L1: Alexandrov Geometry

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- **1. Alexandrov curvature**

Recall that, on a Riemannian manifold with the Levi-Civita covariant derivatives ∇, the Riemann curvature tensor R is defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

- Sectional curvature $K(e_i, e_j) = \langle R(e_i, e_j)e_j, e_i \rangle$, $\forall 1 \leqslant i \neq j \leqslant n$,
- Ricci curvature $Ric(e_i, e_i) = \sum_{j \neq i} K(e_i, e_j)$.

From the view of calculus, curvature measures the diffenence between the change of the second (covariant) derivatieves.

(non-smooth) Function:

- weak or distributional derivatives
- convergence
$$f_j \to f$$
in L^p, L^∞, $W^{k,p}$, C^α,
- Compatibility: If $f_j \to f$ in $W^{1,p}$, then $\partial f_j \to \partial f$ in L^p.

Metric space:

- generalized notions of curvature ??
- convergence
$$M_j \to M$$
in the sense of ?? (Gromov-Hausdorff convergence)
- Compatibility: generelized curvature is stable under GH-convergence.

Metric geometry studies the space by the notions of length, angles, and curvature *synthetically*, etc. Alexandrov (1951) introduced the generalized sectional curvature bounds via **comparison of geodesic triangles**.

Geodesic space

Let (X, d) be a metric space, $|pq| = d(p, q)$.

$$L(\gamma) = \sup_{a=a_0<\cdots<a_m=b} \sum_{i=0}^{m-1} |\gamma(a_i)\gamma(a_{i+1})|.$$

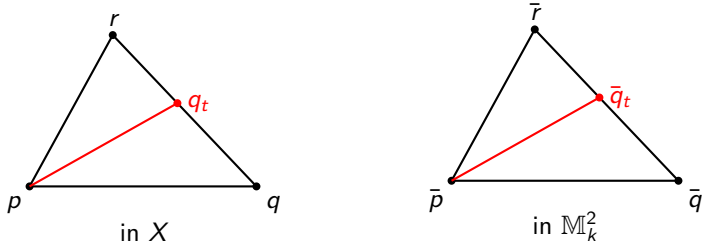
Lower semi-continuity

$\gamma_\nu \to \gamma$ uniformly $\implies \liminf L(\gamma_\nu) \geq L(\gamma)$.

- **Geodesic** (shortest curve): $L(\gamma) = |\gamma(a)\gamma(b)|$.
- **Geodesic space**: every pair of points is joined by a geodesic.

Definition: curvature $\geq k$

- $\mathbb{M}_k^2$: simply connected 2-dim space form of curvature k (the k-plane).
- For $\triangle pqr$ in X, take $\widetilde{\triangle}_k pqr \subset \mathbb{M}_k^2$ with the same side lengths (if $k > 0$: perimeter $< 2\pi/\sqrt{k}$).



Alexandrov convexity

A locally complete geodesic space (X, d) has **curvature** $\geq k$ if every point has a neighborhood U_x such that for every $\triangle pqr \subset U_x$ and $q_t \in qr$ with $|qq_t| = t|qr|$,

$$|pq_t| \geq |\bar{p}\bar{q}_t|.$$

Riemannian case

By Toponogov, (M, d_g) has curvature $\geq k \iff \sec_g \geq k$.

Examples:

1. Riemannian manifolds with $\sec \geq k$ (locally convex boundary allowed).
2. Convex subsets of such manifolds.
3. Convex hypersurfaces.
4. Doubling disk ($k = 0$).

Globalization Theorem

Let (X, d) have curvature $\geq k$, then the Alexandrov convexity holds for any triangle in X.

(we will sketch the proof in this afternoon)

Equivalent conditions

- $\widetilde{\angle}_k rpq$: the k-**comparison angle** at $p =$ the $\angle \overline{rp}\overline{q}$ in $\widetilde{\triangle}_k pqr \subset \mathbb{M}^2_k$.

(i) Monotonicity of comparison angles

$\omega_k(s,t) = \widetilde{\angle}_k q\gamma(s)\sigma(t)$ is non-increasing in each variable.

(ii) Quadruple condition

For any four points $(p; q, r, w)$:

$$\widetilde{\angle}_k qpr + \widetilde{\angle}_k rpw + \widetilde{\angle}_k wpq \leq 2\pi.$$

Consequences:

- Angle $\angle(\gamma, \sigma) = \lim_{s,t\to 0^+} \omega_k(s,t)$ exists; $\angle pqr \geq \widetilde{\angle}_k pqr$.
- Geodesics do not branch.
- The hinge comparison.

Constructions: Products, cones, quotients

Direct product

$d^2_{X\times Y} = |x_1x_2|^2_X + |y_1y_2|^2_Y$.
Curvature $\geq k$ in both factors $\Rightarrow$ product has curvature $\geq \min\{k, 0\}$.

k-cone $C_k(X)$

Vertex o and points (x, r); distance via comparison triangle in the k-plane:

$$\widetilde{\angle}_k a_1 o a_2 = |x_1x_2|.$$

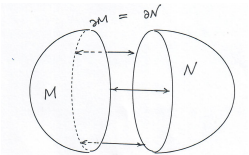
$k = 0$: $|a_1a_2|^2 = r_1^2 + r_2^2 - 2r_1r_2\cos|x_1x_2|$. $\quad C(\mathbb{S}^n) = \mathbb{R}^{n+1}$.

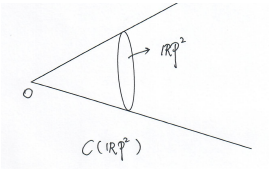
Quotients

Partition into closed equidistant subsets $\Rightarrow$ quotient preserves curvature $\geq k$.

Cones and curvature

gluing and cone:





Theorem

X has curvature ≥ 1 $\iff$ $C(X)$ is not a straight line and has curvature ≥ 0 (similarly for k-cones).

- **2. GH compactness, Hausdorff dimension and tangent cone**

Hausdorff dimension

Lemma

In an Alexandrov space with curvature $\geq k$, the Hausdorff dimension is homogenuous, i.e,

$$\dim_H(X) = \dim_H(B_\epsilon(p))$$

for any ball $B_\epsilon(p)$.

- $k = 0$: rescaling $B_R(p) \to B_r(p)$ is R/r-Lipschitz.

We called it an n-dimensional Alexandrov space. Later, we will show that dim is an integer or $+\infty$.

GH limits and compactness

Theorem

If $(X_i, p_i) \xrightarrow{GH} (X, p)$*, curvature* $\geq k$*, and* X *is boundedly compact, then* X *has curvature* $\geq k$ *and*

$$\dim(X) \leq \liminf_i \dim(X_i).$$

Theorem (Gromov compactness)

For fixed n, k, D*, the family of compact Alexandrov spaces with* $\dim \leq n$*, curvature* $\geq k$*,* $\mathrm{diam} \leq D$ *is compact in the GH topology.*

Space of directions and tangent cone

- Directions Σ'_p: arc-length geodesics from p, identified when $\angle(\gamma, \sigma) = 0$. $\angle(\gamma_1, \gamma_2)$ is a metric on Σ'_p,

$$\angle(\gamma_1, \gamma_3) \leq \angle(\gamma_1, \gamma_2) + \angle(\gamma_2, \gamma_3).$$

  (we will prove it in this afternoon)

- Σ_p: completion under the angle metric; compact.
- Tangent cone $T_p = C(\Sigma_p)$, with $\langle u, v \rangle = ts\cos|\xi\eta|$.

Theorem

$(X, \lambda d, p) \xrightarrow{GH} (T_p, o)$ *as* $\lambda \to \infty$.

Exponential map and boundary

- **Domain** T'_p: vectors realized by geodesics from p; $\exp_p(v) = x$.
- **Difficulty**: T'_p may not contain a neighborhood of the vertex.
  (In Riemannian manifold, it contains a neighborhood of the original).
- **Boundary** defined inductively: $\dim = 1$ manifold boundary; $\dim \geq 2$: $\partial\Sigma_p \neq \emptyset$.
- **Open question**: are boundary components Alexandrov spaces with curvature $\geq k$? Yes for GH limits of manifolds of the same dimension (Alexander-Kapovitch-Petrunin, 08).

- **3. Local structure near the regular points**

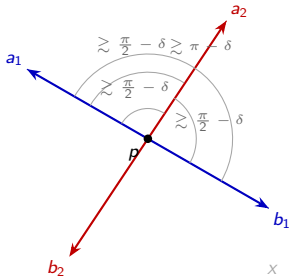
Strainers

Definition

p is (m, δ)-**strained** if there are pairs $\{(a_i, b_i)\}_{i=1}^{m}$, with

$$\widetilde{\angle}_k a_i p b_i > \pi - \delta, \qquad \widetilde{\angle}_k a_i p a_j,\ \widetilde{\angle}_k a_i p b_j,\ \widetilde{\angle}_k b_i p b_j > \tfrac{\pi}{2} - \delta\ (i \neq j).$$

Analogy: an orthonormal frame in $\mathbb{R}^m$.



Theorem

If $\delta \leq \frac{1}{100m}$ and p has an (m, δ)-strainer $\{(a_i, b_i)\}_{i=1}^m$ with $|a_i p|, |b_i p| \geq L$, then

$$\phi(x) = (|a_1 x|, \ldots, |a_m x|)$$

is C_m-open and Lipschitz on $B_{\delta^2 L}(p)$. If no $(m+1, 25\delta)$-strained point is nearby, ϕ is bi-Lipschitz onto an open set of $\mathbb{R}^m$.

To prove the second part of the theorem, it suffices to show that ϕ is one-to-one on $B_{\delta^2 L}(p)$, since ϕ is C_m-open and Lipschitz continuous with Lipschitz constant $\sqrt{m}$. Suppose that it is not true that $\phi(q) = \phi(r)$ for some $q, r \in B_{\delta^2 L}(p)$ with $q \neq r$. Let w be a point in one of the geodesics qr. Then $\{(a_i, b_i), (q, r)\}$ forms an $(m+1, 25\delta)$-strainer at w, which is a contradiction.

Dimension theorem

Theorem

For a complete Alexandrov space, $\dim_H(X) \in \mathbb{N} \cup \{\infty\}$.

- Let $\overline{m} = \sup\{m : \exists\, (m, \delta)\text{-strained point},\ \delta < \frac{1}{100m}\}$.
- $\overline{m} = \infty$: openness $\Rightarrow \dim_H = \infty$.
- $\overline{m} < \infty$: a neighborhood is bi-Lipschitz to an open set of $\mathbb{R}^{\overline{m}}$; dimensional homogeneity gives $\dim_H = \overline{m}$.

Corollaries

(n, δ')-strained points are dense; local compactness.

In fact, thet bi-Lipschitz homeomorphism in the above Theorem can be imporved to the following almost isometry, i.e., the bi-Lipschitz constant is close to 1.

Theorem

For any $\epsilon \in (0,1)$, there is $\delta_{n,k,\epsilon} > 0$ such that the following property holds. Let (X,d) be an n-dimensional Alexandrov space with curvature k. Suppose that $p \in X$ has an (n,δ)-strainer $\{(a_i,b_i)\}$ for some $\delta \le \delta_{n,k,\epsilon}$. Then the map $\phi(x) = (|a_1x|, |a_2x|, \cdots, |a_nx|)$ satisfies

$$1-\epsilon \le \frac{|\phi(x)\phi(y)|_{\mathbb{R}^n}}{|xy|} \le 1+\epsilon, \qquad \forall x,y \in B_{\delta^2 L/(3C_n')}(p), \quad x \ne y,$$

where $L := \min_{i=1,\cdots,n}\{|a_ip|, |b_ip|\}$.

Regular and singular points

Definition

Let (X, d) be an n-dim Alexandrov space. $p \in X$ is called **regular** if it has an δ-regular for all $\delta > 0$ (equivelently, $T_p = \mathbb{R}^n$); otherwise **singular**.

Theorem (Bonnet–Myer type)

Curvature ≥ 1, *n-dim complete:* $\operatorname{diam} \leq \pi$. *If* $\operatorname{diam} = \pi$, *the space is a suspension over a curvature* ≥ 1 *space, or one/two points.*

(We will prove it this afternoon)

Corollary

$\operatorname{rad}(X) = \pi$ *and curvature* $\geq 1 \Rightarrow X \overset{isom}{=} \mathbb{S}^n$. $\qquad \operatorname{rad}(X) := \inf_x \sup_y |xy|$.

Singular set has codimension ≥ 1

Theorem

$\dim_H(S_X) \leq n-1$ *(and* $\leq n-2$ *if* $\partial X = \emptyset$*).*

Sketch:

- $S_\delta(X) = \{x : \mathrm{rad}(\Sigma_x) \leq \pi - \delta\}$, $\qquad S_X = \bigcup_{\delta_i \to 0} S_{\delta_i}(X)$.
- Take dense set $\{p_i\}$: $S_\delta(X) \subset \bigcup_i S_{p_i,\delta/2}$, where

$$S_{p,\delta} = \{x : \widetilde{\angle}_k pxy < \pi - \delta,\ \forall y \neq x\}.$$

- $x \mapsto \uparrow_p^x$ is L-expanding into Σ_p, so $\dim_H(S_{p,\delta}) \leq n-1$.

Now we have, for any small $\delta < (0, \delta_{n,\epsilon,k})$, that

$$X = \mathcal{R}_\delta \cup S_\delta,$$

where $\mathcal{R}_\delta$ is a Lipschitz manifold, and S_δ has dimension $\leq n-1$.

Otsu–Shioya: a 2-dim non-negatively curved space with **dense singular set**.

Open question: Perelman proved for any point $p \in X$, it has a neighborhood which is homeomorphic to a domain of its tangent cone. He conjectured that it is bi-Lipschitz homeomorphism.

- **4. The first and second variations of arc-length**

First variation

Let $n \geq 2$ and $k \in \mathbb{R}$. (X, d) be an n-dimensional Alexandrov space with curvature $\geq k$.

Lemma

Let $a, x \in X$ and fix some geodesic $\gamma(t)$ with unit speed and $\gamma(0) = x$. Then

$$\lim_{t \to 0^+} \frac{|a\gamma(t)| - |ax|}{t} = -\min \cos \angle(\uparrow_x^a, \gamma'(0)),$$

where $\gamma'(0) \in T_x$ is the direction presented by $\gamma(t)$, where $\uparrow_x^a \in \Sigma_x$ is a direction from x to a.

Second variation

Theorem (Petrunin)

*If pq extends beyond both ends, for any sequence $\{\epsilon_j\} \to 0^+$, there is a **parallel transport** isometry $T : T_p \to T_q$ and a subsequence $\{\varepsilon_j\} \subset \{\epsilon_j\}$ with, for $\eta \perp \uparrow_p^q$,*

$$\limsup_{\varepsilon_j \to 0} \frac{|\exp_p(\varepsilon_j \eta) \exp_q(\varepsilon_j T\eta)| - |pq|}{\varepsilon_j^2} \leq -\frac{k}{2}|pq||\eta|^2.$$

- The construction depends on the sequence $\{\epsilon_j\}$.
  Open question: To construct a parallel transport depending only on p, q.
- **Corollary**: the set of regular points is convex.

DC_0 structure

- Open set $X^* \supset \mathrm{Reg}(X)$ with a DC_0-**manifold** structure: transitions are differences of concave functions.
- Riemannian metric at regular points:

$$\langle u, v\rangle_x = |u||v| \cos \angle(u, v),$$

components are differentiable a.e.; (modified) distance functions admit second-order Taylor expansions, a.e..