L2: A Introduction to $\mathrm{RCD}(K, N)$ Spaces

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- **1.** $CD(K, N)$

- Alexandrov space gives a generalized sectional curvature bounded from below.
- Sturm and Lott–Villani introduced the generalized lower bounds of Ricci curvature, the *Curvature–Dimension* condition $\mathrm{CD}(K, N)$, we will discuss in this lecture.
- Gigli refined notion $\mathrm{RCD}(K, N)$, the *Riemannian Curvature–Dimension* condition, by adding the functional analytic notion of *infinitesimal Hilbertianity*.
- Why need to refine? It rules out Finsler spaces from the CD category.

Let (X, d, μ) be a *metric measure space (mm space)*: (X, d) complete and separable, μ a non-negative Borel measure. We always assume

$$0 < \mu(B_r(x)) < +\infty, \qquad \forall B_r(x) \subset X,\ 0 < r < +\infty.$$

Optimal Transport:

$(\mathscr{P}_2(X), W_2)$, L^2-Wasserstein space over (X, d)

- $\mathscr{P}_2(X)$ = all probability measures ν with finite second moment, i.e., $\int_X d^2(x_0, x)d\nu < \infty$ for some (hence for all) $x_0 \in X$.
- Distance W_2: Given $\nu_0, \nu_1 \in \mathscr{P}_2(X)$,

$$W_2(\nu_0, \nu_1) := \inf_q \left\{ \left(\int_{X\times X} d^2(x, y)dq \right)^{1/2} \right\},$$

where q is a coupling of ν_0 and ν_1.
(I.e. a probability measure on $X \times X$ s.t. $q(A \times X) = \nu_0(A)$ and $q(X \times A) = \nu_1(A)$, $\quad \forall A \subset X$.)

Remark:

- $(\mathscr{P}_2(X), W_2)$ is complete;
- $X \hookrightarrow \mathscr{P}_2(X)$ isometrically via $x \mapsto \delta_x$.
- If X is compact, then $\mathscr{P}_2(X)$ is so.
- Given $\nu_0, \nu_1 \in \mathscr{P}_2(X)$, there is an **optimal coupling** $\bar{q}$, i.e., the coupling $\bar{q}$ achieves the Wasserstein distance:

$$W_2^2(\nu_0, \nu_1) = \inf_q \left\{ \int_{X\times X} d^2(x,y) dq \right\} = \int_{X\times X} d^2(x,y) d\bar{q}.$$

Definition (Curvature-Dimension condition)

Let $K \in \mathbb{R}$ and $N \geq 1$. (X, d, μ) is said to be $CD(K, N)$ (generalized notion of "$Ric \geqslant K$ and $\dim \leqslant N$") if it satisfies:

- *for any $\nu_0, \nu_1 \in \mathscr{P}_2(X)$, $\nu_0, \nu_1 \ll \mu$, there is a shortest curve $\nu_{0\leq t\leq 1} \subset \mathscr{P}_2(X)$ from ν_0 to ν_1 such that $\nu_t \ll \mu$ and*

$$\frac{d^2}{dt^2} U(\nu_t) \leq -\frac{K}{N} U(\nu_t) \cdot W_2^2(\nu_0, \nu_1)$$

in the sense of distributions, where

$$U(\nu_t = \rho_t \cdot \mu) := \exp\Big(-\frac{1}{N} Ent(\rho_t)\Big), \qquad Ent(\rho_t) := \int_X \rho_t \log \rho_t d\mu.$$

Examples

- Weighted Riemannian manifolds $(M, g, e^{-\phi} \cdot vol_g)$,
  $$CD(K, N) \iff Ric_g + Hess(\phi) - \frac{d\phi \otimes d\phi}{N} \geqslant K;$$
- Ricci limit spaces = the GH-limits of manifolds with $Ric \geq K$, $dim \leq N$; (Cheeger-Colding)
- Alexandrov spaces: (Petrunin, for $K = 0$; Z-Zhu, for $K \in \mathbb{R}$).
- $(\mathbb{R}^n, \|\cdot\|_p)$. Finler spaces with $Ric \geq K$.

- **1.** $RCD(K, N)$

Let (X, d, μ) be a mm space. $\Omega \subset X$.

- $C_0(\Omega)$, $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$: Continuous, locally Lipschitz functions on Ω.
- $L^p(\Omega) := L^p(\Omega, \mu)$.

For $f \in \mathrm{Lip}_{\mathrm{loc}}(X)$ and x not isolated, the *pointwise Lipschitz constant* (or *slope*) is

$$\mathrm{Lip} f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)} = \lim_{r \to 0^+} \sup_{y \in B_r(x)} \frac{|f(y) - f(x)|}{r}.$$

If x is isolated, $\mathrm{Lip} f(x) = 0$.

Definition

Let $f \in L^2(X)$. A function $G \in L^2(X)$ is a relaxed gradient of f if there exists a sequence of Lipschitz functions $\{f_j\} \subset L^2(X)$ such that

1. $f_j \to f$ *in* $L^2(X)$ *and* $\mathrm{Lip} f_j \rightharpoonup \tilde{G}$ *weakly in* $L^2(X)$;
2. $\tilde{G} \leq G$ μ*-a.e. in* X.

G is minimal if $\|G\|_{L^2}$ is minimal among all relaxed gradients of f.

- The collection of all relaxed gradients of f is a closed convex subset of $L^2(X)$. Hence the minimal relaxed gradient is well-defined.
- $\min\{G_1, G_2\}$ is a relaxed gradient for any two relaxed gradients G_1, G_2. Thus the minimal relaxed gradient is *unique* in $L^2(X)$.

Notation: the minimal relaxed gradient of f, is denoted by $|\nabla f|$.

Definition

The Cheeger energy $\mathrm{Ch} : L^2(X) \to [0, +\infty]$ *is*

$$\mathrm{Ch}(f) := \frac{1}{2} \int_X |\nabla f|^2 \, d\mu.$$

- Ch is *convex* on $L^2(X)$.
- Its domain

$$D(\mathrm{Ch}) := \{f \in L^2(X) : \ \mathrm{Ch}(f) < +\infty\}$$

  is *dense* in $L^2(X)$.
- The *Sobolev space* $W^{1,2}(X)$ is $D(\mathrm{Ch})$ with norm

$$\|f\|^2_{W^{1,2}(X)} := \|f\|^2_{L^2} + 2\mathrm{Ch}(f).$$

Definition (Gigli)

A metric measure space is infinitesimal Hilbertian if the Cheeger energy Ch *satisfies*

$$\mathrm{Ch}(f+g)+\mathrm{Ch}(f-g)=2\big(\mathrm{Ch}(f)+\mathrm{Ch}(g)\big), \qquad \forall f,g\in D(\mathrm{Ch}).$$

Equivalently, $W^{1,2}(X)$ *is a Hilbert space.*

- The parallelogram identity holds a.e.:

$$|\nabla(f+g)|^2+|\nabla(f-g)|^2=2\big(|\nabla f|^2+|\nabla g|^2\big) \quad \mu\text{-a.e.}, \qquad \forall f,g\in W^{1,2}(X).$$

- The limit

$$\langle\nabla f,\nabla g\rangle := \lim_{\epsilon\to 0^+}\frac{|\nabla(f+\epsilon g)|^2-|\nabla f|^2}{2\epsilon}$$

exists in $L^1(X)$. It encodes an *inner product structure* on the space of gradients, allowing one to form $\langle\nabla f,\nabla g\rangle$ and thus to do Riemannian-style calculus on a nonsmooth space.

Definition

A mm space (X, d, μ) is called $RCD(K, N)$ if it satisfies

$$CD(K, N) \quad + \quad \text{infinitesimal Hilbertian.}$$

On an $RCD(K, N)$ space (X, d, μ), there is a canonical Dirichlet form

$$\begin{aligned} \mathscr{E}(f, g) &:= \int_X \langle \nabla f, \nabla g \rangle \, d\mu \\ &= \frac{\mathrm{Ch}(f+g) - \mathrm{Ch}(f-g)}{2}, \qquad \forall f, g \in W^{1,2}(X). \end{aligned}$$

- $\mathscr{E}$ is a symmetric, closed bilinear form on $W^{1,2}(X)$.

Definition

We say $f \in D(\Delta)$ if there exists $h \in L^2(X)$ such that

$$\mathscr{E}(f,g) = -\int_X hg\, d\mu, \qquad \forall g \in W^{1,2}(X).$$

In this case h is unique in $L^2(X)$, and we define

$$\Delta f := h.$$

- Δ is the infinitesimal generator of $\mathscr{E}$.
- It is the metric measure analogue of the Laplace–Beltrami operator.

Some consequences–I

- (X, d) is a geodesic space, i.e., for any pair of points $x, y \in X$, there is a curve γ connecting them s.t. $L(\gamma) = d(x, y)$.
- Stability is a key reason why the theory is powerful: one can obtain RCD spaces as limits of smooth or singular spaces.
- The weak *Bochner inequality*: for any $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X)$,

$$\int_X |\nabla f|^2 \Delta\phi \, d\mu \geq \int_X \Big(\frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2 \Big) \phi \, d\mu$$

  holds for any $\phi \in D(\Delta) \cap L^\infty(X)$ with $\Delta\phi \in L^\infty(X)$ and $\phi \geq 0$.
- For any $f \in W^{1,2}(X)$ with $|\nabla f| \leq 1$ μ-a.e., there is a representative $g \in \mathrm{Lip}(X)$ with Lipschitz constant 1 such that $g = f$ μ-a.e.

Some consequences–II

- Laplacian comparison (Gigli): For $RCD(0,N)$ spaces, $\rho(x) := d(p,x)$,

$$-\int_X \langle \nabla\rho, \nabla\phi\rangle d\mu \le \int_X \frac{(N-1)\phi}{\rho} d\mu, \qquad \forall \phi \in Lip_0(X \setminus \{p\}),\ \phi \ge 0.$$

  For general $K \in \mathbb{R}$, the corresponding property holds.

- Bishop-Gromov comparison theorem (Sturm, Lott-Villani): For $RCD(0,N)$ spaces,

$$\frac{\mu(B_R(x))}{t^N}$$

  is non-increasing in R. For general $K \in \mathbb{R}$, the corresponding property holds.

  (We will prove from Laplacian comparison to Bishop-Gromov comparison in this afternoon)

- **3. Non-collapsing** $RCD(K, N)$

Let (X, d, μ) be an $RCD(K, N)$ space.

- *The rectifiable structure*: There is an $n \in [1, N]$ and $A_n \subset X$ such that

$$m(X \backslash A_n) = 0 \quad \text{and} \quad A_n \ \text{ is an } n\text{−dim rectifiable.}$$

  Here n is called the essenstial dimension.

- *Non-collapsing*: if $n = N$ and $\mu := \mathscr{H}^n$. In this case, the singular part of X has a stratification:

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{N-1} = \mathcal{S},$$

  where for any $0 \leq k \leq N-1$,

$$\mathcal{S}^k := \left\{ x \in X \middle| \text{ no tangent cone at } x \text{ splits off } \mathbb{R}^{k+1} \right\}.$$

  It holds

$$\dim_{\mathscr{H}}(\mathcal{S}^k) \leq k, \quad \forall\ k = 1, 2, \cdots, N-1.$$