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A PROOF OF TOPONOGOV'S THEOREM IN ALEXANDROV GEOMETRY

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Abstract. This note aims to give an elementary proof for Toponogov's theorem in Alexandrov geometry with lower curvature bound. The idea of the proof comes from the fact that, in Riemannian geometry, sectional curvature can be embodied in geodesic variations.

Introduction

An Alexandrov space X with curvature $\geq k$ is roughly defined to be a locally complete intrinsic metric space on which Toponogov's type theorem holds locally ([BGP]). In fact, if X is complete, then a theorem of such type holds globally on X, which is called Toponogov's theorem in Alexandrov geometry. In its proofs ([BGP], [AKP]), to simplify the arguments, it is always assumed that there exists a minimal geodesic (shortest path) between any two points in X.[1] For any two minimal geodesics $[pq]$ and $[qr]$ in X, we can define the angle $\angle pqr$ naturally; and then, Toponogov's theorem in Alexandrov geometry can be formulated as follows ([BGP]).

Theorem A. *Let X be a complete Alexandrov space with curvature $\geq k$. Then for any triangle $\triangle pqr \subset X$ (a union of three minimal geodesics $[pq], [pr], [qr]$), we have that*

$$\angle pqr \geq \tilde{\angle}_k pqr. \tag{0.1}$$

In Theorem A, $\tilde{\angle}_k pqr$ denotes the angle $\angle \tilde{p}\tilde{q}\tilde{r}$ in the comparison triangle $\triangle \tilde{p}\tilde{q}\tilde{r} \subset \mathbb{S}^2_k$ of $\triangle pqr$ (i.e. $|\tilde{p}\tilde{q}| = |pq|, |\tilde{p}\tilde{r}| = |pr|, |\tilde{q}\tilde{r}| = |qr|$), where $\mathbb{S}^2_k$ denotes the complete and simply connected 2-dimensional space form with constant curvature k. And in a proof of Theorem A, one need only consider the case where $|pq|+|pr|+|qr| < \frac{2\pi}{\sqrt{k}}$ (and thus each of $|pq|, |pr|, |qr|$ is less than $\frac{\pi}{\sqrt{k}}$) if $k > 0$ ([BGP]).[2]

So far, there are several proofs for Theorem A ([BGP], [Pl], [Sh]), each of which is skillful and the one in [Pl] is quite beautiful. In Riemannian geometry, sectional

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[1] If X is both complete and locally compact, there is a shortest path between any two points in it.

[2] Once Theorem A has been proven, it can be shown that $|pq| \leq \frac{\pi}{\sqrt{k}}$ and $|pq|+|pr|+|qr| \leq \frac{2\pi}{\sqrt{k}}$ for all $p, q, r \in X$ ([BGP]). Moreover, if $|pq| = \frac{\pi}{\sqrt{k}}$, it is a convention that $\tilde{\angle}_k pqr = 0$.

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curvature can be seen just from geodesic variations (the second variation formula). However, these known proofs have no direct relation with such an idea. The main goal of this paper is to provide a proof of such an idea for Theorem A.

Our strategy is: If $\angle pqr < \tilde{\angle}_k pqr$ for some $\triangle pqr \subset X$, we can split $\triangle pqr$ into two triangles along some $[ps]$ with $s \in [qr]^\circ$ (the interior part of $[qr]$) so that $\angle psq < \tilde{\angle}_k psq$ or $\angle psr < \tilde{\angle}_k psr$. By repeating this process, we can locate an $o \in [qr]$ such that there is information of "curvature $\not\geq k$" around some $[po]$, i.e. there is $\triangle pr_1r_2$ with r_i arbitrarily close to o such that $\angle pr_1r_2 < \tilde{\angle}_k pr_1r_2$. Note that Theorem A holds on a small ball $B(o,\delta)$ (i.e. (0.1) is true for triangles in $B(o,\delta)$) because X is of curvature $\geq k$. Then via Alexandrov's Lemma (Lemma 1.3) on $\triangle pr_1r_2$, we can locate another $o' \in B(o,\delta)$ with $|po'| \leq |po| - \frac{\delta}{3}$ such that there is still information of "curvature $\not\geq k$" around some $[po']$. Step by step, such information can be transmitted to a small neighbourhood of p, a contradiction (because Theorem A holds around p). (The merit of our proof is that the contradiction is gotten just around p, while none of previous proofs locates such a precise neighborhood for a contradiction.)

Remark 0.1. In fact, under the assumption that there exists a minimal geodesic between any two points in X, the condition of 'complete' in Theorem A can be removed ([Pe]). The idea in [Pe] is, for any $[pq] \subset X$, to show a global information of curvature $\geq k$ around $[pq]$ via gluing the local information of curvature $\geq k$ around each point in $[pq]$. Essentially, there is some overlap between the idea of our proof and that in [Pe].

In the rest of the paper, X always denotes the space in Theorem A.

1. Tools of the proof

Note that a proof of Theorem A has to depend only on its local version, where we have the following basic property on angles ([BGP]).

Lemma 1.1. *Let $[pq], [rr'] \subset X$ with $r \in [pq]^\circ$. Then $\angle prr' + \angle qrr' = \pi$.*

Furthermore, we have the following easy observation (cf. [AKP]).

Lemma 1.2. *Let $[pq], [qr] \subset X$. Then for $q_i \in [qr]$ with $q_i \to q$ as $i \to \infty$,*

$$|pq_i| \leq |pq| - \cos\angle pqr \ \cdot |qq_i| + o(|qq_i|).^3 \tag{1.1}$$

Proof. Since Theorem A holds around q, there is $\bar{p} \in [pq]$ near q such that

$$|\bar{p}q_i| \leq |\bar{p}q| - \cos\angle pqr \ \cdot |qq_i| + o(|qq_i|).$$

Then (1.1) follows from that $|pq_i| \leq |p\bar{p}| + |\bar{p}q_i|$. □

Besides Lemmas 1.1 and 1.2, we will also use Alexandrov's lemma ([BGP]).

Lemma 1.3. *Let $\triangle pqr$, $\triangle pqs$, $\triangle abc \subset \mathbb{S}^2_k$ (where $\triangle pqr$ and $\triangle pqs$ are joined to each other in an exterior way along $[pq]$) such that $|ab| = |pr|$, $|ac| = |ps|$, $|bc| = |qr| + |qs|$, and $|ab| + |ac| + |bc| < \frac{2\pi}{\sqrt{k}}$ if $k > 0$. Then $\angle pqr + \angle pqs \leq \pi$ (resp. $\geq \pi$) if and only if $\angle prq \geq \angle abc$ and $\angle psq \geq \angle acb$ (resp. $\angle prq \leq \angle abc$ and $\angle psq \leq \angle acb$).*

[3] In addition, if X is locally compact, then we can select $[pq]$ such that (1.1) is an equality (the first variation formula) once Theorem A has been proven ([BGP], [AKP]).

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2. Proof of Theorem A

Due to the similarity of proofs for $k <, =, > 0$, we only consider the case where $k = 0$. And for simpleness, we denote by $\tilde{\angle}pqr$ the angle $\tilde{\angle}_0 pqr$.

In the proof, we will argue by contradiction, and say that an angle $\angle pqr$ is *bad* if $\angle pqr < \tilde{\angle}pqr$. If each angle of a triangle is not bad, we call the triangle a *good* one.

First of all, we have the following observation about a 'bad' angle just via Lemmas 1.1 and 1.2. (cf. [SSW]).

Lemma 2.1. *For a triangle $\triangle pr_1r_2 \subset X$, if $\angle pr_1r_2$ is bad, then there is $s_0 \in [r_1r_2]^\circ$ such that for any $[ps_0]$*

$$\angle ps_0r_1 \text{ or } \angle ps_0r_2 \text{ is bad;} \tag{2.1}$$

in particular, for each i,

$$\text{if } |r_is_0| \le |r_ip|, \text{ then } \angle ps_0r_i \text{ is bad.} \tag{2.2}$$

Moreover, we have that

$$|ps_0| < \max_{i=1,2}\{|pr_i|\}. \tag{2.3}$$

Proof. Let $\triangle \tilde{p}\tilde{r}_1\tilde{r}_2 \subset \mathbb{R}^2$ be the comparison triangle of $\triangle pr_1r_2$. By Lemma 1.2 (and the first variation formula on $\mathbb{R}^2$), the badness of $\angle pr_1r_2$ implies that the function $|ps| - |\tilde{p}\tilde{s}|$ with $s \in [r_1r_2], \tilde{s} \in [\tilde{r}_1\tilde{r}_2]$ and $|r_is| = |\tilde{r}_i\tilde{s}|$ attains a negative minimum at some $s_0 \in [r_1r_2]^\circ$. By Lemma 1.2 again and the minimality of s_0, we have that

$$\angle ps_0r_i \ge \angle \tilde{p}\tilde{s}_0\tilde{r}_i, \quad i = 1, 2$$

for any $[ps_0]$. It then has to hold that

$$\angle ps_0r_i = \angle \tilde{p}\tilde{s}_0\tilde{r}_i, \quad i = 1, 2$$

because $\angle ps_0r_1 + \angle ps_0r_2 = \pi$ by Lemma 1.1. On the other hand, since $|ps_0| < |\tilde{p}\tilde{s}_0|$, there is $\tilde{p}' \in [\tilde{p}\tilde{s}_0]^\circ$ such that $|\tilde{p}'\tilde{s}_0| = |ps_0|$. It is clear that $|\tilde{p}'\tilde{r}_i| < |\tilde{p}\tilde{r}_i|$ for at least one of i, which implies $\angle \tilde{p}\tilde{s}_0\tilde{r}_i < \tilde{\angle}ps_0r_i$, i.e. (2.1) holds. Especially, it is easy to see that $|\tilde{p}'\tilde{r}_i| < |\tilde{p}\tilde{r}_i|$ if $|r_is_0| \le |r_ip|$;[4] and thus (2.2) follows. Moreover, it is clear that $|\tilde{p}\tilde{s}_0| < \max_{i=1,2}\{|pr_i|\}$,[5] which implies (2.3). □

Corollary 2.2. *For $\triangle pr_1r_2 \subset X$, if $\angle pr_1r_2$ is bad, then there is $\bar{s} \in [r_1r_2]$ such that*

$$|p\bar{s}| \le \max_{i=1,2}\{|pr_i|\}, \tag{2.4}$$

and for any small $\delta > 0$ there exist $s_1, s_2 \in [r_1r_2] \cap B(\bar{s}, \delta)$ such that for any $[ps_i]$

$$\angle ps_1s_2 \text{ or } \angle ps_2s_1 \text{ is bad.} \tag{2.5}$$

[4]For the case where $k > 0$ and $|r_ip| \ge \frac{\pi}{2\sqrt{k}}$, it needs $\angle \tilde{r}_i\tilde{p}\tilde{s}_0 < \frac{\pi}{2}$ besides $|r_is_0| \le |r_ip|$ (note that $\triangle \tilde{p}\tilde{r}_1\tilde{r}_2 \subset \mathbb{S}^2_k$ and $|r_ip| < \frac{\pi}{\sqrt{k}}$, cf. the comments following Theorem A). In fact, '$\angle \tilde{r}_i\tilde{p}\tilde{s}_0 < \frac{\pi}{2}$' holds obviously if $|r_is_0| \ll |r_ip|$ (here, it needs only '$|r_is_0| \le |r_ip|$' for other cases including $k \le 0$).

[5]For the case where $k > 0$ and $\max_{i=1,2}\{|pr_i|\} \ge \frac{\pi}{2\sqrt{k}}$, it needs to be modified to '$|\tilde{p}\tilde{s}_0| < \max_{i=1,2}\{|pr_i|\} + \tau(|r_1r_2|)$', where $\tau(|r_1r_2|)/|r_1r_2| \to 0$ as $|r_1r_2| \to 0$; moreover, the function $\tau(\cdot)$ can be chosen to be the same one (i.e. not depending on $|pr_i|$) if $\max_{i=1,2}\{|pr_i|\} \le c < \frac{\pi}{\sqrt{k}}$ for some constant c.

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Proof. By Lemma 2.1, there is $s_0 \in [r_1r_2]^\circ$ such that for any $[ps_0]$, $\angle ps_0r_1$ or $\angle ps_0r_2$ is bad, say $\angle ps_0r_1$. Then we can apply Lemma 2.1 to $\triangle ps_0r_1$ again. By repeating this infinite times, we can locate a desired $\bar{s}$ if we take into account (2.2) and (2.3). □

In our proof of Theorem A, Lemmas 1.3 and 2.1 shall be the mere keys. For simpleness, we first consider the case where X is, in addition, locally compact.

Proof of Theorem A *where X is locally compact.* We argue by contradiction. Assume that there is a $\triangle pqr$ such that $\angle pqr < \tilde{\angle} pqr$. Then by Corollary 2.2 (see (2.5)), we can consider the nonempty set

$$\mathcal{S} \triangleq \{x \in X |\ \forall\ \delta > 0,\ \exists\ \triangle pr_1r_2 \text{ with } r_i \in B(x,\delta) \text{ s.t. } \angle pr_1r_2 \text{ is bad}\}.$$

It is clear that $p \notin \mathcal{S}$ because Theorem A holds around p, and that $\mathcal{S}$ is closed. Note that a closed and bounded subset in X is compact because X is complete and locally compact (cf. Chapter 2 in [BBI]). So, there is $\bar{o} \in \mathcal{S}$ such that $|p\bar{o}| = \min_{o\in\mathcal{S}}\{|po|\} > 0$.

On the other hand, Theorem A holds around any $o \in \mathcal{S}$, and thus we can define a positive function $\delta(o) \triangleq \min\{\frac{|po|}{2}, \delta_o\}$, where δ_o is the maximal number such that

$$\text{any } \triangle xyz \subset B(o,\delta_o) \text{ is good.} \tag{2.6}$$

Claim. For each $o \in \mathcal{S}$, in $B(o,\delta(o))$, there is another $o' \in \mathcal{S}$ such that $|po'| \le |po| - \frac{\delta(o)}{3}$.

It is obvious that the claim contradicts the existence of $\bar{o}$. Thereby, we just need to verify the claim. Due to $o \in \mathcal{S}$, there is a $\triangle pr_1r_2$ with $r_i \in B(o,\delta(o))$ such that

$$\angle pr_1r_2 \text{ is bad with } |or_i| \ll \delta(o)$$

(in fact, it just needs $|or_i| < c\cdot\delta(o)$ for some small c, e.g. $c = 0.01$). Let $\bar{r}_1 \in [pr_1]$ with $|p\bar{r}_1| = |po| - \frac{\delta(o)}{3}$. Note that for any $[\bar{r}_1r_2]$, $\triangle r_2r_1\bar{r}_1$ is good because it is contained in $B(o,\delta(o))$; so via Lemmas 1.3 and 1.1 on $\triangle pr_1r_2$, the badness of $\angle pr_1r_2$ implies

$$\angle p\bar{r}_1r_2 \text{ is bad.}$$

Then we can apply (2.2) in Lemma 2.1 (note that $|\bar{r}_1r_2| < \delta(o) < |p\bar{r}_1|$ because $\delta(o) \le \frac{|po|}{2}$) to $\triangle p\bar{r}_1r_2$ to locate an $s \in [\bar{r}_1r_2]^\circ$ such that

$$\angle ps\bar{r}_1 \text{ is bad for any } [ps].$$

Put $\bar{r}_2 \triangleq s$ if $|ps| \le |po| - \frac{\delta(o)}{3}$; otherwise, similar to $\bar{r}_1$, we can select $\bar{r}_2 \in [ps]$ with

$$|p\bar{r}_2| = |po| - \frac{\delta(o)}{3}, \text{ and } \angle p\bar{r}_2\bar{r}_1 \text{ is bad for any } [\bar{r}_1\bar{r}_2].$$

(Here, $\bar{r}_1, s, \bar{r}_2$ all lie in $B(o,\epsilon)$ with ϵ very close to $\frac{\delta(o)}{3}$. Note that $|r_2x| + |xp|$ with $x = \bar{r}_1, s, \bar{r}_2$ is very close to $|po|$ because $|or_i| \ll \delta(o)$.) Then by Corollary 2.2, the badness of $\angle p\bar{r}_2\bar{r}_1$ enables us to locate a point $o' \in [\bar{r}_1\bar{r}_2] \cap \mathcal{S}$, which lies in $B(o,\delta(o))$, such that $|po'| \le |po| - \frac{\delta(o)}{3}$ (see (2.4) and note that $|p\bar{r}_i| \le |po| - \frac{\delta(o)}{3}$).[6] □

Eventually, we prove Theorem A for general X, i.e. X might not be locally compact.

[6]For the case where $k > 0$ and $\max_{i=1,2}\{|p\bar{r}_i|\} \ge \frac{\pi}{2\sqrt{k}}$, $\delta(o)$ should be additionally so small that $\tau(\delta(o)) \ll \delta(o)$ (e.g. $\tau(\delta(o)) < 0.01\delta(o)$ when $k = 1$, and refer to Footnote 5 for $\tau(\cdot)$).

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Proof of Theorem A *where X is not locally compact.* When X is not locally compact, there might be no $\bar{o} \in \mathcal{S}$ with $|p\bar{o}| = \min_{o\in\mathcal{S}}\{|po|\}$ in the proof right above. However, starting with an $o_1 \in \mathcal{S}$ and by the claim below (2.6), we can step by step obtain $\{o_i\}_{i=1}^{\infty} \subset \mathcal{S}$ such that

$$|po_{i+1}| \le |po_i| - \frac{\delta(o_i)}{3} \text{ and } |o_io_{i+1}| < \delta(o_i).$$

It follows that

$$|po_{i+1}| \le |po_1| - \sum_{j=1}^{i} \frac{\delta(o_j)}{3}, \text{ and thus } \sum_{j=1}^{i} \frac{\delta(o_j)}{3} \le |po_1|.$$

This implies that $\lim_{i\to\infty} \sum_{j=i}^{\infty} \delta(o_j) = 0$, i.e., $\{o_i\}_{i=1}^{\infty}$ is a Cauchy sequence (note that $|o_io_{i+1}| < \delta(o_i)$). Then by the completeness of X and the closeness of $\mathcal{S}$, $\{o_i\}_{i=1}^{\infty}$ has a limit point $\bar{o} \in \mathcal{S}$. However, it is clear that $\liminf_{i\to\infty} \delta(o_i) \ge \delta(\bar{o}) > 0$, a contradiction. □

We would like to point out that if one can show that $\liminf_{i\to\infty} \delta(o_i) > 0$ in the proof above without the completeness of X, then an alternative proof of [Pe] is gotten.

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