Gradient flows on $\mathrm{CAT}(0)$ spaces and harmonic map heat flows

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Outline

1. $CAT(0)$ space
2. Gradient flows on $CAT(0)$ spaces
3. Application: Harmonic Map Heat Flows between metric spaces
4. Regularity and Bochner formula

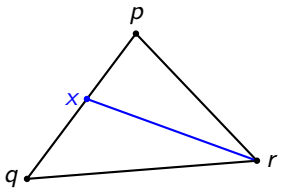
Existence theorem (Eells–Sampson): Let M, N be two smooth manifolds. If M is compact and N is complete with *non-positive sectional curvature*, the harmonic heat flow exists for all $t \geq 0$; if the images remain in a compact subset, it converges to a harmonic map homotopic to f_0.
Consequently, every homotopy class contains a harmonic representative when N has non-positive curvature.

Bochner inequality: Eells and Sampson showed a Bochner identity for harmonic maps, and used is to obtian if $u : M \to N$ is harmonic, $\mathrm{Ric}_M \geq 0$, and N has non-positive sectional curvature, then u is totally geodesic with energy density e_u=constant.

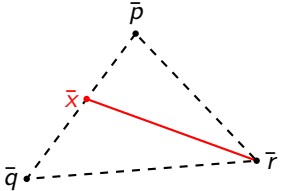
Problem: How to develop a theory of harmonic maps (and their heat flows) between metric spaces? Gromov-Schoen initiated this topic from 1992.

§1. $CAT(0)$ spaces

$CAT(0)$ = Alexandrov spaces with (globally) non-positive curvature.



Triangle in X



Model triangle in $\mathbb{R}^2$

Definition

$$CAT(0) \quad \textit{iff} \quad d(x,r) \leqslant d(\bar{x},\bar{r}) \quad (\textit{iff} \quad \left(d^2(r,x_t)\right)'' \geq 2 \quad \textit{weakly}).$$

Examples:

- Simply connected Riemannian manifolds with non-positive sectional curvature;
- Hilbert space;
- Trees with canonical metric.

Proposition

If (Y, d_Y) be a $CAT(0)$ space, then:

- *$L^2(\Omega, Y)$ is of $CAT(0)$.*
- *$E[u]$ is convex and lower semi-continuous on $L^2(\Omega, Y)$.*

proof:
(1). Let $v, u_0, u_1 \in L^2(\Omega, Y)$. For any $x \in \Omega$, if $u_t(x)$ be the geodesic of $u_0(x)$ to $u_1(x)$ in Y, then u_t is a geodesic from u_0 to u_1 in $L^2(\Omega, Y)$.
(2). d_{L^2} satisfies the $CAT(0)$ condition.
(3). $e_{2,r}[u_{1/2}](x) \leq \frac{1}{2} e_{2,r}[u_0](x) + \frac{1}{2} e_{2,r}[u_1](x)$, $\forall x \in \Omega$.
(4) By using the heat kernel representative:

$$E[u] = \lim_{t \to 0} \int_{\Omega \times \Omega} \frac{1}{2t} d_Y^2(u(x), u(y)) p_t(x, y) d\mu(x) \mu(y),$$

and Fatou's lemma, we have $E[u] \leq \liminf E[u_j]$.

§2. Gradient flows on $CAT(0)$ spaces

- Recall in $\mathbb{R}^n$ that the heat flow $\partial_t u = \Delta u$ is a gradient flow of energy $\int_{\mathbb{R}^n} |\nabla u|^2 dx$ in $L^2(\mathbb{R}^n)$.

- In 1998, Mayer and Jost extended to the setting of convex functional in $CAT(0)$ spaces.

Notations

- (L, D): complete $\mathrm{CAT}(0)$ space.
- $G : L \to [0, +\infty]$: *convex* and *l.s.c.*
- Domain: $D(G) := \{ u \in L : G(u) < +\infty \}$.

Goal: construct the gradient flow of G starting at $u_0 \in D(G)$.

Definition (Jost, Mayer)

For $h > 0$ and $u_0 \in \overline{D(G)}$, the resolvent J_h of G is the unique minimizer of

$$u \mapsto E(u; u_0, h) := G(u) + \frac{1}{2h} D^2(u, u_0), \qquad u \in L.$$

Well-posedness:

- $u \mapsto G(u) + \frac{1}{2h} D^2(u, v)$ is *strictly convex*: convexity of G + strict convexity of D^2 in $\mathrm{CAT}(0)$.
- The l.s.c and strict convexity imply the infimum can be achieved. For this, we need to show any minimizing sequence $\{u_j\}$ is a Cauchy sequence. Consider such sequence such that

$$F(u_j) \leq \inf F(u) + 2^{-j}, \quad \text{where} \quad F(u) := E(u; u_0, h).$$

  we have

$$\frac{1}{2} F(u_j) + \frac{1}{2} F(u_k) \geq F(\frac{u_j + u_k}{2}) + \frac{1}{4} D^2(u_j, u_k).$$

- Strict convexity $\Rightarrow$ uniqueness of the minimizer.

Proposition

1. **(Resolvent identity)** For $\mu > \lambda > 0$,

$$J_\mu(v) = J_\lambda\left(\frac{\mu-\lambda}{\mu}\, J_\mu(v) + \frac{\lambda}{\mu}\, v\right),$$

   where the inner argument is the point at fraction λ/μ along the geodesic from $J_\mu(v)$ to v.
2. **(Contraction)** For all $v_0, v_1 \in L$,

$$D\big(J_h(v_0),\, J_h(v_1)\big) \leqslant D(v_0, v_1).$$

Proof Sketch (1): Resolvent Identity: Set $w := \dfrac{\mu - \lambda}{\mu} J_\mu(v) + \dfrac{\lambda}{\mu} v$.

Minimality of $J_\lambda(w)$ and $J_\mu(v)$:

$$G(J_\lambda(w)) + \tfrac{1}{2\lambda} D^2(J_\lambda(w), w) \leqslant G(J_\mu(v)) + \tfrac{1}{2\lambda} D^2(J_\mu(v), w),$$
$$G(J_\mu(v)) + \tfrac{1}{2\mu} D^2(J_\mu(v), v) \leqslant G(J_\lambda(w)) + \tfrac{1}{2\mu} D^2(J_\lambda(w), v).$$

Adding and using the definition of w:

$$D^2(J_\lambda(w), v) \geqslant \frac{\mu}{\lambda} D^2(J_\lambda(w), w) + \frac{\mu}{\mu - \lambda} D^2(w, v).$$

The triangle inequality in $\mathrm{CAT}(0)$ gives the reverse inequality, so *equality* holds. Then equality in the first minimality inequality plus *uniqueness* of the minimizer of $E(\cdot; w, \lambda)$ yield

$$J_\lambda(w) = J_\mu(v).$$

Proof Sketch (2): Contraction

Let $u_i := J_h(v_i)$, $i = 0, 1$, and $u_t := (1-t)u_0 + tu_1$.

Convexity of G + $\mathrm{CAT}(0)$ inequality:

$$E(u_t; v_0, h) + E(u_{1-t}; v_1, h) \leq E(u_0; v_0, h) + E(u_1; v_1, h) - \frac{t}{2h}\Big(D^2(u_0, u_1) - D^2(v_0, v_1)\Big).$$

Since u_i minimizes $E(\cdot; v_i, h)$, the left side dominates $E(u_0; v_0, h) + E(u_1; v_1, h)$; hence

$$D^2(u_0, u_1) \leq D^2(v_0, v_1).$$

Gradient Flows:

Theorem (Mayer)

Let $G : L \to [0, +\infty]$ be convex and l.s.c. on a $\mathrm{CAT}(0)$ space (L, D). For any $t > 0$ and $u_0 \in D(G)$, the limit

$$u(t) := \lim_{n\to\infty} J_{t/n}^n(u_0), \qquad J_{t/n}^n := \underbrace{J_{t/n} \circ \cdots \circ J_{t/n}}_{n \text{ times}},$$

exists, $u(t) \to u_0$ as $t \to 0$, and $G(u(t)) \leqslant G(u_0)$.
Moreover, if G is strict convex, then $\lim t_j \to +\infty u_{t_j}$ converges to a min $G(u)$.

$u(t)$ is the gradient flow of G starting at u_0.

Proof Sketch (1): Key Recurrence:
For $n \geqslant m$, $\mu \geqslant \lambda > 0$, set $\alpha = \lambda/\mu$, $\beta = (\mu - \lambda)/\mu$, and

$$a_{m,n} := D^2\big(J_\mu^m(u_0),\, J_\lambda^n(u_0)\big).$$

Combining the *resolvent identity*, *contraction*, and $\mathrm{CAT}(0)$:

$$\begin{aligned} a_{m,n} &= D^2\big(J_\lambda\big(\beta J_\mu^m(u_0) + \alpha J_\mu^{m-1}(u_0)\big),\, J_\lambda^n(u_0)\big) \\ &\leqslant D^2\big(\beta J_\mu^m(u_0) + \alpha J_\mu^{m-1}(u_0),\, J_\lambda^{n-1}(u_0)\big) \\ &\leqslant \beta\, D^2\big(J_\mu^m(u_0), J_\lambda^{n-1}(u_0)\big) + \alpha\, D^2\big(J_\mu^{m-1}(u_0), J_\lambda^{n-1}(u_0)\big) \\ &= \alpha\, a_{m-1,n-1} + \beta\, a_{m,n-1}. \end{aligned}$$

Proof Sketch (2): Bounds from Minimality:
Energy bound. For $u_j := J_\mu(u_{j-1})$, minimality of u_j gives

$$D^2(u_0, u_j) \leqslant 2j\mu\big(G(u_0) - G(u_j)\big) \leqslant 2j\mu\, G(u_0),$$

since $G \geqslant 0$. Hence

$$a_{m-j,0} \leqslant 2G(u_0)\,\mu\,(m-j), \qquad a_{0,n-j} \leqslant 2G(u_0)\,\lambda\,(n-j).$$

Unrolling the recurrence:

$$a_{m,n} \leqslant 2G(u_0)\left\{\sum_{j=0}^{m-1} \alpha^j \beta^{n-j} \binom{n}{j}(m-j)\mu + \sum_{j=m}^{n} \alpha^m \beta^{j-m} \binom{j-1}{m-1}(n-j)\lambda\right\}.$$

Proof Sketch (3): Cauchy Property and Limits:
Cauchy estimate. With $\mu = t/m$, $\lambda = t/n$:

$$a_{m,n} \leqslant 4G(u_0)\, t \left(\frac{1}{m} - \frac{1}{n}\right)^{1/2} \xrightarrow[m,n\to\infty]{} 0.$$

So $\{J^n_{t/n}(u_0)\}$ is Cauchy; its limit is $u(t)$.

Initial datum. From $a_{m,0} \leqslant 2G(u_0)\mu m = 2G(u_0)t$:

$$D^2(u(t), u_0) \leqslant 2G(u_0)\, t, \qquad \text{hence } u(t) \to u_0 \text{ as } t \to 0.$$

Energy decay. If u_λ lies on the geodesic from $J_h(u_0)$ to u_1, convexity of $u \mapsto G(u) + \frac{1}{2h}D^2(u_0, u)$ gives

$$D^2(J_h(u_0), u_1) \leqslant D^2(u_0, u_1) - 2h\big(G(J_h(u_0)) - G(u_1)\big).$$

Taking $u_0 = u_1$: $G(J_h(u_0)) \leqslant G(u_0)$, hence $G(u(t)) \leqslant G(u_0)$.

Fundamental Properties:

Theorem

Let $u_0 \in D(G)$ and $u(t)$ be the gradient flow of G starting at u_0. Set $F_t(u_0) := u(t)$. Then:

- **(Contraction)** $D(F_t(u), F_t(v)) \leqslant D(u, v)$; in particular, the flow is unique.
- **(Semigroup)** $F_{t+s}(u_0) = F_t \circ F_s(u_0)$ for all $s, t > 0$.
- **(Regularity)** $u(t) \in \mathrm{Lip}_{\mathrm{loc}}\big((0, +\infty), L\big) \cap C^{1/2}\big([0, +\infty), L\big)$.
- **(EVI)** For every $u_1 \in L$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D^2(u(t), u_1) + G(u(t)) \leqslant G(u_1) \quad \text{in } \mathscr{D}'(0, +\infty).$$

Proof Sketch: Contraction and Semigroup:

(1) Contraction. Immediate from the contraction of each J_h.

(2) Semigroup.

- *Step 1.* Show $F_{2t}(u_0) = F_t(F_t(u_0))$ by proving

$$\lim_{n\to\infty} J^n_{t/n}\big(J^n_{t/n}(u_0)\big) = \lim_{n\to\infty} J^{2n}_{2t/(2n)}(u_0).$$

- *Step 2.* Iterate: $F_{kt} = (F_t)^k$ for $k = 2, 3, \ldots$.
- *Step 3.* Extend $F_{s+t} = F_s \circ F_t$ from $s, t \in \mathbb{Q}^+$ to all of $[0, +\infty)$ by density and the contraction property.

Proof Sketch: Regularity Hölder continuity. From $D(u(t), u_0) \leqslant \sqrt{2G(u_0)}\, t^{1/2}$ and the semigroup property,

$$D(u(t+s), u(s)) \leqslant \sqrt{2G(u(s))}\, t^{1/2} \leqslant \sqrt{2G(u_0)}\, t^{1/2},$$

so $u \in C^{1/2}([0, +\infty), L)$.

Local Lipschitz. Iterating $D^2(J^n_{s/n}(u_0), u_1) \leqslant D^2(u_0, u_1) - 2s\big(G(J^n_{s/n}(u_0)) - G(u_1)\big)$ and passing to the limit, using the semigroup,

$$D^2(u(t+s), u_1) \leqslant D^2(u(t), u_1) - 2s\big(G(u(t+s)) - G(u_1)\big).$$

Taking $u_1 = u(t)$ and dividing by s^2:

$$\frac{D^2(u(t+s), u(t))}{s^2} \leqslant 2\,\frac{G(u(t)) - G(u(t+s))}{s}.$$

Monotonicity $t \mapsto G(u(t)) \Rightarrow$ a.e. differentiability $\Rightarrow u \in \mathrm{Lip}_{\mathrm{loc}}((0, +\infty), L)$.

Proof Sketch: EVI

Starting from

$$D^2(u(t+s),u_1) \leqslant D^2(u(t),u_1) - 2s\big(G(u(t+s)) - G(u_1)\big), \qquad \forall\, u_1 \in L,$$

divide by $2s > 0$:

$$\frac{D^2(u(t+s),u_1) - D^2(u(t),u_1)}{2s} + G(u(t+s)) \leqslant G(u_1).$$

Since $t \mapsto D^2(u(t),u_1) \in \mathrm{Lip}_{\mathrm{loc}}((0,+\infty))$ and $s \mapsto G(u(t+s))$ is monotone, letting $s \to 0^+$ in $\mathscr{D}'$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}D^2(u(t),u_1) + G(u(t)) \leqslant G(u_1) \quad \text{in } \mathscr{D}'(0,+\infty).$$

This is the *Evolution Variational Inequality* (EVI_0), a powerful metric characterization of gradient flows.

§3. Application: Harmonic Map Heat Flows between metric spaces

The Energy Setting

- Target (Y, d_Y): a $\mathrm{CAT}(0)$ space.
- $L^2(\Omega, Y)$ is itself a $\mathrm{CAT}(0)$ space w.r.t. $D := d_{L^2}$.
- $W^{1,2}(\Omega, Y)$: Korevaar–Schoen Sobolev space of maps.

Energy class with boundary data $\psi \in W^{1,2}(\Omega, Y)$

$$W^{1,2}_{\psi}(\Omega, Y) := \Big\{ u \in W^{1,2}(\Omega, Y) : d_Y(u(x), \psi(x)) \in W^{1,2}_0(\Omega),\ E[u] \le E[\psi] \Big\}.$$

- Convex subset of $L^2(\Omega, Y)$.
- Closed w.r.t. D (l.s.c. of E).
- Hence $(W^{1,2}_{\psi}(\Omega, Y), D)$ is itself a $\mathrm{CAT}(0)$ space.
- If $\Omega = M$ compact without boundary: take $KS^{1,2}_{\psi} = W^{1,2}$.

Harmonic Map Heat Flow as a Gradient Flow

- Functional: $G(u) := \dfrac{E[u]}{2}$ (half Dirichlet energy).
- G is convex and l.s.c. on $(L^2(\Omega, Y), D)$.
- Apply the gradient-flow theorem on $(W^{1,2}_{\psi}(\Omega, Y), D)$.

Theorem (Mayer '98; Jost '98)

For every $u_0 \in W^{1,2}_{\psi}(\Omega, Y)$ there exists a unique gradient flow

$$u(\cdot, t) : \ \Omega \times (0, +\infty) \to Y$$

starting at u_0 with boundary data ψ, called a *weak solution* of the *harmonic map heat flow*. It enjoys:

- long-time existence;
- regularity

$$t \mapsto u(x, t) \in \mathrm{Lip}_{\mathrm{loc}}\big((0, +\infty), L^2(\Omega, Y)\big) \cap C^{1/2}\big([0, +\infty), L^2(\Omega, Y)\big).$$

§4. Regularity and Bochner inequality

Theorem (Han-Zhang-Zhu, 26+)

Let $\Omega \subset X$ be a bounded domain of an RCD(K, N) space, and (Y, d_Y) be a CAT(0) space, let $u(x,t)$ be a weak solution to the harmonic map heat flow, with initial-boundary data $u_0, \phi \in W^{1,2}(\Omega, Y)$. Suppose that the image of u_0 is bounded in Y. Then

(1) *$u(x,t)$ is locally Lipschitz continuous in $\Omega \times (0, +\infty)$,*

(2) *Eells-Sampson-type Bochner inquality holds*

$$(\Delta - \partial_t)(\mathrm{lip}_x u)^2 \geqslant 2|\nabla \mathrm{lip}_x u|^2 + 2K(\mathrm{lip}_x u)^2$$

on $\Omega \times (0, +\infty)$ in the sense of distributions, where the pointwise spatial Lipschitz constant

$\mathrm{lip}_x u(x,t) := \limsup_{y \to x,\ y \neq x} \frac{d_Y\left(u(x,t), u(y,t)\right)}{d(y,x)}$.