Problem Sheet: Lectures 3–4

Alexey A. Tuzhilin's minicourse — Lectures 3–4: problems and theory

Intersection property

Write $U_r(x) = \{v \in V : \|v - x\| < r\}$ and $B_t(a) = \{v \in V : \|v - a\| \le t\}$. A finite-dimensional normed space V has the intersection property (with origin 0) if for every $\rho > 0$ there is $t > 0$ such that, whenever $t' > t$ and $r_i \le \rho$, the set

$$(V \setminus B_{t'}(0)) \cap \bigcap_{i=1}^{m} U_{r_i}(x_i)$$

is empty or contractible. Contractible spaces are connected.

1. Prove directly that $\mathbb{R}$ with its usual norm has the intersection property. Give an explicit admissible threshold $t = t(\rho)$ and use that an intersection of finitely many open intervals is an interval.

2. Show that the definition is independent of the chosen origin: if it holds with the deleted balls centered at 0, then it also holds with $B_{t'}(a)$ in place of $B_{t'}(0)$ for every fixed $a \in V$. Track how the threshold changes.

3. A normed space satisfies the convex cone property if for any closed ball $B = \{x : \|x - x_B\| \le r_B\}$ and open ball $A = \{x : \|x - x_A\| < r_A\}$, with $r_A < r_B$, such that the difference $A \setminus B$ is non-empty, the cone of directions (direction of a ray is a corresponding unit vector) from the center of B through the set $A \setminus B$ is convex. Prove that a normed space satisfying the convex cone property, satisfies a finite intersection property.

Simplicial complexes

For a metric space X and $r > 0$, the Vietoris–Rips complex $\mathrm{VR}_r(X)$ has vertex set X and contains a finite set σ exactly when $\operatorname{diam}\sigma < r$. A clique complex of a graph is the simplicial complex whose simplices are precisely the finite vertex sets in which every pair is joined by an edge. Thus $\mathrm{VR}_r(X)$ is the clique complex of the graph with edges between pairs at distance $< r$. A simplicial complex is called a flag complex if every finite set of vertices that are pairwise joined by edges is a simplex; equivalently, it is the clique complex of its 1-skeleton.
If $X \subset Z$, the external Čech complex $\check{C}_r(X; Z)$ contains σ exactly when some $z \in Z$ satisfies $d(z, x) < r$ for every $x \in \sigma$. All inequalities are strict.

1. Let P_m be the vertices of a regular m-gon on the unit circle with the chord metric. For $m = 4, 5$, determine every value of r at which the combinatorial type of $\mathrm{VR}_r(P_m)$ changes, and describe the complex on each interval between consecutive critical values.

2. Prove that the factor 2 in $\check{C}_r(X; Z) \subset \mathrm{VR}_{2r}(X)$ is sharp: for every $c < 2$, construct $r > 0$, a geodesic space Z, and a two-point set $X \subset Z$ that is a simplex of $\check{C}_r(X; Z)$ but not of $\mathrm{VR}_{cr}(X)$.

3. Let Z be a metric space, let $X \subseteq Z$, and let $r > 2 \cdot d_{\mathrm{GH}}(X, Z)$. Then for any $\varepsilon > 0$, there exist simplicial maps

$$\check{C}(Z, \varepsilon) \xrightarrow{\overline{h}} \check{C}(X, r + \varepsilon) \xrightarrow{\overline{g}} \check{C}(Z, 3r + 2\varepsilon)$$

such that the composition $\overline{g} \circ \overline{h}$ is contiguous to the inclusion $\iota : \check{C}(Z, \varepsilon) \hookrightarrow \check{C}(Z, 3r + 2\varepsilon)$, where all complexes are ambient Čech complexes using balls in Z.

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Jung constant and its variations

For bounded nonempty $A \subset V$, set

$$R(A) = \inf_{v \in V} \sup_{a \in A} \|v - a\|, \qquad R_{\operatorname{conv} A}(A) = \inf_{v \in \operatorname{conv} A} \sup_{a \in A} \|v - a\|.$$

The Jung and relative Jung constants are the suprema of these quantities divided by $\operatorname{diam} A$. In ℓ_∞^n, an axis-parallel box containing A has Also acoordinate ranges $M_j - m_j$, and its smallest enclosing ℓ_∞-ball has radius $\frac{1}{2}\max_j(M_j - m_j)$; hence $J(\ell_\infty^n) = 1/2$, while $J_s(\ell_\infty^n) = (n-1)/n$. If $X \subset V$, then $d_{GH}(V, X) \le d_H(V, X)$ and

$$d_{GH}(V, X) \ge \frac{d_H(V, X)}{2J_s(V)}.$$

If V has the intersection property, $J_s(V)$ may be replaced by $J(V)$. The spaces ℓ_∞^n and Euclidean spaces have this property; for the Euclidean plane, $J = 1/\sqrt{3}$.

1. In $V = \ell_\infty^n$, $n \ge 3$, let $A = \{e_1, \dots, e_n\}$ (first n vectors of the standard basis). Compute $\operatorname{diam} A$, $R(A)$, and $R_{\operatorname{conv} A}(A)$. Prove the last value by minimizing over $\operatorname{conv} A = \{x_i \ge 0 : \sum x_i = 1\}$, and show that this set realizes both stated Jung constants.

2. Let $X = a\mathbb{Z}^n \subset \ell_\infty^n$, where $a > 0$. Compute $d_H(\ell_\infty^n, X)$ exactly. Then use both GH inequalities and $J(\ell_\infty^n) = 1/2$ to determine $d_{GH}(\ell_\infty^n, X)$.

3. (a) Using the inequalities between the Hausdorff and the Gromov-Hausdorff distances, prove that for a subset X of an arbitrary finite-dimensional normed space V, the following conditions are equivalent: $d_{\mathrm{GH}}(X, V) < \infty \iff d_{\mathrm{H}}(X, V) < \infty$.
(b) Are these conditions equivalent if V is infinite-dimensional?