1 Part I. Intersection property

Throughout we take $m \geq 1$ balls. For $m = 0$ the empty intersection would be V, and $V \setminus B_{t'}(0)$ is disconnected for $V = \mathbb{R}$, so the definition must intend $m \geq 1$.

1.1 Problem 1. $\mathbb{R}$ has the intersection property, with threshold $t(\rho) = \rho$

Let $\rho > 0$, $t = \rho$, $t' > t$, $r_1, \ldots, r_m \leq \rho$ and $x_1, \ldots, x_m \in \mathbb{R}$. - The set $I = \bigcap_i U_{r_i}(x_i) = \bigcap_i (x_i - r_i, x_i + r_i)$ is an intersection of finitely many open intervals. So it is an open interval (α, β) or empty. - Since $I \subset U_{r_1}(x_1)$, its length satisfies $\beta - \alpha \leq 2r_1 \leq 2\rho$. - We have $\mathbb{R} \setminus B_{t'}(0) = (-\infty, -t') \cup (t', \infty)$. Hence

$$(\mathbb{R} \setminus B_{t'}(0)) \cap I = \big((\alpha, \beta) \cap (-\infty, -t')\big) \cup \big((\alpha, \beta) \cap (t', \infty)\big).$$

- Both pieces can be non-empty only if $\alpha < -t'$ and $\beta > t'$. That would force $\beta - \alpha > 2t' > 2\rho$, which is impossible. - So the set equals one of the two pieces. Each piece is an interval, possibly empty. A non-empty interval is convex and hence contractible.

So $t(\rho) = \rho$ is admissible.

Optimality of the threshold. No $t < \rho$ works. Take $t' \in (t, \rho)$, $m = 1$, $x_1 = 0$, $r_1 = \rho$. Then $(\mathbb{R} \setminus B_{t'}(0)) \cap (-\rho, \rho) = (-\rho, -t') \cup (t', \rho)$, which is disconnected and so not contractible. Hence the set of admissible thresholds is exactly $[\rho, \infty)$.

1.2 Problem 2. Independence of the origin

Suppose the property holds with deleted balls centred at 0 and threshold $t(\rho)$. Fix $a \in V$, and let $\tau_a(v) = v - a$. This translation is an isometry of V, so in particular a homeomorphism. It maps $B_{t'}(a)$ onto $B_{t'}(0)$ and $U_{r_i}(x_i)$ onto $U_{r_i}(x_i - a)$. Hence

$$\tau_a\Big((V \setminus B_{t'}(a)) \cap \bigcap_{i=1}^{m} U_{r_i}(x_i)\Big) = (V \setminus B_{t'}(0)) \cap \bigcap_{i=1}^{m} U_{r_i}(x_i - a).$$

The right-hand side is an admissible configuration for the origin 0: it has the same radii $r_i \leq \rho$ and the same $t' > t(\rho)$, with centres $x_i - a$. So it is empty or contractible. Being empty or contractible is preserved by homeomorphisms, so the left-hand side is empty or contractible too.

The threshold does not change: $t_a(\rho) = t(\rho)$ for every a. Running the argument with τ_{-a} gives the converse. So the property, and the optimal threshold function, are the same for every origin.

1.3 Problem 3. Convex cone property ⇒ (finite) intersection property

Assume V is finite-dimensional with the convex cone property. For $K \subset V$ write

$$\mathrm{Cone}(K) = \{sk : \ s \geq 0, \ k \in K\}$$

for the cone over K from the origin. The convex cone property says: if $A = U_{r_A}(x_A)$, $B = B_{r_B}(0)$, $r_A < r_B$ and $A \setminus B \neq \varnothing$, then $\mathrm{Cone}(A \setminus B)$ is convex. This is the same as convexity of the set of directions through $A \setminus B$. By Problem 2 it is enough to treat deleted balls centred at 0.

Claim. $t(\rho) = \rho$ is an admissible threshold.

Let $t' > \rho$, $r_i \leq \rho$, $A_i = U_{r_i}(x_i)$, $W = \bigcap_{i=1}^m A_i$, $B = B_{t'}(0)$, and $S = W \setminus B$. Suppose $S \neq \varnothing$. Then every $A_i \setminus B \neq \varnothing$, and $r_i \leq \rho < t'$, so the convex cone property applies to each pair (A_i, B).

Step 1: $C = \mathrm{Cone}(S)$ is convex. - Let $p, q \in S$ and let $z \in [p, q]$ with $z \neq 0$. Since W is convex, $z \in W$. - Fix i. Since $p, q \in A_i \setminus B$ and $\mathrm{Cone}(A_i \setminus B)$ is convex, $z \in \mathrm{Cone}(A_i \setminus B)$. So $\lambda_i z \in A_i \setminus B$ for some $\lambda_i > 0$. - Along the ray, $\|sz\| = s\|z\|$. So $\lambda_i z \notin B$ means $\lambda_i > t'/\|z\|$. - The set $\{s : \ sz \in A_i\}$ is convex and contains

1 and λ_i. - Put $\lambda = \max\{1, \min_i \lambda_i\}$. Then λ lies between 1 and λ_i for every i, so $\lambda z \in A_i$ for all i, i.e. $\lambda z \in W$. - Also $\lambda \geq \min_i \lambda_i > t'/\|z\|$, so $\|\lambda z\| > t'$. - Hence $\lambda z \in S$ and $z \in C$. So C contains every segment between points of S. - For general points $sp, s'q \in C$ with $s, s' \geq 0$, not both zero, a convex combination $\alpha sp + (1-\alpha)s'q$ equals $(\alpha s + (1-\alpha)s')\, z$ for some $z \in [p, q]$. So it lies in C. Thus C is convex.

Step 2: C contains no line. - Suppose $v \neq 0$ and both sv and $-s'v$ $(s, s' > 0)$ lie in $S \subset A_1$. - Then $\|sv - (-s'v)\| = \|sv\| + \|s'v\| > 2t'$. - But two points of the open ball A_1 are at distance $< 2r_1 \leq 2\rho < 2t'$. This is a contradiction. - So $C \setminus \{0\}$ contains no pair $\pm v$.

Step 3: $C \setminus \{0\}$ is contractible. - Fix $v \in C \setminus \{0\}$ and let $H(x, s) = (1-s)x + sv$. By convexity, $H(x, s) \in C$. - If $H(x, s) = 0$ with $0 < s < 1$, then $-v = \frac{1-s}{s} x \in C$, which contradicts Step 2. - So H is a contraction of $C \setminus \{0\}$ to v. - Let $D = \{x/\|x\| : \ x \in S\}$ be the set of directions of S. Then $C \setminus \{0\} \cong D \times (0, \infty)$ via $x \mapsto (x/\|x\|, \|x\|)$. So D is contractible.

Step 4: $S \cong D \times (0, 1)$. - For $u \in D$, the set $\{s > 0 : \ su \in W\}$ is a bounded open interval $(a'(u), b(u))$, because W is open, convex and bounded. - Hence $\{s > 0 : \ su \in S\} = (a(u), b(u))$ with $a(u) = \max\{a'(u), t'\} < b(u)$. - b **is continuous on D.** - b is lower semicontinuous because W is open. - Suppose $u_k \to u$ in D and $b(u_k) \to s^* > b(u)$. Then $s^* u \in \overline{W}$, and $s_1 u \in W$ for some $s_1 < b(u)$. - For a convex open W, the half-open segment from a point of W to a point of $\overline{W}$ lies in W. So $su \in W$ for $s \in (b(u), s^*)$, contradicting the definition of $b(u)$. - The same argument shows that a' is continuous, and hence so is a. - Therefore $(u, \theta) \mapsto \big((1-\theta)a(u) + \theta b(u)\big)u$ is a homeomorphism $D \times (0, 1) \to S$. Its inverse is $x \mapsto \big(x/\|x\|, \ (\|x\| - a)/(b - a)\big)$.

By Steps 3 and 4, S is contractible. So for every ρ, the threshold $t = \rho$ works for every finite family of balls of radius $\leq \rho$. That is the (finite) intersection property. □

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2 Part II. Simplicial complexes

2.1 Problem 1. $\mathrm{VR}_r(P_m)$ for $m = 4, 5$

The chord between vertices k steps apart has length $2\sin(k\pi/m)$. A set σ is a simplex iff all its pairwise distances are $< r$.

$m = 4$**.** The side is $\sqrt{2}$ and the diagonal is 2. The critical values are $r = \sqrt{2}$ and $r = 2$.
cccc

r	$\mathrm{VR}_r($
$0 < r \leq \sqrt{2}$	
$\sqrt{2} < r \leq 2$	
$r > 2$	

$m = 5$**.** The side is $s = 2\sin\frac{\pi}{5} = \sqrt{(5-\sqrt{5})/2} \approx 1.1756$, and the diagonal is $d = 2\sin\frac{2\pi}{5} = \sqrt{(5+\sqrt{5})/2} \approx 1.9021$. The critical values are $r = s$ and $r = d$.
cccc

r	$\mathrm{VR}_r($
$0 < r \leq s$	
$s < r \leq d$	
$r > d$	

All inequalities in the definition are strict. So at a critical value itself the complex is still the one from the interval *below* it, and the intervals are $(\cdot,\cdot]$. There are no other changes, because the only distances that occur are the side and the diagonal.

2.2 Problem 2. The factor 2 in $\check{C}_r(X;Z) \subset \mathrm{VR}_{2r}(X)$ is sharp

Inclusion. If $d(z,x) < r$ for all $x \in \sigma$, then $d(x,y) \le d(x,z) + d(z,y) < 2r$. So $\operatorname{diam}\sigma < 2r$.
Sharpness. Let $c < 2$. Take the geodesic space $Z = [0,2] \subset \mathbb{R}$ and $X = \{0,2\}$. Choose $r = 4/(2+c)$ if $1 < c < 2$, and $r = 2$ if $c \le 1$. - Then $r > 1$, so the centre $z = 1 \in Z$ satisfies $d(1,0) = d(1,2) = 1 < r$. Hence $X \in \check{C}_r(X;Z)$. - Also $cr \le 2 = \operatorname{diam} X$. For $1 < c < 2$ this is $4c \le 2(2+c) \iff c \le 2$. For $c \le 1$ it is $2c \le 2$. So $X \notin \mathrm{VR}_{cr}(X)$.
So no constant $c < 2$ can replace 2.

2.3 Problem 3. Interleaving maps between Čech complexes

All complexes are ambient: $\check{C}(Y,s)$, for $Y \subset Z$, has vertex set Y, and a finite $\sigma \subset Y$ is a simplex iff $d(p,y) < s$ for all $y \in \sigma$, for some $p \in Z$.
Since $2d_{GH}(X,Z) < r$, there is a correspondence $R \subset X \times Z$ with $\operatorname{dis} R < r$. Here X carries the restricted metric. Using the axiom of choice, choose: - $h : Z \to X$ with $(h(z), z) \in R$ for every $z \in Z$; - $g : X \to Z$ with $(x, g(x)) \in R$ for every $x \in X$.
For $(x,z),(x',z') \in R$ we have $\left|d(x,x') - d(z,z')\right| \le \operatorname{dis} R < r$.
h **is simplicial** $\check{C}(Z,\varepsilon) \to \check{C}(X, r+\varepsilon)$**.** Let σ have centre $w \in Z$, i.e. $d(w,z) < \varepsilon$ for all $z \in \sigma$. Then for $z \in \sigma$,

$$d(h(w), h(z)) \le d(w,z) + \operatorname{dis} R < \varepsilon + r.$$

So $h(w) \in X \subset Z$ is a centre for $h(\sigma)$, and $h(\sigma) \in \check{C}(X, r+\varepsilon)$.
g **is simplicial** $\check{C}(X, r+\varepsilon) \to \check{C}(Z, 3r+2\varepsilon)$**.** Let σ have centre $p \in Z$, so $d(p,x) < r+\varepsilon$ for $x \in \sigma$. Fix $x_0 \in \sigma$. For every $x \in \sigma$, $d(x_0,x) < 2(r+\varepsilon)$. Hence

$$d(g(x_0), g(x)) \le d(x_0,x) + \operatorname{dis} R < 2r + 2\varepsilon + r = 3r + 2\varepsilon,$$

so $g(x_0)$ is a centre for $g(\sigma)$.
$g \circ h$ **is contiguous to the inclusion** ι**.** For every $z \in Z$, both $(h(z), z)$ and $(h(z), g(h(z)))$ lie in R. So

$$d(z, gh(z)) \le d(h(z), h(z)) + \operatorname{dis} R < r.$$

Let $\sigma \in \check{C}(Z,\varepsilon)$ have centre w. For $z \in \sigma$ we get $d(w,z) < \varepsilon$ and

$$d(w, gh(z)) \le d(w,z) + d(z, gh(z)) < \varepsilon + r \le 3r + 2\varepsilon.$$

So w is a common centre of $\sigma \cup gh(\sigma)$, and $\sigma \cup gh(\sigma) \in \check{C}(Z, 3r+2\varepsilon)$. This holds for every simplex, which is exactly contiguity of $g \circ h$ and ι. (In fact $\sigma \cup gh(\sigma) \in \check{C}(Z, r+\varepsilon)$.) $\square$
Remark. The hypothesis bounds only d_{GH}, not d_H. So g must come from the correspondence and not from the inclusion $X \hookrightarrow Z$: a point of Z need not be close to X inside Z.

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3 Part III. Jung constant and its variations

3.1 Problem 1. $A = \{e_1, \dots, e_n\} \subset \ell_\infty^n$, $n \geq 3$

- $\operatorname{diam} A = 1$. For $i \neq j$, $e_i - e_j$ has entries $1, -1, 0$, so $\|e_i - e_j\|_\infty = 1$.
- $R(A) = \frac{1}{2}$. The point $v = (\frac{1}{2}, \dots, \frac{1}{2})$ satisfies $\|v - e_i\|_\infty = \frac{1}{2}$ for all i. Conversely, for any v, $1 = \|e_1 - e_2\| \leq \|v - e_1\| + \|v - e_2\|$, so $\sup_a \|v - a\| \geq \frac{1}{2}$.
- $R_{\operatorname{conv} A}(A) = \frac{n-1}{n}$.
  - Let $x \in \operatorname{conv} A = \{x_i \geq 0,\ \sum x_i = 1\}$. Then $x - e_i$ has i-th entry $x_i - 1$ and j-th entry x_j for $j \neq i$.
  - Since $x_j \leq \sum_{k \neq i} x_k = 1 - x_i$, we get $\|x - e_i\|_\infty = 1 - x_i$.
  - Hence $\max_i \|x - e_i\|_\infty = 1 - \min_i x_i$.
  - Since $\sum x_i = 1$, $\min_i x_i \leq \frac{1}{n}$, with equality iff x is the barycentre $(\frac{1}{n}, \dots, \frac{1}{n})$.
  - So the infimum over $\operatorname{conv} A$ is $1 - \frac{1}{n} = \frac{n-1}{n}$. It is attained only at the barycentre.

Both Jung constants are realized.

$$\frac{R(A)}{\operatorname{diam} A} = \frac{1}{2} = J(\ell_\infty^n), \qquad \frac{R_{\operatorname{conv} A}(A)}{\operatorname{diam} A} = \frac{n-1}{n} = J_s(\ell_\infty^n).$$

So A attains both suprema. For completeness, here are proofs of the two upper bounds that make these suprema equal to the stated values.

- $J(\ell_\infty^n) \leq \frac{1}{2}$. If $A \subset \prod_j [m_j, M_j]$ is the smallest box, then $M_j - m_j \leq \operatorname{diam} A$. The box's centre c satisfies $\|c - a\|_\infty \leq \frac{1}{2} \max_j (M_j - m_j) \leq \frac{1}{2} \operatorname{diam} A$. Together with $R(A) \geq \frac{1}{2} \operatorname{diam} A$, which holds in every normed space, this gives $J = \frac{1}{2}$.
- $J_s(\ell_\infty^n) \leq \frac{n-1}{n}$ **for** $n \geq 2$.
  - Normalize $\operatorname{diam} A = 1$ and replace A by its closure. This changes neither the diameter nor the value of the infimum, since $\operatorname{conv} \overline{A} \subset \overline{\operatorname{conv} A}$. Write $w_j = M_j - m_j \leq 1$ and $c = \frac{n-1}{n}$.
  - We need a point of $\operatorname{conv} \overline{A}$ in the box $Q = \prod_j [M_j - c,\ m_j + c]$, which is non-empty since $w_j \leq 1 \leq 2c$. Then every $a \in A$ is within c of that point.
  - If no such point exists, the compact convex sets $\operatorname{conv} \overline{A}$ and Q can be strictly separated: there is $f = \sum f_j x_j$ with $\sum |f_j| = 1$ and $f(a) > \max_Q f$ for all $a \in \overline{A}$.
  - A direct computation gives
$$f(a) - \max_Q f = \sum_{f_j > 0} f_j (a_j - m_j) + \sum_{f_j < 0} |f_j| (M_j - a_j) - c.$$
  - Pick j^* with $|f_{j^*}|$ maximal, so $|f_{j^*}| \geq \frac{1}{n}$. Let $a \in \overline{A}$ minimize the j^*-th coordinate if $f_{j^*} > 0$, and maximize it if $f_{j^*} < 0$.
  - Then the j^* term vanishes, and every other term is at most $|f_j| w_j \leq |f_j|$. So $f(a) - \max_Q f \leq (1 - \frac{1}{n}) - c = 0$. This is a contradiction.

3.2 Problem 2. $X = a\mathbb{Z}^n \subset \ell_\infty^n$

- $d_H(\ell_\infty^n, X) = \frac{a}{2}$.
  - Rounding each coordinate, $z_j = \operatorname{round}(v_j / a)$, gives $\|v - az\|_\infty \leq \frac{a}{2}$ for every v.
  - The point $v = (\frac{a}{2}, \dots, \frac{a}{2})$ satisfies $|\frac{a}{2} - ak| \geq \frac{a}{2}$ for every integer k, so $\operatorname{dist}(v, X) = \frac{a}{2}$.
  - Since $X \subset V$, we get $d_H = \sup_v \operatorname{dist}(v, X) = \frac{a}{2}$.

- **Upper bound.** $d_{GH}(\ell_\infty^n, X) \le d_H = \frac{a}{2}$.
- **Lower bound.** ℓ_∞^n has the intersection property (sheet), so J_s may be replaced by $J(\ell_\infty^n) = \frac{1}{2}$:

$$d_{GH}(\ell_\infty^n, X) \ge \frac{d_H(\ell_\infty^n, X)}{2J(\ell_\infty^n)} = \frac{a/2}{1} = \frac{a}{2}.$$

Therefore

$$\boxed{d_{GH}(\ell_\infty^n, a\mathbb{Z}^n) = d_H(\ell_\infty^n, a\mathbb{Z}^n) = \tfrac{a}{2}.}$$

Independent check of the lower bound. Ultrametrization is 1-Lipschitz for d_{GH} (see Section E of Solutions_Seminar_Partitions_MST_Jung.md). Chains with small steps join any two points of ℓ_∞^n, so $\mathcal{U}(\ell_\infty^n)$ is a point. In $a\mathbb{Z}_\infty^n$, distinct points are at distance $\ge a$ and are joined by steps of length exactly a, so $\mathcal{U}(a\mathbb{Z}^n) = a\Delta_{\aleph_0}$. Hence

$$d_{GH} \ge d_{GH}(\Delta_1, a\Delta_{\aleph_0}) = \tfrac{1}{2}\operatorname{diam}(a\Delta_{\aleph_0}) = \tfrac{a}{2}.$$

3.3 Problem 3. $d_{GH}(X, V) < \infty \iff d_H(X, V) < \infty$

(a) V finite-dimensional, $\varnothing \ne X \subset V$. - ($\Leftarrow$) $d_{GH}(X, V) \le d_H(X, V)$. This holds in every metric space: use the identity embedding. - ($\Rightarrow$) By the sheet's inequality, $d_H(X, V) \le 2J_s(V)\, d_{GH}(X, V)$. Moreover $J_s(V) \le 1$: for any bounded A and any $a_0 \in A \subset \operatorname{conv} A$, $\sup_{a \in A} \|a_0 - a\| \le \operatorname{diam} A$. (And $J_s \ge J \ge \frac{1}{2} > 0$, so the division on the sheet is legitimate.) Hence $d_H(X, V) \le 2\, d_{GH}(X, V) < \infty$.

(b) Infinite-dimensional V: not equivalent in general. The implication $\Leftarrow$ still holds. The implication $\Rightarrow$ can fail. - Take $V = \ell^2$ and $X = \{x \in \ell^2 :\ x_1 = 0\}$. The right shift $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ is a linear isometry of V onto X, so $d_{GH}(X, V) = 0$. - But $\operatorname{dist}(te_1, X) = t$ for every $t > 0$, so $d_H(X, V) = \infty$. - The same works in ℓ^p $(1 \le p \le \infty)$ and c_0.

The finite-dimensional argument breaks because in infinite dimensions there is no inequality $d_H \le C\, d_{GH}$ for subsets: a proper subspace can be isometric to the whole space.