Sobolev Maps between Metric Spaces

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Outline

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2. Other equivalent definitions
3. Heat-Kernel Approximation

§1. Dirichlet energy of maps

Let (X, d, μ) be an $\mathrm{RCD}(K, N)$ space with $N \in (1, \infty)$, $K \in \mathbb{R}$.

- $\Omega \subset X$ is a bounded open domain.
- (Y, d_Y) is a complete metric space.
- $u \in L^2(\Omega, Y)$ is a (measurable) map.

When is u a "Sobolev map" into Y, and what is its energy?

Definition

For $p \in [1, \infty]$ and $Q \in Y$, a Borel map $u : \Omega \to Y$ belongs to $L^p(\Omega, Y, Q)$ if:

1. u has *separable range*;
2. $d_Y\big(u(\cdot), Q\big) \in L^p(\Omega, \mu)$.

If $\mu(\Omega) < +\infty$, then $L^p(\Omega, Y, Q)$ is independent of Q.

From now on we assume $\mu(\Omega) < +\infty$ and write

$$L^p(\Omega, Y) := L^p(\Omega, Y, Q) \quad \text{for some (hence all) } Q \in Y.$$

It is equipped with

$$d^p_{L^p}(u, v) := \int_\Omega d^p_Y\big(u(x), v(x)\big) d\mu(x).$$

Remark: (1) Here the range is separable is a technical condition.

We need an intrinsic definition for the energy.

- Y cannot in general be embedded in a Euclidean space.
- Classical Sobolev spaces use linear structure (derivatives, distributions).

Several notions in the literature.

Korevaar–Schoen, Jost, Reshetnyak, Hajłasz–Koskela, Ohta, Gigli–Tyulenev.

Idea: When $u \in C^1(\mathbb{R}^n)$, then

$$u(x) - u(0) = \nabla u(0) \cdot x + o(|x|),$$

$$\frac{1}{|B_r(0)|} \int_{B_r(0)} |u(x) - u(0)|^2 = \frac{1}{|B_r(0)|} \int_{B_r(0)} |\nabla u(0) \cdot x|^2 + o(r^2) = \frac{r^2}{n+2} + o(r^2),$$

where we used

$$\frac{1}{|B_r(0)|} \int_{B_r(0)} (e \cdot x)^2 = \frac{r^2}{n+2}.$$

Remark: If you average over $\partial B_r(0)$, the constant is $\frac{1}{n}$.

Approximating energy density: For $u \in L^2(\Omega, Y)$ and $r > 0$, define the *approximating energy density*

$$e_{2,r}[u,\Omega](x) := \begin{cases} \dfrac{n+2}{\mu(B_r(x))} \displaystyle\int_{B_r(x)} \frac{d_Y^2\big(u(x),u(y)\big)}{r^2}\,\mathrm{d}\mu(y), & B_r(x) \subset \Omega, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where n is the *essential dimension* of (X, d, μ), the constant $n \in [1, N]$ such that

$$\mu(X \setminus A_n) = 0,$$

and A_n is all points with tangent cone is $\mathbb{R}^n$.

Remark: $e_{p,r}[u,\Omega]$ is similar.

To consider the convergence as $r \to 0$, we consider the functional:
For $\phi \in C_0(\Omega)$, define the *approximating energy at scale* r:

$$E_{2,r}[u, \Omega](\phi) := \int_\Omega \phi(x)\, e_{2,r}[u, \Omega](x)\, \mathrm{d}\mu(x).$$

The *total energy* is

$$E_2[u] = E_2[u, \Omega] := \sup_{\substack{\phi \in C_0(\Omega) \\ 0 \le \phi \le 1}} \left(\limsup_{r \to 0} E_{2,r}[u, \Omega](\phi) \right).$$

Definition (Sobolev maps $W^{1,2}(\Omega, Y)$)

A map $u \in L^2(\Omega, Y)$ belongs to $W^{1,2}(\Omega, Y)$ if its total energy is finite:

$$E_2[u] < \infty.$$

Theorem

Let $u \in W^{1,2}(\Omega, Y)$. Then:

1. *For every $\phi \in C_0(\Omega)$, the limit*

$$E_2[u](\phi) := \lim_{r \to 0} E_{2,r}[u, \Omega](\phi)$$

exists (independently the choice of $r \to 0$). It is called the energy measure of u. Moreover, $E_2[u] \ll \mu$: there is $e_u \in L^1(\Omega)$, the energy density, such that

$$E_2[u](\phi) = \int_\Omega \phi \, e_u \, \mathrm{d}\mu, \qquad E_2[u] = \int_\Omega e_u \, \mathrm{d}\mu.$$

2. *As $r \to 0$,*

$$e_{2,r}[u, \Omega] \to e_u \quad \mu\text{-a.e. and in } L^1. \tag{2}$$

§2. Other equivalent definitions

Jost's approximating functionals:
For $x \in \Omega$ and $\epsilon > 0$, set $\mu_x^\epsilon := \mu|_{B_\epsilon(x)}$ and

$$J_\epsilon^p(u) := \inf_\Omega \frac{\displaystyle\int_\Omega d_Y^p\big(u(x), u(y)\big)\,\mathrm{d}\mu_x^\epsilon(y)}{\displaystyle\int_\Omega d^p(x,y)\,\mathrm{d}\mu_x^\epsilon(y)}\,\mathrm{d}\mu(x).$$

The denominator is a normalization depending only on (X, d, μ), not on u.

Define J^p as the Γ-limit of J^p_ϵ as $\epsilon \to 0$ (or along a subsequence $\epsilon_j \to 0$) in the $L^p(\Omega, Y)$-topology:

- *liminf inequality:* for every $u_\epsilon \to u$ in $L^p(\Omega, Y)$,

$$\liminf_{\epsilon \to 0} J^p_\epsilon(u_\epsilon) \geq J^p(u);$$

- *recovery sequence:* there exists $u_\epsilon \to u$ in $L^p(\Omega, Y)$ with

$$\lim_{\epsilon \to 0} J^p_\epsilon(u_\epsilon) = J^p(u).$$

Definition: For $1 < p < \infty$ and $u \in L^p(\Omega, Y)$:

$$u \in J^{1,p}(\Omega, Y) \iff J^p(u) < \infty.$$

For $1 < p < \infty$ and $u \in L^p(\Omega, Y)$:

Definition (Reshetnyak)

$u \in R^p(\Omega, Y)$ if:

1. for every Lipschitz $\psi : Y \to \mathbb{R}$, the composition $\psi \circ u \in W^{1,p}(\Omega)$;
2. there exists $w \in L^p(\Omega)$ (independent of ψ) such that, for every Lipschitz $\psi : Y \to \mathbb{R}$,

$$|\nabla(\psi \circ u)|(x) \leq Lip(\psi) \cdot w(x), \quad \mu\text{-a.e. } x \in \Omega. \tag{3}$$

Remark

When Ω is a bounded domain of a smooth manifold, Chiron proved

$$W^{1,p}(\Omega, Y) = J^{1,p}(\Omega, Y) = R^{1,p}(\Omega, Y).$$

Natural question. Does this equality continue to hold when Ω is a bounded domain of a *singular* metric space?

Lemma (Contraction property)

Let $1 < p < \infty$ and $u \in W^{1,p}(\Omega, Y)$. Let (Z, d_Z) be another complete metric space and $\psi : Y \to Z$ Lipschitz. Then $\psi \circ u \in W^{1,p}(\Omega, Z)$ and

$$E_p^{\psi \circ u}(\phi) \leq Lip^p(\psi)\, E_p^u(\phi), \qquad \forall\, 0 \leq \phi \in C_0(\Omega), \tag{4}$$

where

$$Lip(\psi) := \sup_{\substack{y, y' \in Y \\ y \neq y'}} \frac{d_Z\big(\psi(y), \psi(y')\big)}{d_Y(y, y')}.$$

The follows from the definition of $W^{1,p}(\Omega, Y)$ and the pointwise inequality

$$d_Z\big(\psi \circ u(x), \psi \circ u(y)\big) \leq Lip(\psi) \cdot d_Y\big(u(x), u(y)\big), \quad \forall x, y \in \Omega.$$

- For every $Q \in Y$,
$$d_Y\big(Q, u(\cdot)\big) \in W^{1,p}(\Omega, \mathbb{R})$$
and
$$E_p^{d_Y(Q,u(\cdot))}(\phi) \leq E_p^u(\phi), \qquad \forall\, 0 \leq \phi \in C_0(\Omega). \tag{5}$$
- Taking ψ Lipschitz and bounded, we get $\psi \circ u \in W^{1,p}(\Omega, \mathbb{R})$ with controlled energy.
- In particular, the contraction lemma gives
$$W^{1,p}(\Omega, Y) \subset R^{1,p}(\Omega, Y)$$
whenever the scalar Sobolev theory is available.

§3. Heat-Kernel method

Linear case ($Y = \mathbb{R}$). By the Beurling–Deny formula,

$$\int_X |\nabla \phi|^2 \, \mathrm{d}\mu = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X |\phi(x) - \phi(y)|^2 p_t(x, y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y), \tag{6}$$

for $\phi \in W^{1,2}(X)$ **Question.** Does the same hold for maps into a metric space Y?

Heat-kernel approximation for metric targets

Theorem

Let $u \in W^{1,2}(\Omega, Y) \cap C(\Omega, Y)$. Then, for μ-a.e. $x \in \Omega$,

$$\lim_{t \to 0} \frac{1}{2t} \int_X d_Y^2\big(u(x), u(y)\big)\, p_t(x, y)\, \mathrm{d}\mu(y) = e_u(x), \tag{7}$$

where $y \mapsto d_Y^2(u(x), u(y))$ is understood as the zero extension of $\Omega \ni y \mapsto d_Y^2(u(x), u(y))$.

So the energy density can be recovered by *heat-kernel averaging* of distance-squared, exactly as in the linear case.

Proof idea for the heat-kernel approximation

Consider the convergence of normalized measures

$$\frac{\mu}{\mu(B_r(x))} \to c_d\,\mathcal{L}^d \qquad (r \to 0)$$

by the convergence of heat kernels on the rescaled spaces

$$p_1^{X_j} \to p_1^{\mathbb{R}^d} \qquad (j \to \infty),$$

where

$$X_j = \left(X,\ r_j^{-1} d,\ \frac{\mu}{\mu(B_{r_j}(x))},\ x\right)$$

and $p_t^{X_j}$ is the heat kernel on X_j.

- The heat kernel is stable under pmGH convergence (for RCD spaces).