Bochner inequality on Alexandrov Spaces and its applications

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Contents

1. Sobolev spaces and weak Laplacian on Alexandrov spaces
2. Bochner inequality on Alexandrov spaces
3. Li-Yau's estimate and heat kernel

§1. Sobolev spaces and weak Laplacian

Sobolev spaces

- (X, d): n-dim. Alexandrov space, curvature $\geqslant k$, $k \in \mathbb{R}$. Recall that

$$X = \mathcal{R}_\delta \cup \mathcal{S}_\delta,$$

  where $\mathcal{R}_\delta$ is a Lipschitz manifold with full measure, and $\mathcal{S}_\delta := \{x \in X : \mathrm{rad}(\Sigma_x) \leq \pi - \delta\}$.
- Suppose that (U_ϕ, ϕ) is a Lipschitz atlas on $\mathcal{R}_\delta$, it admits a L^∞-Riemannian metric $g = (g_{ij})$.
- For $f \in \mathrm{Lip}_{\mathrm{loc}}(X)$, the gradient $\nabla_i f$ is well-defined in $L^\infty_{\mathrm{loc}}(\mathcal{R}_\delta)$, and also in $L^\infty_{\mathrm{loc}}(X)$.

Definition

Let $\Omega \subset X$. For $1 \leqslant p < \infty$ and $f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$:

$$\|f\|_{W^{1,p}(\Omega)} := \left(\|f\|^p_{L^p(\Omega)} + \||\nabla f|\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Sobolev space $W^{1,p}(\Omega)$ is the closure of $\{f \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) : \|f\|_{W^{1,p}} < \infty\}$.

Distributional Laplacian

Definition

For $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, its distributional Laplacian $\mathbf{\Delta} u$ is defined as a functional on $\mathrm{Lip}_0(\Omega)$:

$$\mathbf{\Delta} u(\phi) = -\int_\Omega \langle \nabla u, \nabla \phi \rangle \, \mathrm{d}\mu, \quad \forall \phi \in Lip_0(\Omega),$$

where $\mu := \mathscr{H}^n$ is the n-dimensional Hausdorff measure on X.

Definition (Sub/supersolutions)

Let ν be a measure on Ω,

- $\mathbf{\Delta} u \geqslant \nu$ (called a subsolution) means $\mathbf{\Delta} u(\phi) \geqslant \int_\Omega \phi \, d\nu$ for all $\phi \geqslant 0$ in Lip_0. In this case, $\mathbf{\Delta} u$ is a measure.
- $\mathbf{\Delta} u \leqslant \nu$ (called a supersolution) means $\mathbf{\Delta}(-u) \geqslant -\nu$.
- $\mathbf{\Delta} u = \nu$ means both $\mathbf{\Delta} u \geqslant \nu$ and $\mathbf{\Delta} u \leqslant \nu$.
- $\mathbf{\Delta} u = 0$ is called to be harmonic.

Remarks:

- (Comparison property of sub/super-harmonic): If $u \in W^{1,2}_{\mathrm{loc}} \cap C(\Omega)$, then $\mathbf{\Delta} u \geqslant 0$ if and only if for any subdomain $B \Subset \Omega$ it holds $u_B \geqslant u$ on B, where u_B is the harmonic function on B with boundary data u.

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- Chain rule and Leibniz rule still holds

§2. Bochner inequality

Recall the Bochner formula on a smooth manifold (M^n, g) with $Ric \geqslant k$:

$$\begin{aligned}\frac{1}{2}\Delta|\nabla u|^2 &= |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u)\\ &\geqslant \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla \Delta u\rangle + k|\nabla u|^2,\end{aligned}$$

The proof is based on the Ricci identity (commute derivatives):

$$u_{ijj} = u_{jij} = u_{jji} + R_{ijji},$$

which does not make sense for Alexandrov spaces.

We will introduce a proof of (Bochner inequality)

$$\frac{1}{2}\Delta|\nabla u|^2 \geqslant \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla \Delta u\rangle + k|\nabla u|^2,$$

without the Ricci identity.

- Assuming $u \in Lip_{\mathrm{loc}}$ and $k = 0$, we consider the Hamilton–Jacobi (or Moreau-Yoshida, or Hopf-Lax) semigroup

$$u_t(x) := \min_{y\in M}\left\{u(y) + \frac{|xy|^2}{2t}\right\}, \qquad x \in M,\ t > 0.$$

Then:

- For almost any $x \in M$,

$$\lim_{t\to 0^+}\frac{u_t(x) - u(x)}{t} = -\frac{|\nabla u|^2}{2}.$$

- For almost all $x \in M$, there exists a unique $F_t(x) := \exp_x(-t\nabla u_t)$ satisfying

$$u_t(x) = u(F_t(x)) + \frac{|xF_t(x)|^2}{2t}.$$

Lemma (Key lemma)

Assume that $\Delta u = f$ for some $f \in C$, and assume $Ric(M) \geqslant K$. For any sufficiently small $t > 0$, $a > 0$,

$$a^2 \Delta u_t(x) \leq f \circ F_t(x) + \frac{n(a-1)^2}{t} + \frac{Kt}{3}(a^2 + a + 1)|\nabla u_t|^2. \qquad (*)$$

This lemma implies the Bochner inequality: Indeed, for $K = 0$, by suitably choosing a in $(*)$,

$$\frac{\Delta u_t(x) - f(x)}{t} \leq \frac{f(F_t(x)) - f(x)}{t} - \frac{1}{n} f(x) f(F_t(x)) + C|f(F_t(x)) - f(x)| + Ct$$

for some constant $C > 0$. Assume that f is Lipschitz, and notice that $F_t(x) = \exp_x(-t\nabla u_t)$, letting $t \to 0^+$,

$$\frac{f \circ F_t - f}{t} \to -\langle \nabla f, \nabla u \rangle, \qquad \frac{1}{2}\Delta |\nabla u|^2 \geq \frac{f^2}{n} + \langle \nabla u, \nabla f \rangle.$$

Proof of the key lemma: We consider only $a = 1$.

1. If the assertion fails for u_t. By the comparison property, there exists a subdomain B and small $\delta_1 > 0$ such that $\Delta u_t \geqslant f \circ F_t + \delta_1$ on B. Then

$$v + u_t$$

has a strict minimum in B, where v solves $\Delta v = -f \circ F_t - \delta_1$ on B with boundary dat $-u_t$. Say x_0.

2. From the definition of $u_t(x_0)$, the function

$$H(x, y) := v(x) + u(y) + \frac{|xy|^2}{2t}$$

has a strict minimum $(x_0, y_0 := F_t(x_0))$ in $B \times \Omega$.

3. Let $T : T_{x_0} \to T_{y_0}$ be the parellel transport, and $\varepsilon > 0$,

$$H\Big(\exp_{x_0}(\eta), \exp_{y_0}(T\eta)\Big) - H(x_0, y_0)$$

$$= v(\exp_{x_0}(\eta)) - v(x_0) + u(\exp_{y_0}(T\eta)) - u(y_0) + \frac{1}{2t}\Big(|\exp_{x_0}(\eta), \exp_{y_0}(T\eta)|^2 - |x_0, y_0|^2\Big).$$

- Integrating over $B_\varepsilon(o) \subset T_{x_0}$, we have LHS $\geqslant 0$,

$$0 \leqslant \Delta v(x_0) \cdot \varepsilon^2 + \Delta u(y_0) \cdot \varepsilon^2 + o(\varepsilon^2) = -\delta_1 \cdot \varepsilon^2 + o(\varepsilon^2).$$

which is impossible as $\varepsilon \to 0$, where we have used:

- mean value property:

$$\frac{1}{|B_\varepsilon(o)|} \int_{B_\varepsilon(o)} \left(v(\exp_{x_0}(\eta)) - v(x_0)\right) = \Delta v(x_0) \cdot \varepsilon^2 + o(\varepsilon^2),$$

- $Ric \geqslant 0$ and the second variation of arc-length implies

$$\frac{1}{|B_\varepsilon(o)|} \int_{B_\varepsilon(o)} \left(|\exp_{x_0}(\eta) \exp_{y_0}(T\eta)|^2 - |x_0 y_0|^2 \right) d\eta \leq o(\varepsilon^2).$$

Theorem (Bochner inequality)

Let X, d be an Alexandrov space with curv $\geq k$. Suppose $f \in Lip_{\mathrm{loc}}(X)$ and u satisfies $u = f \cdot \mu$. Then $|\nabla u|^2 \in W^{1,2}_{\mathrm{loc}}(M)$, and

$$-\frac{1}{2}\int_X \langle \nabla |\nabla u|^2, \nabla \phi \rangle \, \mathrm{d}\mu \geq \int_X \phi \left(\frac{f^2}{n} + \langle \nabla u, \nabla f \rangle + (n-1)k|\nabla u|^2 \right) \mathrm{d}\mu$$

for every $0 \leq \phi \in W^{1,2}_0(X) \cap L^\infty(X)$.

Remark:

- The above result assumes $f \in Lip_{\mathrm{loc}}(X)$. The general $f \in W^{1,2}$ can be approximated: take $f_j \in \mathrm{Lip}_{\mathrm{loc}}$ with $f_j \to f$ in $W^{1,2}$, solve $\boldsymbol{\Delta} u_j = f_j$ with $u_j - u \in W^{1,2}_0$; Poincaré inequality gives $u_j \to u$ in $W^{1,2}$; pass to the limit.
- This proves for any Alexandrov space (X, d) with curvature $\geq k$, the mm space $(X, d, \mathscr{H}^n)$ is an $RCD((n-1)k, n)$ spaces.

For the Alexandrov spaces, the above mean value property and second variation of arc-length hold only almost everywhere. But the minimum point (x_0, y_0) maybe lie in the zero-measured subset.

Petrunin's perturbation trick

Let $U \subset X$ be a domain and let $u \in W^{1,2}(U) \cap C(U)$ satisfy

$$\boldsymbol{\Delta} u \leqslant \lambda \quad \text{on } U, \qquad \lambda \in \mathbb{R}.$$

Assume $A \subset U$ has *full measure*. Suppose $x^* \in U$ is a *minimal point* of u. Then, for any $\delta > 0$, there exist a small domain $D_\delta \subset U$ and a δ-concave function f_δ on D_δ such that $u + f_\delta$ has a minimal point $x_\delta \in A \cap D_\delta$ and $|x^* x_\delta| < \delta$.

The *perturbation* $u + f_\delta$ moves the minimizer of u into the given "good" full-measure set A.

§3. Li-Yau's estimate and heat kernels

As an application, we consider the Li-Yau estimate to the heat equations. Recall that in the smooth setting, we have

Theorem (Li-Yau)

Let M^n be an n-dimensional Riemannian manifold with $Ric \geqslant 0$, and let B_{2R} be a geodesic ball of radius $2R$. If $u(x,t)$ is a smooth positive solution of $\Delta u = \partial_t u$ on $M \times (0,\infty)$, then the following Li-Yau estimate holds

$$|\nabla \log u|^2 - \partial_t \log u \leqslant \frac{n}{2t}.$$

The proof is based on:

- Bochner inequality and
- Maximum principle:
  Let f be a C^2-function on a smooth manifold. If f achieves one of its local maximum at x_0, then $\nabla f(x_0) = 0$ and $\Delta f(x_0) \leqslant 0$.

Sketch of the proof (assume M is compact).

- Bochner inequality $\Longrightarrow$
$$\Big(\Delta - \frac{\partial}{\partial t}\Big)F + \langle \nabla f, \nabla F\rangle \geqslant \frac{2}{nt}F^2 - \frac{F}{t},$$
  where $f = \log u$ and $F = t(|\nabla f|^2 - \partial_t f)$.

- Maximum principle $\Longrightarrow$
$$maxF \leqslant \frac{n}{2}.$$

Theorem (Maximum principle)

Let Ω be a bounded domain in (X, d). Let $f(x) \in W^{1,2}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ such that $\mathbf{\Delta} f$ is a signed Radon measure with $\mathbf{\Delta}^{\mathrm{sing}} f \geqslant 0$, where $\mathbf{\Delta}^{\mathrm{sing}} f$ is the singular part w.r.t. μ. Suppose that f achieves one of its strict maximum in Ω in the sense that: there exists a neighborhood $U \subset\subset \Omega$ such that

$$\sup_{U} f > \sup_{\Omega \setminus U} f.$$

Then, given any $w \in H^1(\Omega) \cap L^{\infty}(\Omega)$, there is a sequence $\{x_j\}_{j \in \mathbb{N}} \subset U$ such that they are the approximate continuity points of $\mathbf{\Delta}^{\mathrm{ac}} f$ and $\langle \nabla f, \nabla w \rangle$, and that

$$f(x_j) \geqslant \sup_{\Omega} f - \frac{1}{j} \qquad \text{and} \qquad \mathbf{\Delta}^{\mathrm{ac}} f(x_j) + \langle \nabla f, \nabla w \rangle (x_j) \leqslant \frac{1}{j}.$$

Li-Yau's estimates for heat kernel:

For $\forall \epsilon > 0$, there are $\exists C_1, C_2 > 0$ depending on N, K, ϵ such that

$$\frac{1}{C_1 m\left(B_{\sqrt{t}}(x)\right)} \exp\left\{-\frac{d^2(x,y)}{(4-\epsilon)t} - C_2 t\right\} \leq p_t(x,y) \leq \frac{C_1}{m\left(B_{\sqrt{t}}(x)\right)} \exp\left\{-\frac{d^2(x,y)}{(4+\epsilon)t} + C_2 t\right\},$$

$$|\nabla p_t(\cdot, y)|(x) \leq \frac{C_1}{\sqrt{t} \cdot m\left(B_{\sqrt{t}}(x)\right)} \exp\left\{-\frac{d^2(x,y)}{5t} + C_2 t\right\},$$

$$|\frac{\partial}{\partial t} p_t(\cdot, y)|(x) \leq \frac{C_1}{t \cdot m\left(B_{\sqrt{t}}(x)\right)} \exp\left\{-\frac{d^2(x,y)}{5t} + C_2 t\right\}.$$

Remark: The above Li-Yau estimates hold for *RCD* spaces.