$u : B_{10}(p) \longrightarrow \mathbb{R}^n, \qquad u(p) = 0.$$

The analytic data for the coordinate map

Write $u = (u_1, \dots, u_n)$ and $G_{ij} = \langle \nabla u_i, \nabla u_j \rangle$.

- **Harmonicity and unit gradients:**

$$\Delta u_i = 0, \qquad \sup_{B_{10}} |\nabla u| \leq 1 + \kappa.$$

- **Almost orthonormal gradients and small Hessian:**

$$⨍_{B_{10}} \sum_{i,j} |G_{ij} - \delta_{ij}| \leq C(n)\kappa, \qquad ⨍_{B_{10}} \sum_{i} |\nabla^2 u_i|^2 \leq C(n)\kappa.$$

The radius 10 is fixed, so its scaling factors are absorbed into the constants.

Metric control from n splitting coordinates

Metric lemma

If $u : B_2(q) \to \mathbb{R}^n$ is an (n, δ)-splitting map and $\mathrm{Ric} \geq -(n-1)\delta$

$$\big||u(y) - u(z)| - d(y,z)\big| \leq \Psi(\delta \mid n), \qquad y, z \in B_1(q).$$

- **Why we need distance distortion.**

The next 2 slides show the proof. Image containment

$$B_{1-\Psi}(0^n) \subset u(B_1(p))$$

will come later from degree method, rather than from approximate density alone.

The metric lemma: choose a good segment

Write $G_{ij} = \langle \nabla v_i, \nabla v_j \rangle$ and $H = \sum_i |\nabla^2 v_i|$.

- **The segment inequality controls averaged errors.**

$$h := H + \sum_{i,j} |G_{ij} - \delta_{ij}|, \qquad \fint_{B_2(q)} h \le \Psi(\delta \mid n),$$

$$\fint_{B_1(q)\times B_1(q)} \int_{\gamma_{yz}} h \le C(n) \fint_{B_2(q)} h \le \Psi(\delta \mid n).$$

- **Perturb arbitrary endpoints slightly.**

Relative volume comparison and the averaged Gram bound allow y, z to be replaced by nearby y', z' so that

$$d(y, y') + d(z, z') \le \Psi(\delta \mid n), \quad |G(y') - I| \le \Psi(\delta \mid n), \quad \int_{\gamma_{y'z'}} H \le \Psi(\delta \mid n).$$

Use small endpoint neighborhoods and discard pairs with large segment error. This uses relative volume ratios, not noncollapse.

The metric lemma: integrate the Hessian

Let γ be the selected unit-speed segment, of length ℓ, with initial direction $w = \dot\gamma(0)$.

- **Taylor's formula along a geodesic:**

$$v_i(z') - v_i(y') = \ell\langle \nabla v_i(y'), w\rangle + \int_0^\ell (\ell - t)\nabla^2 v_i(\dot\gamma(t), \dot\gamma(t))\, dt.$$

- **The initial differential is almost an isometry.**

$$G(y') \approx I \quad \Longrightarrow \quad \big||Dv_{y'}w| - 1\big| \le \Psi(\delta \mid n),$$

Here $Dv_{y'}w = (\langle \nabla v_i(y'), w\rangle)_{i=1}^n$ is a vector in $\mathbb{R}^n$; its norm is close to 1.

$$|v(z') - v(y') - \ell Dv_{y'}w| \le \Psi(\delta \mid n)\ell \quad \Longrightarrow \quad \big||v(z') - v(y')| - \ell\big| \le C(n)\Psi(\delta \mid n).$$

Move back to y, z using the Lipschitz bound. This proves the metric lemma independently of volume convergence.

A simpler all-scale good set

Put the Gram and Hessian errors into the same maximal function.

$$F := \sum_i |\nabla^2 u_i|^2 + \sum_{i,j} |G_{ij} - \delta_{ij}|, \qquad \fint_{B_{10}(p)} F \le C(n)\kappa.$$

- **Declare a point good when every centered average is small.**

$$\Omega_\lambda := \left\{ x \in B_1(p) : \sup_{0<s\le 4} \fint_{B_s(x)} F \le \lambda \right\}.$$

- **At a good point, all smaller balls are splitting balls.**

$$\fint_{B_s(x)} \sum_{i,j} |G_{ij} - \delta_{ij}| \le \lambda, \qquad s^2 \fint_{B_s(x)} \sum_i |\nabla^2 u_i|^2 \le 16\lambda.$$

Harmonicity and gradient bound are inherited from $B_{10}(p)$. Thus u is (n, η)-splitting on every $B_s(x)$, $s \le 4$, with $\eta = C(n)(\lambda + \kappa)$.

Why the good set has almost full measure

- **Apply the Vitali-covering lemma to the bad balls.**

For each bad point choose $s \le 4$ with $\fint_{B_s(x)} F > \lambda$. Select pairwise disjoint balls $B_{s_i}(x_i)$ so that $\{B_{5s_i}(x_i)\}$ cover $B_1 \setminus \Omega_\lambda$.

$$\begin{aligned}\operatorname{Vol}(B_1 \setminus \Omega_\lambda) &\le \sum_i \operatorname{Vol} B_{5s_i}(x_i) \le C(n) \sum_i \operatorname{Vol} B_{s_i}(x_i) \\ &\le \frac{C(n)}{\lambda} \sum_i \int_{B_{s_i}(x_i)} F \le \frac{C(n)}{\lambda} \int_{B_5(p)} F.\end{aligned}$$

- **Normalize using relative volume comparison.**

$$\frac{\operatorname{Vol}(B_1 \setminus \Omega_\lambda)}{\operatorname{Vol} B_1} \le C(n) \frac{\kappa}{\lambda}.$$

$$\lambda = \sqrt{\kappa} \quad \Longrightarrow \quad \eta \le C(n)\sqrt{\kappa}, \qquad \frac{\operatorname{Vol}(B_1 \setminus \Omega_\lambda)}{\operatorname{Vol} B_1} \le C(n)\sqrt{\kappa}.$$

Choose a good point near the center

Fix $0 < a < 1/10$. We will choose a before making κ small.

- **Bishop–Gromov supplies a relative lower bound.**

$$\frac{\operatorname{Vol} B_a(p)}{\operatorname{Vol} B_1(p)} \geq c(n)a^n.$$

- **The bad set cannot occupy this entire small ball B_a.**

$$C(n)\sqrt{\kappa} < c(n)a^n \quad \Longrightarrow \quad \exists x \in \Omega_\lambda \cap B_a(p).$$

- **At this point, the differential is invertible.**

$$\fint_{B_s(x)} |G - I| \leq C(n)\eta \quad (s > 0) \quad \Longrightarrow \quad |G(x) - I| \leq C(n)\eta.$$

Let $s \downarrow 0$ and by smoothness, for small η, Du_x has full rank. We will use this single good point to determine the degree.

A unique regular fiber

Take $y \in B_1(p)$, $y \neq x$, and put $d = d(x, y)$ and $r = 3d/2$.

- **The good-point estimates apply at the required scale.**

$$y \in B_r(x), \qquad 2r = 3d < 3(1 + a) < 4.$$

- **Apply the metric lemma centered at x.**

$$\begin{aligned} |u(y) - u(x)| &\geq d - \Psi(\eta \mid n) r \\ &= \left(1 - \frac{3}{2}\Psi(\eta \mid n)\right) d > 0. \end{aligned}$$

$$u^{-1}(u(x)) \cap B_1(p) = \{x\}, \qquad Du_x \text{ is invertible.}$$

Thus $u(x)$ is a regular value with exactly 1 preimage in $B_1(p)$.

Degree modulo 2 method

Let $D \Subset M^n$ have smooth boundary, and let $f : \overline{D} \to \mathbb{R}^n$ be smooth. For $z \notin f(\partial D)$, the degree $\deg_2(f, D, z) \in \{0, 1\}$ is defined.

- **At a regular value, count preimages modulo 2.**

$$\deg_2(f, D, z) \text{ is defined to be } \#\big(f^{-1}(z) \cap D\big) \pmod 2.$$

z is a regular value means that Df_y is invertible at every $y \in f^{-1}(z) \cap D$.

- **Homotopy invariance: avoid 0 on the boundary.**

If $H_t : \overline{D} \to \mathbb{R}^n$, $0 \le t \le 1$, is a smooth homotopy satisfying

$$H_t(y) \neq 0 \qquad (y \in \partial D,\ 0 \le t \le 1),$$

$$\text{then } \deg_2(H_0, D, 0) = \deg_2(H_1, D, 0).$$

Keep the boundary image outside the target ball

Recall the previous metric lemma estimate, using $u(p) = 0$ and $\Psi = \Psi(\delta|n)$:

$$\big||u(y)| - d(p,y)\big| \le \Psi, \qquad y \in B_2(p).$$

- **Choose a smooth domain inside the unit ball.**

$$\overline{B}_{1-a}(p) \subset D \Subset B_1(p), \qquad 0 < \Psi < a < 1/10.$$

- **Every boundary point maps outside an inner ball.**

For $y \in \partial D$, we have $d(p,y) \ge 1 - a$, hence

$$|u(y)| \ge d(p,y) - \Psi \ge 1 - a - \Psi.$$

- **Define the target region.**

$$Q := B_{1-a-\Psi}(0^n) \quad \Longrightarrow \quad Q \cap u(\partial D) = \varnothing.$$

Thus every target point in Q is admissible for the degree.

Compute the degree at a reference value

The good-point argument supplied $x \in B_a(p)$ such that

$$u^{-1}(u(x)) \cap B_1(p) = \{x\} \quad \text{is a single point,} \qquad Du_x \text{ is invertible.}$$

- **The reference value lies in Q.**

Set $z_0 := u(x)$. By previous estimate and $\Psi < a < 1/10$,

$$|z_0| \le d(p, x) + \Psi < a + \Psi < 1 - a - \Psi.$$

Therefore $z_0 \in Q$. Also $x \in B_a(p) \subset D$.

- **There is exactly 1 regular preimage in D.**

$$u^{-1}(z_0) \cap D = \{x\}, \qquad Du_x \text{ is invertible.}$$

$$\deg_2(u, D, z_0) = \#(u^{-1}(z_0) \cap D) \pmod 2 = 1.$$

Move the target point by a boundary-avoiding homotopy

- **Connect the reference value to an arbitrary target.**

Fix $z \in Q$. Since Q is convex,

$$\gamma(t) := (1-t)z_0 + tz \in Q, \qquad 0 \le t \le 1.$$

- **Translate u by this moving target.**

$$H_t(y) := u(y) - \gamma(t).$$

Since $\gamma(t) \in Q$ and $Q \cap u(\partial D) = \varnothing$,

$$H_t(y) \neq 0 \qquad (y \in \partial D,\ 0 \le t \le 1).$$

- **Apply homotopy invariance of mod 2 degree.**

$$\begin{aligned} \deg_2(u, D, z) &= \deg_2(H_1, D, 0) \\ &= \deg_2(H_0, D, 0) = \deg_2(u, D, z_0) = 1. \end{aligned}$$

Why nonzero degree forces an actual preimage

We have proved that $\deg_2(u, D, z) = 1$ for every $z \in Q$.

- **Suppose a target point has no preimage in D.**

If some $z \in Q$ were not in $u(D)$, then

$$u^{-1}(z) \cap D = \varnothing.$$

- **An empty fiber has degree 0.**

A value with no preimages is automatically a regular value: there are no preimages at which regularity could fail. Hence

$$\deg_2(u, D, z) = \#\varnothing \pmod 2 = 0,$$

contradicting $\deg_2(u, D, z) = 1$.

$$B_{1-a-\Psi}(0^n) = Q \subset u(D) \subset u(B_1(p)).$$

Every point of the inner Euclidean ball therefore has at least 1 actual preimage in D.

The area formula converts image into volume

- **Use the n-dimensional Jacobian.**
$$J_u = \sqrt{\det(G_{ij})} \leq (1+\Psi)^n.$$
- **Count multiplicity by the area formula.**
$$\int_D J_u \, dV_g = \int_{\mathbb{R}^n} N(u, D, z) \, dz \geq \mathcal{L}^n(u(D)),$$

Here N counts preimages. It is at least 1 on the image.

$$\omega_n(1-a-\Psi)^n \leq \mathcal{L}^n(u(D)) \leq (1+\Psi)^n \operatorname{Vol} B_1(p).$$

Finish the prepared large-ball estimate

$$\operatorname{Vol} B_1(p) \geq \omega_n \frac{(1-a-\Psi)^n}{(1+\Psi)^n}, \qquad \Psi = \Psi(\delta \mid n).$$

Given a target error, first choose a small. Then choose Ψ small so that a good point exists in $B_a(p)$ and $\Psi < a$.

- **Bishop–Gromov supplies the opposite inequality.**

$$\operatorname{Vol} B_1(p) \leq V^n_{-\delta}(1) \leq \omega_n + C(n)\delta.$$

$$\left|\operatorname{Vol} B_1(p) - \omega_n\right| \leq \Psi(\delta \mid n).$$

This proves weak version of volume convergence. It remains to pass from a large comparison ball to the radius-2 hypothesis by a covering argument.

Part 2: Local curvature estimate

Local curvature estimate

There exist $0 < \varepsilon_0 = \varepsilon_0(n) < 1/2$ and $C_0 = C_0(n) > 0$ such that the following holds. Let (M^n, g, p) be a complete Ricci-flat manifold. If

$$\frac{\mathrm{Vol}_g B_1(p)}{\omega_n} \geq 1 - \varepsilon_0,$$

then

$$\sup_{B_{\varepsilon_0}(p)} |\mathrm{Rm}_g| \leq C_0.$$

- **Almost Euclidean volume prevents curvature concentration.**

We prove the estimate by contradiction: a normalized curvature blow-up would have a smooth Euclidean limit with curvature 1 at its base point.

The proof uses Bishop–Gromov comparison and smooth compactness for Ricci-flat metrics.

Local estimate: a hypothetical curvature blow-up

- **Assume that no dimensional pair (ε_0, C_0) works.**

For integers $i \geq 3$, take the candidate constants $\varepsilon_0 = i^{-1}$ and $C_0 = i^4$. There would be complete Ricci-flat (M_i, g_i, p_i) with

$$\mathrm{Vol}_{g_i} B_1(p_i) \geq (1 - i^{-1})\omega_n, \qquad \mathrm{Ric}_{g_i} = 0.$$

- **Choose a point of very large curvature near p_i.**

$$q_i \in B_{i^{-1}}(p_i), \qquad K_i(q_i) > i^4, \qquad K_i := |\,\mathrm{Rm}_{g_i}\,|.$$

We will move the center slightly to obtain a neighborhood on which the curvature is bounded by a fixed multiple of its value at the new center.

- **Prepare a distance cutoff.**

$$\rho_i := \frac{2}{i}, \qquad F_i(z) := K_i(z)\big(\rho_i - d_{g_i}(p_i, z)\big)^2.$$

Point selection by a weighted maximum

- **Maximize F_i on $\overline{B}_{\rho_i}(p_i)$.**

Completeness makes this closed ball compact. Since $F_i = 0$ on its boundary and $F_i(q_i) > 0$, a maximum occurs at an interior point x_i. Set

$$Q_i := K_i(x_i), \qquad s_i := \rho_i - d_{g_i}(p_i, x_i) > 0.$$

- **The selected point has a large curvature scale.**

$$Q_i s_i^2 = F_i(x_i) \geq F_i(q_i) > i^4 \left(\frac{1}{i}\right)^2 = i^2.$$

- **The cutoff stays comparable near x_i.**

$$z \in B_{s_i/2}(x_i) \quad \Longrightarrow \quad \rho_i - d_{g_i}(p_i, z) \geq \frac{s_i}{2}.$$

$$K_i(z)\frac{s_i^2}{4} \leq F_i(z) \leq Q_i s_i^2 \quad \Longrightarrow \quad K_i(z) \leq 4Q_i.$$

Rescale to curvature 1 on an expanding ball

- **Use the curvature scale at the selected point.**

$$h_i := Q_i g_i, \qquad d_{h_i} = \sqrt{Q_i}\, d_{g_i}, \qquad |\operatorname{Rm}_{h_i}| = Q_i^{-1} |\operatorname{Rm}_{g_i}|.$$

Since $s_i \le 2/i$, we also have $Q_i > i^4/4 \to \infty$. Ricci-flatness is preserved, and the curvature at x_i is now 1.

$$\operatorname{Ric}_{h_i} = 0, \qquad |\operatorname{Rm}_{h_i}|(x_i) = 1.$$

- **The controlled neighborhood becomes arbitrarily large.**

$$L_i := \frac{s_i \sqrt{Q_i}}{2} > \frac{i}{2} \longrightarrow \infty.$$

$$\sup_{B^{h_i}_{L_i}(x_i)} |\operatorname{Rm}_{h_i}| \le 4.$$

Thus every fixed-radius ball around x_i eventually has a uniform curvature bound. We next obtain a volume lower bound there.

Transfer the almost maximal volume to the new center

- **The new center is still close to p_i.**

$$a_i := d_{g_i}(p_i, x_i) < \frac{2}{i}, \qquad B_1^{g_i}(p_i) \subset B_{1+a_i}^{g_i}(x_i).$$

- **Bishop–Gromov controls every smaller radius.**

Since $\mathrm{Ric}_{g_i} = 0$, for $0 < r \leq 1 + a_i$ we have

$$1 \geq \frac{\mathrm{Vol}_{g_i} B_r(x_i)}{\omega_n r^n} \geq \frac{\mathrm{Vol}_{g_i} B_{1+a_i}(x_i)}{\omega_n (1+a_i)^n} \geq \frac{1 - i^{-1}}{(1+a_i)^n} =: \alpha_i \longrightarrow 1.$$

- **Apply this at each fixed rescaled radius.**

For a fixed $R > 0$, put $r = R/\sqrt{Q_i}$. Since $Q_i \to \infty$, this radius is eventually smaller than $1 + a_i$. Rescaling preserves volume ratios.

$$\frac{\mathrm{Vol}_{h_i} B_R(x_i)}{\omega_n R^n} \longrightarrow 1 \qquad \text{for every fixed } R > 0.$$

The compactness input for Ricci-flat metrics

- **Curvature and volume bounds control injectivity radius.**

The Cheeger–Gromov–Taylor estimate, on a complete manifold, says

$$\sup_{B_r(z)} |\operatorname{Rm}| \le r^{-2}, \qquad \operatorname{Vol} B_r(z) \ge v r^n \qquad (v > 0)$$

$$\Longrightarrow \qquad \operatorname{inj}(z) \ge c(n, v) r.$$

With $r = 1/2$, the bounds above give $\operatorname{inj}_{h_i}(x_i) \ge \iota(n) > 0$.

- **Ricci-flat elliptic regularity gives smooth compactness.**

In harmonic coordinates, Ricci-flatness is elliptic. Curvature and injectivity bounds give interior estimates for every derivative. Passing to a subsequence,

$$(M_i, h_i, x_i) \longrightarrow (M_\infty, h_\infty, x_\infty) \qquad \text{smoothly on compact sets.}$$

The expanding controlled balls ensure that the limit is complete and Ricci-flat.

The Euclidean limit contradicts normalized curvature

- **Smooth convergence preserves the normalized curvature.**

$$|\operatorname{Rm}_{h_\infty}|(x_\infty) = \lim_{i\to\infty} |\operatorname{Rm}_{h_i}|(x_i) = 1.$$

- **Every limit ball has exactly Euclidean volume.**

$$\operatorname{Vol}_{h_\infty} B_R(x_\infty) = \lim_{i\to\infty} \operatorname{Vol}_{h_i} B_R(x_i) = \omega_n R^n, \qquad R > 0.$$

This volume continuity follows from smooth convergence. It does not use the GH volume convergence theorem from Part 1.

- **Apply the equality case of Bishop–Gromov.**

The complete limit has $\operatorname{Ric}_{h_\infty} = 0$ and maximal volume at every radius, so it is isometric to $\mathbb{R}^n$.

$$|\operatorname{Rm}_{h_\infty}|(x_\infty) = 0 \neq 1.$$

This contradiction proves the local curvature estimate: there exist dimensional $\varepsilon_0, C_0 > 0$.