Chapter 3

Topological methods in GH-distance estimates

In this chapter we outline the topological tools needed to obtain estimates of the Gromov–Hausdorff distance. The main idea is to construct simplicial complexes (Čech and Vietoris–Rips) from a metric space, whose homology reflects the topology of the space. Then we establish that spaces that are close in the GH sense have close complexes, which allows us to transfer topological invariants from one space to another. At the end of the chapter we prove a lemma about the connection between the Čech and Vietoris–Rips complexes through the Jung constant defined in Chapter 2.

The material in this chapter is based on the work of [9] and [8]. The exposition of the basics of algebraic topology follows the monographs [4] and [6].

3.1 Simplicial complexes and their homology

Let us recall the necessary concepts of algebraic topology.

Definition 3.1. An ***abstract simplicial complex*** is a pair (V, Σ), where V is a set of vertices, and $\Sigma \subset 2^V$ is a family of non-empty finite subsets of V, called ***simplices***, such that if $\sigma \in \Sigma$ and $\tau \subset \sigma$, $\tau \neq \varnothing$, then $\tau \in \Sigma$. A simplex consisting of $k+1$ vertices is called ***k-dimensional***; if at the same time $\tau \subset \sigma$, $\tau \neq \varnothing$, then the simplex τ is called ***face of the simplex*** σ.

Definition 3.2. Let Σ be an abstract simplicial complex on the vertex set V. Consider the vector space $\mathbb{R}^V$ of all functions $f\colon V \to \mathbb{R}$ with pointwise operations. The ***geometric realization*** $|\Sigma|$ is a subset in $\mathbb{R}^V$ consisting of functions f such that

$$f(v) \geq 0 \text{ for all } v \in V, \qquad \sum_{v \in V} f(v) = 1,$$
$$\operatorname{supp} f := \big\{v \in V : f(v) > 0\big\} \in \Sigma.$$

Note that the sum in the condition contains only a finite number of non-zero terms.

In other words, the realization point is a convex combination of vertices of some simplex of the complex, written by its barycentric coordinates $f(v)$; for the simplex $\sigma = \{v_0, \dots, v_k\}$, $|\sigma|$ denotes the set of f such that $\operatorname{supp} f \subset \sigma$; the mapping $f \mapsto \big(f(v_0), \dots, f(v_k)\big)$ defines a bijection from $|\sigma|$ onto the standard k-simplex

$$\Delta^k = \big\{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : t_i \geq 0,\ \sum_{i=0}^{k} t_i = 1\big\}.$$

At the same time $|\tau| \subset |\sigma|$ for $\tau \subset \sigma$, so $|\Sigma| = \bigcup_{\sigma \in \Sigma} |\sigma|$.

Remark 3.3. On $|\Sigma|$ we consider ***weak topology***, that is, the CW topology defined by the family of subspaces $|\sigma|$. Like any CW-complex, it is determined by the condition on ***closed*** sets: a subset $A \subset |\Sigma|$ is closed if and only if each of the sets $A \cap |\sigma|$ is closed in $|\sigma|$.

The condition for open sets is obtained from here by passing to complements: a subset $U \subset |\Sigma|$ is open if and only if each of the sets $U \cap |\sigma|$ is open in $|\sigma|$. In other words, the cells of the complex are realizations of simplexes, and the gluing maps are embeddings of the faces $|\tau| \hookrightarrow |\sigma|$. Continuity of the mapping $g\colon |\Sigma| \to Z$ is also convenient to check by cells: g is continuous if and only if all its restrictions $g|_{|\sigma|}$ are continuous, that is, if and only if for every closed $B \subset Z$ and every simplex σ the set $g^{-1}(B) \cap |\sigma|$ is closed in $|\sigma|$.

Note that the weak topology coincides with the subspace topology induced from $\mathbb{R}^V$ with the product topology, that is, the topology of pointwise convergence, for locally finite complexes: it is sufficient that each simplex is contained in only a finite number of simplices, as is the case in a finite complex. For non-locally finite complexes, the weak topology may differ from that induced from $\mathbb{R}^V$. All complexes encountered in this course are finite, so for them both descriptions of topology are equivalent.

Definition 3.4. Let $\sigma = \{v_0, \dots, v_k\}$ be a simplex of dimension $k \geq 1$. An ***orientation*** of a simplex σ is the equivalence class of its orderings $(v_0, \dots, v_k)$, in which two orderings $(v_0, \dots, v_k)$ and $(v_{\pi(0)}, \dots, v_{\pi(k)})$ are considered equivalent if the permutation π even. The notation $[v_0, \dots, v_k]$ denotes the simplex σ together with the orientation given by the ordering $(v_0, \dots, v_k)$: the ordering $(v_{\pi(0)}, \dots, v_{\pi(k)})$ specifies the same orientation if the permutation π is even, and the opposite if π is odd. In other words, the symbol $[v_0, \dots, v_k]$ is defined up to even permutations of its indices.

Remark 3.5. A simplex of dimension $k \geq 1$ has exactly two orientations: even orderings of vertices form one equivalence class, odd ones form another, and any two orderings from different classes differ in the transposition of two vertices and therefore define opposite orientations. Geometrically, orientation is the choice of the direction of traversal of the simplex: for a segment $\{a, b\}$ this is the choice of the beginning and end, and then the orientations of $[a, b]$ and $[b, a]$ are opposite, and for a triangle it is the choice of the order of traversal of its vertices, say, clockwise or counterclockwise. The zero-dimensional simplex $\{v\}$ has only one ordering, so it has exactly one orientation.

Definition 3.6. Before defining chains, recall that in any Abelian group H multiplication by integers is defined: for $n \geq 0$ we put $nh = h + \dots + h$ (n terms), and for $n < 0$ we put $nh = -\big(|n|h\big)$, where $-x$ denotes the only element of the group for which $x + (-x) = 0$. In particular, $1 \cdot h = h$ and $(-1) \cdot h = -h$: one acts identically, and minus one transforms the element into the opposite. Thus, every Abelian group is a module over the ring $\mathbb{Z}$, and the numbers 1 and -1 are not elements of the group H, but integers acting on its elements in the indicated way. This is precisely the action that is meant throughout below when an integer is multiplied by an element of a group. Now let Σ be an abstract simplicial complex, and G be an Abelian group of coefficients, written additively (for example, $G = \mathbb{Z}$ or $G = \mathbb{Z}/2$). ***A group of k-dimensional chains*** $C_k(\Sigma; G)$ is an Abelian group defined by generators and relations. The generators are the formal symbols $g[v_0, \dots, v_k]$, where g runs through the group G, and $\{v_0, \dots, v_k\}$ are all possible k-simplices of the complex Σ with all possible orientations of their vertices. Relations are equalities

$$(g_1 + g_2)[v_0, \dots, v_k] = g_1[v_0, \dots, v_k] + g_2[v_0, \dots, v_k],$$

$$g[v_{\pi(0)}, \dots, v_{\pi(k)}] = \operatorname{sgn}(\pi)\, g[v_0, \dots, v_k]$$

for all $g, g_1, g_2 \in G$ and all permutations π of the vertices of the simplex. In the second of these, $\operatorname{sgn}(\pi) = \pm 1$ is an integer multiplying the element of the free Abelian group generated

by all the symbols $g[v_0, \dots, v_k]$ (such a multiplication is defined as explained above); if the permutation π is even, both sides of this equality are the same element of the group, and if the permutation is odd, they are opposite elements. The number $\operatorname{sgn}(\pi)$ is not an element of the group G, so no unit in G is required to write these relations. We emphasize that the symbol $[v_0, \dots, v_k]$ without a coefficient is not an element of the group $C_k(\Sigma; G)$: the elements are the symbols $g[v_0, \dots, v_k]$ and their sums, so, for example, $0_G[v_0, \dots, v_k]$ is the zero of the group. The notation $[v_0, \dots, v_k]$ is used below only as a shorthand within expressions in which the coefficient is already specified. It follows from the relations that if in each k-simplex σ we choose one orientation and denote the symbol corresponding to it by $[\sigma]$, then each element of the group $C_k(\Sigma; G)$ is uniquely written in the form of a finite sum $\sum_\sigma g_\sigma[\sigma]$, where $g_\sigma \in G$. In other words, $C_k(\Sigma; G)$ is isomorphic to the direct sum of copies of the group G, numbered by k-simplices of the complex Σ: each simplex corresponds to exactly one term, and replacing the chosen orientation with the opposite one replaces the corresponding coefficient g_σ with $-g_\sigma$ and does not change the chain itself. For $k = 0$, the ordering of the vertex is unique, the second relation does not arise, and $C_0(\Sigma; G)$ is the direct sum of copies of the group G, numbered by vertices. For $G = \mathbb{Z}$ the group $C_k(\Sigma; \mathbb{Z})$ is free commutative, and its rank is equal to the number of k-simplices of the complex Σ.

Remark 3.7. We emphasize that two orientations of one simplex do not give two independent terms of the group $C_k(\Sigma; G)$. Indeed, for an odd permutation π the elements $g[v_{\pi(0)}, \dots, v_{\pi(k)}]$ and $g[v_0, \dots, v_k]$ are opposite, so each k-simplex corresponds to exactly one copy of the group G in the direct sum decomposition, and not two. In other words, the choice of a specific orientation of a simplex is only a choice of notation for the corresponding term: when the orientation changes, all the coefficients of this simplex change sign. Note that everything essentially depends on the group of coefficients: if $G = \mathbb{Z}/2$, then $-g = g$ for any $g \in G$, so changing the orientation does not change the elements of the group at all, and $C_k(\Sigma; \mathbb{Z}/2)$ is a vector space over $\mathbb{Z}/2$ with a basis of unoriented k-simplices.

Definition 3.8. The ***boundary operator*** $\partial_k \colon C_k(\Sigma; G) \to C_{k-1}(\Sigma; G)$ is given on the generators by the formula

$$\partial_k\big(g[v_0, \dots, v_k]\big) = \sum_{i=0}^{k} (-1)^i g[v_0, \dots, \widehat{v_i}, \dots, v_k], \qquad g \in G,$$

where $\widehat{v_i}$ means that the vertex v_i is missing, and the factors $(-1)^i$ are integers acting on the elements of the group $C_{k-1}(\Sigma; G)$ as explained in Definition 3.6. The formula defines a homomorphism of groups: it defines a homomorphism of a free Abelian group generated by symbols, which vanishes on both families of relations from Definition 3.6 — when permuting π vertices of a simplex, both sides of the equality are multiplied by the integer $\operatorname{sgn}(\pi)$, and by the coefficient g the right-hand side is additive.

Lemma 3.9. *For each k the composition of two consecutive boundary operators is zero*:

$$\partial_{k-1} \circ \partial_k = 0.$$

Proof. The operators ∂_k and ∂_{k-1} are homomorphisms of Abelian groups, so it suffices to verify the equality $(\partial_{k-1} \circ \partial_k)(g[v_0, \dots, v_k]) = 0$ on the generators of the group $C_k(\Sigma; G)$, where $g \in G$ and $\{v_0, \dots, v_k\}$ is a k-simplex of the complex Σ with a chosen orientation of its vertices. Applying the formula from Definition 3.8 twice, we obtain

$$\partial_{k-1}\partial_k\big(g[v_0, \dots, v_k]\big) = \sum_{i<j} \Big((-1)^i (-1)^{j-1} + (-1)^j (-1)^i \Big) g[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k],$$

where the summation is over all pairs of indices $i < j$ from the set $\{0, \dots, k\}$. For a fixed pair $i < j$ the simplex $[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k]$ occurs in the double sum exactly twice: when the vertex v_i is removed first and the vertex v_j is removed afterwards, it enters with the coefficient $(-1)^i(-1)^{j-1}$, and when the vertex v_j is removed first and the vertex v_i is removed afterwards, with the coefficient $(-1)^j(-1)^i$. The exponents in these two integers differ by one, so the integers themselves are opposite, and the contribution of the pair $i < j$ is zero. Consequently, the whole double sum is zero, that is, $(\partial_{k-1} \circ \partial_k)(g[v_0, \dots, v_k]) = 0$. Since the equality is verified on all generators of the group $C_k(\Sigma; G)$, it holds for all its elements. □

It follows from the lemma just proved that the image $\operatorname{im} \partial_{k+1}$ of the boundary operator ∂_{k+1} is contained in the kernel $\ker \partial_k$. Indeed, for any chain $c \in C_{k+1}(\Sigma; G)$ its image $\partial_{k+1}c$ lies in $\ker \partial_k$, since $\partial_k(\partial_{k+1}c) = (\partial_k \circ \partial_{k+1})c = 0$. The group $C_k(\Sigma; G)$ is Abelian, so the subgroup $\operatorname{im} \partial_{k+1}$ is normal in $\ker \partial_k$, and the quotient group $\ker \partial_k / \operatorname{im} \partial_{k+1}$ is defined.

Definition 3.10. The ***k-dimensional homology group*** of a complex Σ with coefficients in G is defined as

$$H_k(\Sigma; G) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

The elements of $\ker \partial_k$ are called ***cycles***, the elements of $\operatorname{im} \partial_{k+1}$ are called ***boundaries***.

Known fact 3.11 (Homotopy invariance of homology)**.** Continuous mapping $f\colon |\Sigma| \to |\Delta|$ induces a homomorphism $f_*\colon H_k(\Sigma; G) \to H_k(\Delta; G)$ for each k. If $f, g\colon |\Sigma| \to |\Delta|$ are homotopic, then $f_* = g_*$. *Source*: [4, Chapter 2, Theorem 2.10].

3.2 Čech complex and covering nerve

Definition 3.12. Let $\{U_\alpha\}_{\alpha \in A}$ be a family of subsets of the topological space Y. The ***nerve*** of this family is an abstract simplicial complex $\mathcal{N}(\{U_\alpha\})$ with a set of vertices A, in which $\{\alpha_0, \dots, \alpha_k\} \subset A$ is a simplex if and only if $\bigcap_{i=0}^k U_{\alpha_i} \neq \varnothing$.

Definition 3.13. Let Z be a metric space, $X \subset Z$ a subset, and $r > 0$. The ***external Čech complex*** $\check{C}_r(X; Z)$ is an abstract simplicial complex with a set of vertices X, in which the set $\{x_0, \dots, x_k\} \subset X$ is a simplex if and only if there exists a point $z \in Z$ such that $|zx_i|_Z < r$ for all $i = 0, \dots, k$.

In other words, $\check{C}_r(X; Z)$ is the nerve of the family of open balls $\big\{U_r(x)\big\}_{x \in X}$ in the space Z.

Definition 3.14. An open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a topological space Y is called ***good*** if any finite non-empty intersection $\bigcap_{i=1}^k U_{\alpha_i}$ is contractible.

Recall that a family $\{W_\gamma\}_{\gamma \in \Gamma}$ of subsets of a space Y is called ***locally finite*** if every point $y \in Y$ has a neighbourhood intersecting only finitely many elements of this family. A topological space Y is called ***paracompact*** if every open covering $\{V_\beta\}_{\beta \in B}$ of Y admits a locally finite open refinement, that is, there exist a locally finite family of open sets $\{W_\gamma\}_{\gamma \in \Gamma}$ and a mapping $\varphi\colon \Gamma \to B$ such that $Y = \bigcup_{\gamma \in \Gamma} W_\gamma$ and $W_\gamma \subset V_{\varphi(\gamma)}$ for all $\gamma \in \Gamma$.

Known fact 3.15 (Nerve Lemma)**.** Let Y be a paracompact topological space, and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a good open covering of Y. Then the nerve $\mathcal{N}(\mathcal{U})$ is homotopy equivalent to Y. *Source*: [4, Corollary 4G.3].

Remark 3.16. Every metric space is paracompact. This fact was proven in the work of [7, Corollary 1]. Therefore, the nerve lemma applies to any metric space.

3.3 Vietoris–Rips complex

Definition 3.17. The ***Vietoris–Rips complex*** $\mathrm{VR}(X,r)$ of a metric space X for $r>0$ is an abstract simplicial complex with a set of vertices X, in which $\{x_0,\dots,x_k\}\subset X$ is a simplex if and only if $\operatorname{diam}\{x_0,\dots,x_k\}<r$.

Lemma 3.18. *For any metric space Z, its subset $X\subset Z$ and any $r>0$ there are inclusions*:

$$\mathrm{VR}(X,r)\xrightarrow{\iota_1}\check{C}_r(X;Z)\xrightarrow{\iota_2}\mathrm{VR}(X,2r).$$

Proof. Let us prove the first inclusion. Let $\sigma=\{x_0,\dots,x_k\}\in\mathrm{VR}(X,r)$. By Definition 3.17, $\operatorname{diam}\sigma<r$. For any point $x_i\in\sigma$ we have $|x_ix_i|_Z=0<r$. Consequently, there is a point $x_i\in X$ such that $|x_ix_j|_Z<r$ for all j. By Definition 3.13, this means that $\sigma\in\check{C}_r(X;Z)$.

Let us prove the second inclusion. Let $\sigma=\{x_0,\dots,x_k\}\in\check{C}_r(X;Z)$. Then there exists a point $z\in Z$ such that $|zx_i|_Z<r$ for all i. For any i,j we apply the triangle inequality in Z:

$$|x_ix_j|_Z\le|x_iz|_Z+|zx_j|_Z<r+r=2r.$$

Therefore, $\operatorname{diam}\sigma<2r$. By Definition 3.17, this means that $\sigma\in\mathrm{VR}(X,2r)$. □

3.4 Proximity of the Vietoris–Rips complexes

The following lemma is the key result of this chapter. It shows that spaces that are close in the GH sense have close Vietoris–Rips complexes.

Definition 3.19. Two simplicial maps $f,g\colon\Sigma\to\Delta$ are called ***close*** (contiguous) if for any simplex $\sigma\in\Sigma$ the set $f(\sigma)\cup g(\sigma)$ is contained in some simplex of the complex Δ.

Known fact 3.20. Close simplicial mappings induce homotopic mappings on geometric realizations and, therefore, identical mappings in homology. *Source*: [9, section 2].

Lemma 3.21 ([49, Lemma 5.2])**.** *Let $X,Y\in\mathcal{M}$ and $2\,d_{GH}(X,Y)<r$. By the description of the Gromov–Hausdorff distance via correspondences, there exists a correspondence $R\in\mathcal{R}(X,Y)$ with $\operatorname{dis}R<r$. Then for any such correspondence R, any mappings $h\colon Y\to X$ and $g\colon X\to Y$ satisfying*

$$\big(h(y),y\big)\in R\text{ for all }y\in Y,\qquad\big(x,g(x)\big)\in R\text{ for all }x\in X,$$

and any $\varepsilon>0$, $\nu\ge0$, the mappings h and g induce simplicial mappings

$$\mathrm{VR}(Y,\varepsilon)\xrightarrow{\bar h}\mathrm{VR}(X,r+\varepsilon)\xrightarrow{\iota_\nu}\mathrm{VR}(X,r+\varepsilon+\nu)\xrightarrow{\bar g}\mathrm{VR}(Y,2r+\varepsilon+\nu),$$

where ι_ν is the inclusion, and the composition $\bar g\circ\iota_\nu\circ\bar h$ is close to the inclusion $\iota\colon\mathrm{VR}(Y,\varepsilon)\hookrightarrow\mathrm{VR}(Y,2r+\varepsilon+\nu)$.

Proof. Step 1. *Proof that h is a simplicial mapping.* Let $\sigma=\{y_0,\dots,y_k\}\in\mathrm{VR}(Y,\varepsilon)$. By Definition 3.17, $\operatorname{diam}\sigma<\varepsilon$. Since $\operatorname{dis}R<r$, for any i,j we have:

$$\big|h(y_i)h(y_j)\big|_X\le|y_iy_j|_Y+\operatorname{dis}R<\varepsilon+r.$$

Therefore, $\operatorname{diam}h(\sigma)<r+\varepsilon$, which means $h(\sigma)\in\mathrm{VR}(X,r+\varepsilon)$.

Step 2. *Proof that g is a simplicial mapping.* Let $\tau=\{x_0,\dots,x_k\}\in\mathrm{VR}(X,r+\varepsilon+\nu)$. By Definition 3.17, $\operatorname{diam}\tau<r+\varepsilon+\nu$. Since $\operatorname{dis}R<r$, for any i,j we have:

$$\big|g(x_i)g(x_j)\big|_Y\le|x_ix_j|_X+\operatorname{dis}R<(r+\varepsilon+\nu)+r=2r+\varepsilon+\nu.$$

Therefore, $\operatorname{diam} g(\tau) < 2r + \varepsilon + \nu$, which means $g(\tau) \in \mathrm{VR}(Y, 2r + \varepsilon + \nu)$.

Step 3. *Proof of the closeness of the composition.* Let us show that $\bar{g} \circ \iota_\nu \circ \bar{h}$ is close to ι. Let $\sigma = \{y_0, \dots, y_k\} \in \mathrm{VR}(Y, \varepsilon)$. Recall that $\big(h(y), y\big) \in R$ for all $y \in Y$ and $\big(x, g(x)\big) \in R$ for all $x \in X$. In particular, for any i and j both pairs

$$\big(h(y_i), y_i\big), \qquad \big(h(y_j), g(h(y_j))\big)$$

belong to R: the first by the choice of the mapping h, the second by the choice of the mapping g, since $h(y_j) \in X$. Let us apply the definition of distortion to these two pairs:

$$\Big| \big|h(y_i)h(y_j)\big|_X - \big|y_i\,(g \circ h)(y_j)\big|_Y \Big| \leq \operatorname{dis} R < r.$$

From here

$$\big|y_i\,(g \circ h)(y_j)\big|_Y < \big|h(y_i)h(y_j)\big|_X + r.$$

Let us evaluate the first term. The pairs $\big(h(y_i), y_i\big)$ and $\big(h(y_j), y_j\big)$ also lie in R, so by definition of distortion

$$\big|h(y_i)h(y_j)\big|_X < |y_i y_j|_Y + r < \varepsilon + r,$$

since $\operatorname{diam} \sigma < \varepsilon$. Therefore,

$$\big|y_i\,(g \circ h)(y_j)\big|_Y < (\varepsilon + r) + r = \varepsilon + 2r.$$

Finally, applying the definition of distortion to the pairs $\big(h(y_i), g(h(y_i))\big)$ and $\big(h(y_j), g(h(y_j))\big)$, which also lie in R, we obtain

$$\big|g(h(y_i))g(h(y_j))\big|_Y < \big|h(y_i)h(y_j)\big|_X + r < \varepsilon + 2r.$$

So, pairwise distances between points from σ are less than ε, and all other pairwise distances in the set $\sigma \cup (g \circ h)(\sigma)$ are less than $\varepsilon + 2r$. This means that for any $\nu \geq 0$ this set is contained in some simplex of the complex $\mathrm{VR}(Y, 2r + \varepsilon + \nu)$. Therefore, the mappings are close. □

3.5 Jung's constant and the connection of complexes

In this section we establish a connection between the Čech and Vietoris–Rips complexes through Jung's constant. Recall that the radius of the circumscribed ball and the Jung constant of the metric space are defined in Chapter 2 (Definitions 2.19 and 2.20), and the estimate $1/2 \leq J(Z) \leq 1$ is given in Remark 2.21.

Lemma 3.22 ([49, Lemma 5.3])**.** *For any metric space Z, its nonempty subset X, and positive numbers $s, r > 0$:*

(1) *every simplex of the complex* $\mathrm{VR}(X, s)$ *belongs to* $\check{C}_{J(Z)s}(X; Z)$;

(2) *every simplex of the complex* $\check{C}_r(X; Z)$ *belongs to* $\mathrm{VR}(X, 2r)$.

Proof. Item 1. Let $\sigma = \{x_0, \dots, x_k\} \in \mathrm{VR}(X, s)$. By Definition 3.17, $\operatorname{diam} \sigma < s$. By the definition of Jung's constant (Definition 2.20), there is a point $z \in Z$ such that $\sup_{x \in \sigma} |zx|_Z \leq J(Z) \cdot \operatorname{diam} \sigma < J(Z)s$. Therefore, by Definition 3.13, $\sigma \in \check{C}_{J(Z)s}(X; Z)$.

Point 2. This is exactly Lemma 3.18. □

Corollary 3.23 ([49, Lemma 5.3])**.** *For any metric space Z and its subset X there are inclusions*:

$$\mathrm{VR}(X, s) \xrightarrow{\iota} \check{C}_{J(Z)s}(X; Z) \xrightarrow{\iota'} \mathrm{VR}\big(X, 2J(Z)s\big).$$

Proof. This is an immediate consequence of Lemma 3.22: the first inclusion is obtained from item 1, the second from item 2 with $r = J(Z)s$. □

3.6 Higher homology of connected manifolds

The following fact will be used in Chapter 4 to obtain a contradiction in homology.

Known fact 3.24. Let M be a connected n-dimensional manifold without boundary. Then its n-dimensional homology with coefficients in $\mathbb{Z}_2$ is equal to

$$H_n(M;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & \text{if } M \text{ compact,} \\ 0, & \text{if } M \text{ is not compact.} \end{cases}$$

Source: [4, Theorem 3.26] (compact case), [4, Proposition 3.29] (non-compact case).

Corollary 3.25. *Let M be a connected closed n-dimensional manifold, and $U \subset M$ be any proper open subset. Then $H_n(U;\mathbb{Z}_2) = 0$.*

Proof. Let us decompose U into the union of all its connected components. Each component of C is open in M as a component of an open subset of the manifold, so C is a connected n-dimensional manifold without boundary. The image of any singular simplex $\Delta^n \to U$ is connected and therefore lies entirely in one component, which means that groups of chains are decomposed into a direct sum over the components: $C_\bullet(U;\mathbb{Z}_2) = \bigoplus_C C_\bullet(C;\mathbb{Z}_2)$, whence

$$H_n(U;\mathbb{Z}_2) = \bigoplus_C H_n(C;\mathbb{Z}_2).$$

Each component of C is ***non-compact***: if a component C_i of C were compact, then it would be closed in the Hausdorff M and at the same time open, that is, clopen; in a connected M this would give $C_i = M$, which means $M \subset U$ is a contradiction with the properness of U. It remains to apply to each component the non-compact case of Fact 3.24: $H_n(C;\mathbb{Z}_2) = 0$, and the direct sum of zeros is zero. For $U = \varnothing$ the statement is trivial. □

Chapter Summary

This chapter proves:

(1) definitions of a simplicial complex, geometric realization and topology on it, orientation of a simplex, chain groups, boundary operator and homology (Definitions 3.1, 3.2, 3.3, 3.4, 3.6, 3.8, 3.10);

(2) definitions of the Čech and Vietoris–Rips complexes (Definitions 3.13, 3.17);

(3) connection between the Čech and Vietoris–Rips complexes (Lemma 3.18);

(4) closeness of Vietoris–Rips complexes for spaces that are close in the GH sense (Lemma 3.21);

(5) connection of complexes through the Jung constant (Lemma 3.22, Corollary 3.23).

(6) vanishing of the top homology of a proper open subset of a manifold (Corollary 3.25).

Accepted as a known fact: homotopy invariance of homology ([4, Chapter 2, Theorem 2.10]); nerve lemma (Leray) ([4, Corollary 4G.3]); paracompactness of metric spaces ([7, Corollary 1]); close simplicial mappings ([9, section 2]); higher homology of connected manifolds ([4, Theorem 3.26, Proposition 3.29]). Left out of scope: sharpness of the estimates in terms of homology.

Chapter 4

Estimation of the GH-distance to a normed space through Jung's constant

In this chapter we consider the following problem.

Let V be a finite-dimensional normed space, and let $X \subseteq V$ be a non-empty subset for which the Hausdorff distance $d_H(V, X)$ is finite. It is required to estimate the Gromov–Hausdorff distance $d_{GH}(V, X)$ in terms of $d_H(V, X)$ and geometric characteristics of V.

We will obtain two estimates of the Gromov–Hausdorff distance in terms of the Hausdorff distance: a weak one, valid for any finite-dimensional normed space, and a stronger one, valid for spaces with the intersection property (Section 4.4). The exact formulations are given in Theorems 4.10 and 4.20; they involve the Jung constant and the relative Jung constant (Definitions 2.20 and 4.3). As a consequence, for spaces with sup-norm we obtain the exact equality $d_{GH}(V, X) = d_H(V, X)$ (Corollary 4.21).

The material in this chapter is based on the work of [49].

4.1 Problem statement and basic definitions

Definition 4.1. Let V be a finite-dimensional normed space over $\mathbb{R}$ of dimension $n \geq 1$ with norm $\|\cdot\|$. The distance between points $x, y \in V$ is denoted

$$|xy| := \|x - y\|.$$

Let $X \subseteq V$ be a non-empty subset. The Hausdorff distance is assumed to be finite:

$$d_H(V, X) < \infty.$$

Let us recall the standard formula for the Hausdorff distance:

$$d_H(V, X) = \inf\bigl\{r > 0 : V \subseteq U_r(X),\ X \subseteq U_r(V)\bigr\},$$

where $U_r(A) = \bigl\{v \in V : \exists a \in A,\ |va| < r\bigr\}$.

We also need a definition of correspondence distortion and a standard formula for expressing the GH distance in terms of correspondences:

$$d_{GH}(V, X) = \frac{1}{2} \inf\bigl\{\operatorname{dis} R : R \in \mathcal{R}(V, X)\bigr\}.$$

4.2 Jung's constant and relative Jung's constant

In this section, we recall the definition of Jung's constant, and introduce the relative Jung constant and the Gulevich number.

Definition 4.2. Let $A \subseteq V$ be a non-empty bounded subset. As for any metric space (see Definition 2.19), $R(A)$ denotes the ***circumscribed ball radius*** of the set A:

$$R(A) = \inf_{v \in V} \sup_{a \in A} \|v - a\|.$$

If $Y \subseteq V$ is a non-empty subset, then the ***relative circumscribed ball radius*** of the set A relative to Y is the quantity

$$R_Y(A) := \inf_{y \in Y} \sup_{a \in A} \|y - a\|.$$

In particular, $R_V(A) = R(A)$.

Recall that the ***Jung constant*** of a normed space V is the quantity

$$J(V) = \sup_{\substack{A \subseteq V \\ \operatorname{diam} A > 0}} \frac{R(A)}{\operatorname{diam} A}$$

(see Definition 2.20 in Chapter 2).

Definition 4.3. The ***relative Jung constant*** of the normed space V is the quantity

$$J_s(V) := \sup_{\substack{A \subseteq V \\ \operatorname{diam} A > 0}} \frac{R_{\operatorname{conv} A}(A)}{\operatorname{diam} A},$$

where $\operatorname{conv} A$ denotes the convex hull of the set A in V.

Remark 4.4. In the work [51] the quantity $J_s(V)$ was denoted by $J(V)$ and called the Jung constant. However, subsequent work established the distinction between $J(V)$ and $J_s(V)$. We follow this terminology.

For what follows we need the Gulevich number and its connection with the measure of non-convexity.

Definition 4.5. For a non-empty bounded subset $A \subseteq V$ we define ***non-convexity measure***:

$$\lambda(A) := \sup_{c \in \operatorname{conv} A} \inf_{a \in A} \|c - a\|.$$

Definition 4.6. The ***Gulevich number*** of the normed space V is the quantity

$$G(V) := \sup_{\substack{A \subseteq V \\ \operatorname{diam} A > 0}} \frac{\lambda(A)}{\operatorname{diam} A}.$$

Known fact 4.7 (Gulevich's theorem)**.** For any Banach space V the following equality holds:

$$G(V) = J_s(V).$$

Source: [49, Theorem 2.6].

Known fact 4.8 (Estimates of Jung's constant)**.** For any normed space V:

$$\frac{1}{2} \le J_s(V) \le 1, \qquad \frac{1}{2} \le J(V) \le 1.$$

If V is a Hilbert space of dimension n, then

$$J(V) = J_s(V) = \sqrt{\frac{n}{2(n+1)}}.$$

If $V = \mathbb{R}^n$ with norm $\|\cdot\|_\infty$ (sup-norm), then

$$J(V) = \frac{1}{2}, \qquad J_s(V) = \frac{n-1}{n} \quad (n \geq 2).$$

Source: [50] for $J(V)$ in the Euclidean case; [53] for $J_s(V)$ in sup-norm; [52] for general estimates.

Known fact 4.9 (Equality criterion $J(V) = J_s(V)$)**.** In a normed space V the equality $J(V) = J_s(V)$ holds if and only if V is a Hilbert space or $\dim V \leq 2$. *Source*: [54] and [55].

4.3 Weak estimate of GH-distance

In this section we prove a weak estimate that is valid for any finite-dimensional normed space.

Theorem 4.10 ([49, Theorem 3.1])**.** *Let V be a finite-dimensional normed space, and $X \subseteq V$ be a non-empty subset. Then*

$$d_{GH}(V, X) \geq \frac{1}{2J_s(V)} d_H(V, X).$$

For the proof we need several auxiliary lemmas.

4.3.1 Lemma on triangulation

Recall that a ***convex polyhedron*** in $\mathbb{R}^N$ is the convex hull of a finite set of points. ***Polyhedral complex*** (in the geometric sense) is a union of convex polyhedra in the space $\mathbb{R}^N$, generally speaking, infinite-dimensional, such that the intersection of any two polyhedra is either empty or is a face of each of them; this union is called the ***body*** of the complex. We will consider only complexes in finite-dimensional spaces $\mathbb{R}^N$, so the topology on the body of the complex is induced from $\mathbb{R}^N$.

Center of a polyhedron is the arithmetic mean of all its vertices; the center lies in the relative interior of the polyhedron.

We construct the triangulation of the polyhedral complex by analogy with the barycentric subdivision, increasing at each step the dimension of the divided cells.

Step 1. Zero-dimensional faces (points) are not divided, and each one-dimensional face (segment) is divided by its center.

Step 2. Let the partition have already been constructed for all polyhedra of dimension less than d, and on the common face of two polyhedra the partitions coming from them coincide. Then each d-dimensional polytope P is divided into simplices $\operatorname{conv}(b_P \cup T)$, where b_P is the center of P, and T are the simplices into which the proper faces of P are divided. Each such simplex lies in P: the center b_P does not belong to the affine hull of its own face and therefore is affinely independent with the vertices T.

By sorting out the cells of the complex in order of increasing dimension, we obtain a partition of its body into simplexes, and on the common faces of neighboring cells the partitions coincide — simply by construction, since the partition of a face is determined only by the face itself.

Lemma 4.11 ([49, section 3]). *Let V be an n-dimensional normed space. For any $\varepsilon > 0$ there is a triangulation Σ of the space V such that the diameter of each simplex is less than ε.*

Proof. Let us choose a basis $e_1, \dots, e_n$ of the space V such that $\|e_i\| < \varepsilon/n$ for all i: in an arbitrary basis $f_1, \dots, f_n$ it is sufficient to replace each vector f_i by $\lambda_i f_i$ with a suitable small $\lambda_i > 0$.

Consider a lattice

$$\Lambda := \left\{ \sum_{i=1}^{n} m_i e_i : m_i \in \mathbb{Z} \right\}$$

and for each $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ the parallelepiped

$$K_m := \left\{ \sum_{i=1}^{n} (m_i + t_i) e_i : 0 \le t_i \le 1 \right\}.$$

For any $x = \sum_{i=1}^{n} (m_i + t_i) e_i$ and $y = \sum_{i=1}^{n} (m_i + s_i) e_i$ with $0 \le t_i, s_i \le 1$ we have

$$\|x - y\| \le \sum_{i=1}^{n} |t_i - s_i| \cdot \|e_i\| \le \sum_{i=1}^{n} \|e_i\| < n \cdot \frac{\varepsilon}{n} = \varepsilon,$$

that is, $\operatorname{diam} K_m < \varepsilon$.

Parallelepipeds K_m are convex polyhedra, and their family is a polyhedral complex: the intersection of two of them is either empty or a common face of both. The described construction is applicable to each cell K_m, and on the common face of two parallelepipeds the partitions coincide. Therefore, the union of all partitions K_m is a triangulation Σ of the space V, and each of its simplex lies in some K_m, whence $\operatorname{diam} \Delta < \varepsilon$. □

4.3.2 Lemma on continuous extension

Lemma 4.12 ([49, Lemmas 3.2, 3.3]). *Let V be a finite-dimensional normed space, and let $f\colon V \to X \subset V$ be an arbitrary mapping (not necessarily continuous) with $\operatorname{dis} f < C < \infty$. Then for any $\varepsilon > 0$ there is a continuous mapping $f^*\colon V \to V$ such that the following three conditions are satisfied*:

(1) *$f^*(y) = f(y)$ for each vertex y of the triangulation Σ from Lemma 4.11*;

(2) $\operatorname{dis} f^* \le 3C + 4\varepsilon$;

(3) *for any simplex $\Delta \in \Sigma$ we have* $\operatorname{diam} f^*(\Delta) < C + \varepsilon$.

Proof. Let Σ be the triangulation from Lemma 4.11 with $\operatorname{diam} \Delta < \varepsilon$ for all $\Delta \in \Sigma$. Let Y denote the set of vertices of Σ.

Step 1. *Definition of f^* on vertices.* For each vertex $y \in Y$ we put $f^*(y) := f(y)$.

Step 2. *Definition of f^* on simplexes.* For an arbitrary point $v \in V \setminus Y$ we choose the unique simplex $\Delta \in \Sigma$ such that v belongs to the relative interior of Δ. Let $\{y_0, \dots, y_n\} \subset Y$ be the set of Δ vertices. By the definition of the simplex, there are unique barycentric coordinates $\lambda_0, \dots, \lambda_n > 0$ that satisfy the conditions

$$\sum_{i=0}^{n} \lambda_i = 1, \qquad v = \sum_{i=0}^{n} \lambda_i y_i.$$

Let us put

$$f^*(v) := \sum_{i=0}^{n} \lambda_i f(y_i).$$

This definition is correct, since there are only one barycentric coordinates in the simplex. The continuity of f^* follows from the fact that on each simplex Δ the mapping $f^*|_\Delta$ is an affine extension of the values at the vertices. An affine mapping on a compact simplex is continuous, and consistency on the common faces of simplexes is ensured by the uniqueness of barycentric coordinates.

Step 3. *Proof of point* 3. Let $\Delta \in \Sigma$ with vertices $y_0, \dots, y_n$. Let us take arbitrary $v, w \in \Delta$ with barycentric coordinates λ_i and μ_i, respectively:

$$v = \sum_{i=0}^{n} \lambda_i y_i, \qquad w = \sum_{i=0}^{n} \mu_i y_i, \qquad \sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} \mu_i = 1.$$

Let us calculate $\big\|f^*(v) - f^*(w)\big\|$:

$$f^*(v) - f^*(w) = \sum_{i=0}^{n} \lambda_i f(y_i) - \sum_{j=0}^{n} \mu_j f(y_j).$$

Let us transform this expression. Since $\sum_{j=0}^{n} \mu_j = 1$, then

$$\sum_{i=0}^{n} \lambda_i f(y_i) = \left(\sum_{j=0}^{n} \mu_j\right) \sum_{i=0}^{n} \lambda_i f(y_i) = \sum_{i,j} \lambda_i \mu_j f(y_i).$$

Likewise,

$$\sum_{j=0}^{n} \mu_j f(y_j) = \left(\sum_{i=0}^{n} \lambda_i\right) \sum_{j=0}^{n} \mu_j f(y_j) = \sum_{i,j} \lambda_i \mu_j f(y_j).$$

Therefore,

$$f^*(v) - f^*(w) = \sum_{i,j} \lambda_i \mu_j \big(f(y_i) - f(y_j)\big).$$

By the triangle inequality for the norm:

$$\big\|f^*(v) - f^*(w)\big\| \le \sum_{i,j} \lambda_i \mu_j \big\|f(y_i) - f(y_j)\big\|.$$

Since $\operatorname{dis} f < C$ and $\operatorname{diam} \Delta < \varepsilon$, for all i, j we have:

$$\big\|f(y_i) - f(y_j)\big\| \le |y_i y_j| + \operatorname{dis} f < \varepsilon + C.$$

Substituting:

$$\big\|f^*(v) - f^*(w)\big\| < \sum_{i,j} \lambda_i \mu_j (C + \varepsilon) = (C + \varepsilon) \left(\sum_{i=0}^{n} \lambda_i\right) \big(\sum_{j=0}^{n} \mu_j\big) = C + \varepsilon.$$

Thus, $\operatorname{diam} f^*(\Delta) < C + \varepsilon$. Point 3 has been proven.

Step 4. *Proof of point* 2. Take arbitrary $v, w \in V$. Let us choose simplices $\Delta_v, \Delta_w \in \Sigma$ containing v and w, respectively. Let $y_v \in \Delta_v$ and $y_w \in \Delta_w$ be arbitrary vertices of these simplices.

Let us estimate $\big\|f^*(v) - f^*(w)\big\| - |vw|$. Let us apply the triangle inequality:

$$\big\|f^*(v) - f^*(w)\big\| - |vw| \le \big\|f^*(v) - f(y_v)\big\| + \big\|f(y_v) - f(y_w)\big\| + \big\|f(y_w) - f^*(w)\big\| - |y_v y_w| + |v y_v| + |y_w w|.$$

By point 3, $\big\|f^*(v) - f(y_v)\big\| < C + \varepsilon$ and $\big\|f(y_w) - f^*(w)\big\| < C + \varepsilon$. Also, by definition of distortion, $\big\|f(y_v) - f(y_w)\big\| \le |y_v y_w| + \operatorname{dis} f < |y_v y_w| + C$. And, since $v, y_v \in \Delta_v$, we have $|v y_v| < \varepsilon$; similarly $|y_w w| < \varepsilon$.

Substituting these estimates:

$$\begin{aligned}\big\|f^*(v) - f^*(w)\big\| - |vw| &< (C+\varepsilon) + \big(|y_v y_w| + C\big) + (C+\varepsilon) \\ &\quad - |y_v y_w| + \varepsilon + \varepsilon \\ &= 3C + 4\varepsilon.\end{aligned}$$

Similarly, swapping v and w, we get:

$$|vw| - \big\|f^*(v) - f^*(w)\big\| < 3C + 4\varepsilon.$$

Therefore,

$$\Big|\big\|f^*(v) - f^*(w)\big\| - |vw|\Big| < 3C + 4\varepsilon.$$

Passing to the supremum over all $v, w \in V$, we obtain $\operatorname{dis} f^* \le 3C + 4\varepsilon$. Point 2 has been proven. □

4.3.3 Webster's Lemma

Lemma 4.13 (Webster's)**.** *Let $f\colon V \to W$ be a continuous map between finite-dimensional normed spaces,* $\dim V \ge \dim W$*, and* $\operatorname{dis} f < \infty$*. Then* $\dim V = \dim W$ *and f is surjective. Source*: [56, Proposition 4.1].

4.3.4 Proof of the weak estimate theorem

Now we are ready to prove Theorem 4.10.

Proof. If $d_{GH}(V, X) = \infty$, then the inequality $d_{GH}(V, X) \ge \frac{1}{2J_s(V)} d_H(V, X)$ is satisfied automatically. Suppose that $d_{GH}(V, X) < \infty$.

Let $C > 0$ be an arbitrary number satisfying the condition $C > 2d_{GH}(V, X)$. According to the formula for the Gromov–Hausdorff distance through correspondences, there is a correspondence $R \in \mathcal{R}(V, X)$ such that $\operatorname{dis} R < C$. Let us choose a mapping $f\colon V \to X$ with the property: for each $v \in V$, $\big(v, f(v)\big) \in R$ holds. Then, by definition of distortion:

$$\operatorname{dis} f \le \operatorname{dis} R < C.$$

Let us fix an arbitrary $\varepsilon > 0$. By Lemma 4.12 there is a continuous mapping $f^*\colon V \to V$ such that $\operatorname{dis} f^* \le 3C + 4\varepsilon$. By Lemma 4.13 applied to $f^*\colon V \to V$ (here $W = V$ and $\dim V = \dim W$), the mapping f^* is surjective.

Therefore,

$$V = \bigcup_{\Delta \in \Sigma} f^*(\Delta).$$

Let us take an arbitrary point $v \in V$. Then there exists a simplex $\Delta \in \Sigma$ such that $v \in f^*(\Delta)$. By point 3 of Lemma 4.12, $\operatorname{diam} f^*(\Delta) < C + \varepsilon$. Moreover, $f^*(\Delta) \subset \operatorname{conv}(f(\text{vertices } \Delta))$, and all points of $f(\text{vertices } \Delta)$ lie in X.

Let us denote $A := f(\text{vertices } \Delta) \subset X$. Then $v \in \operatorname{conv} A$. By definition of the measure of nonconvexity (Definition 4.5):

$$\inf_{a \in A} \|v - a\| \leq \lambda(A) \leq J_s(V) \cdot \operatorname{diam} A,$$

where the last inequality follows from Gulevich's theorem (Fact 4.7).

Since $A \subset f^*(\Delta)$, we have $\operatorname{diam} A \leq \operatorname{diam} f^*(\Delta) < C + \varepsilon$. Therefore, there is a point $a \in A \subset X$ such that

$$\|v - a\| \leq J_s(V)(C + \varepsilon).$$

Since $v \in V$ was chosen arbitrarily, we get:

$$\sup_{v \in V} |vX| \leq J_s(V)(C + \varepsilon),$$

where $|vX| = \inf_{x \in X} \|v - x\|$.

According to the standard maximum formula for the Hausdorff distance:

$$d_H(V, X) = \max\{\sup_{v \in V} |vX|,\ \sup_{x \in X} |xV|\}.$$

Since $X \subseteq V$, then $\sup_{x \in X} |xV| = 0$. Therefore,

$$d_H(V, X) \leq J_s(V)(C + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we go to the limit $\varepsilon \to 0$:

$$d_H(V, X) \leq J_s(V)C.$$

Since $C > 0$ was an arbitrary number with the condition $C > 2d_{GH}(V, X)$, we get:

$$d_H(V, X) \leq 2J_s(V)d_{GH}(V, X).$$

Dividing both sides by $2J_s(V)$ (note that $J_s(V) > 0$ by Fact 4.8), we obtain the required inequality:

$$d_{GH}(V, X) \geq \frac{1}{2J_s(V)} d_H(V, X).$$

□

4.4 Intersection property

To strengthen the estimate, we need an additional property of a normed space.

Definition 4.14. A finite-dimensional normed space V satisfies the ***intersection property*** if for any $\rho > 0$ there is a $t > 0$ such that for any $t' > t$ and for any finite family of open balls $\{U_{r_i}(x_i)\}_{i=1}^m$ with $r_i \leq \rho$

$$\big(V \setminus B_{t'}(0)\big) \cap \big(\bigcap_{i=1}^m U_{r_i}(x_i)\big)$$

is either empty or contractible (in particular, connected).

Remark 4.15. Here $B_{t'}(0) = \{v \in V : \|v\| \leq t'\}$ is a closed ball of radius t' with center at zero, and $U_{r_i}(x_i) = \{v \in V : \|v - x_i\| < r_i\}$ is an open ball.

Known fact 4.16 (Euclidean spaces satisfy the intersection property)**.** Every finite-dimensional Euclidean space satisfies the intersection property. *Source*: [49, Lemma 4.1].

Known fact 4.17 (Spaces with sup-norm satisfy the intersection property)**.** Every finite-dimensional space $V = \mathbb{R}^n$ with norm $\|\cdot\|_\infty$ satisfies the intersection property. *Source*: [49, Lemma 4.2].

Known fact 4.18 (Cylindrical norms satisfy the intersection property)**.** Let B^2 be a unit ball of arbitrary norm in $\mathbb{R}^2$, and let $K^{n-2} \subseteq \mathbb{R}^{n-2}$ be a $(n-2)$-dimensional cube. Let us define the unit ball in $\mathbb{R}^n$ as $B^n = B^2 \times K^{n-2}$. Then the norm in $\mathbb{R}^n$ with the unit ball B^n (such norms are called ***cylindrical***) satisfies the intersection property. *Source*: [49, Lemma 4.4].

Known fact 4.19 (Not all norms satisfy the intersection property)**.** There are finite-dimensional normed spaces that do not satisfy the intersection property. Counterexamples can be constructed in $\mathbb{R}^3$ with a unit ball in the form of an icosahedron, dodecahedron or octahedron. *Source*: [49, Remark 4.5].

4.5 Strengthened GH-distance estimation

We now formulate and prove the main result of this chapter. The proof uses the Čech and Vietoris–Rips complexes from Chapter 3, as well as the nerve lemma (Fact 3.15).

Theorem 4.20 ([49, Theorem 6.1])**.** *Let V be a finite-dimensional normed space,* $\dim V \geq 1$*, satisfying the intersection property (Definition 4.14). Let $X \subseteq V$ be a non-empty subset with finite $d_H(V, X)$. Then*

$$d_{GH}(V, X) \geq \frac{1}{2J(V)} d_H(V, X),$$

where $J(V)$ is Jung's constant (Definition 2.20).

Proof. Assume the opposite: $d_{GH}(V, X) < \frac{1}{2J(V)} d_H(V, X)$.

Step 1. *Choice of parameters r and ε.* There are positive numbers r and ε such that

$$2d_{GH}(V, X) < r < r + \varepsilon < \frac{1}{J(V)} d_H(V, X) \tag{4.1}$$

and, in addition,

$$(r + \varepsilon)J(V) < 2r. \tag{4.2}$$

The inequality (4.2) holds for sufficiently small $\varepsilon > 0$, since $rJ(V) < 2r$ (since $J(V) \leq 1$).

Step 2. *Construction of correspondences and mappings.* According to the formula for the Gromov–Hausdorff distance, through the correspondences from $2d_{GH}(V, X) < r$, it follows that there is a correspondence $C \in \mathcal{R}(V, X)$ with $\operatorname{dis} C < r$. Let us choose mappings $h\colon V \to X$ and $g\colon X \to V$ such that $\big(v, h(v)\big) \in C$ for all $v \in V$ and $\big(g(x), x\big) \in C$ for all $x \in X$. Then $\operatorname{dis} h < r$ and $\operatorname{dis} g < r$.

Step 3. *Application of the lemma on the proximity of mappings.* By Lemma 3.21, the mappings h and g induce simplicial mappings between Vietoris–Rips complexes:

$$\mathrm{VR}(V; \varepsilon) \xrightarrow{\bar{h}} \mathrm{VR}(X; r + \varepsilon) \xrightarrow{\iota} \mathrm{VR}\big(X; 2(r + \varepsilon)J(V)\big) \xrightarrow{\bar{g}} \mathrm{VR}\big(V; r + 2(r + \varepsilon)J(V)\big),$$

where ι is an inclusion, and the composition $\bar{g} \circ \iota \circ \bar{h}$ is close to inclusion

$$\iota_V\colon \mathrm{VR}(V; \varepsilon) \hookrightarrow \mathrm{VR}\big(V; r + 2(r + \varepsilon)J(V)\big).$$

Step 4. *Transition to Čech complexes.* Recall that $\check{C}_r(X;Z)$ is an outer Čech complex (Definition 3.13); here and below, the ambient space everywhere is V. By Lemma 3.22 we have inclusions

$$\check{C}_{\varepsilon/2}(V) \overset{\beta'}{\hookrightarrow} \mathrm{VR}(V;\varepsilon) \subseteq \check{C}_{J(V)\varepsilon}(V),$$

$$\mathrm{VR}\big(X;r+\varepsilon\big) \overset{\iota'}{\hookrightarrow} \check{C}_{(r+\varepsilon)J(V)}(X;V) \overset{\iota''}{\hookrightarrow} \mathrm{VR}\big(X;2(r+\varepsilon)J(V)\big),$$

$$\mathrm{VR}\big(V;r+2(r+\varepsilon)J(V)\big) \overset{\beta}{\hookrightarrow} \check{C}_{r+2(r+\varepsilon)J(V)}(V)$$

(the last inclusion is carried out by Definition 3.13: the diameter of the simplex of the Vietoris–Rips complex is less than or equal to the radius of the corresponding Čech complex). Set $\rho := r + 2(r+\varepsilon)J(V)$. Then the mappings from step 3 together with these inclusions give a short diagram

$$\check{C}_{\varepsilon/2}(V) \xrightarrow{h_0} \check{C}_{(r+\varepsilon)J(V)}(X;V) \xrightarrow{g_0} \check{C}_\rho(V),$$

where $h_0 := \iota' \circ \bar{h} \circ \beta'$ and $g_0 := \beta \circ \bar{g} \circ \iota''$. Since $\iota'' \circ \iota' = \iota$, composition

$$g_0 \circ h_0 = \beta \circ \bar{g} \circ \iota \circ \bar{h} \circ \beta'$$

close to inclusion

$$\kappa_0 \colon \check{C}_{\varepsilon/2}(V) \hookrightarrow \check{C}_\rho(V).$$

Step 5. *Good covering.* By Definition 3.13, each of the three Čech complexes from step 4 is the nerve of the family of balls centered at its vertices: the complexes $\check{C}_{\varepsilon/2}(V)$, $\check{C}_{(r+\varepsilon)J(V)}(X;V)$ and $\check{C}_\rho(V)$ are the nerves of the families

$$\mathcal{U}_1 := \big\{U_{\varepsilon/2}(v)\big\}_{v\in V}, \qquad \mathcal{U}_2 := \big\{U_{(r+\varepsilon)J(V)}(x)\big\}_{x\in X}, \qquad \mathcal{U}_3 := \big\{U_\rho(v)\big\}_{v\in V}$$

respectively. The balls of a normed space are convex, so non-empty finite intersections of balls are contractible, that is, all three families $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ are good (Definition 3.14), and by the nerve lemma (Fact 3.15)

$$\big|\check{C}_{\varepsilon/2}(V)\big| \simeq V, \qquad \big|\check{C}_\rho(V)\big| \simeq V :$$

the families $\mathcal{U}_1$ and $\mathcal{U}_3$ cover V, since every point $v \in V$ lies in the balls of radii $\varepsilon/2$ and ρ centered at v.

The middle family $\mathcal{U}_2$ is structured differently: it does not cover V. Indeed, $X \subseteq V$, therefore $d_H(V,X) = \sup_{v\in V} d(v,X)$, and from the inequality (4.1) it follows that $(r+\varepsilon)J(V) < d_H(V,X)$. This means there is a point $y \in V$ such that

$$d(y,X) > (r+\varepsilon)J(V), \qquad \text{that is} \qquad y \notin \bigcup_{x\in X} U_{(r+\varepsilon)J(V)}(x).$$

Let us shift the origin of coordinates to point y: the shift does not change any distance, so we can assume that

$$0 = y \in V \qquad \text{and} \qquad \|x\| > (r+\varepsilon)J(V) \quad \text{for all } x \in X.$$

The space V is convex and therefore contractible, so $H_n(V) = 0$ for $n \geq 1$. Thus, for the original Čech complexes the top homology is trivial, and the contradiction does not follow from the step diagram 4: a complex with non-trivial top homology is needed. We obtain such a complex by replacing the ambient space V with a sphere S^n_R of the same dimension (steps 6–7); this is possible because V and $S^n_R \setminus \{N\}$ are homeomorphic.

Step 6. *Intersection property.* Let us apply the intersection property (Definition 4.14) to the number $\rho = r + 2(r+\varepsilon)J(V)$: there is $t > 0$ such that for any $t' > t$ the set

$V \setminus B_{t'}(0)$ intersects with any finite intersection of balls of radius at most ρ is either empty or contractible. Let us put

$$U_1 := V \setminus B_{t+9r}(0), \qquad U_2 := V \setminus B_{t+6r}(0), \qquad U_3 := V \setminus B_t(0).$$

Here $t + 9r > t + 6r > t$, and all balls from the three families of step 5 have a radius of no more than ρ: in Fact 4.8 we have $J(V) \geq 1/2$, therefore

$$\varepsilon/2 < \varepsilon + 2r \leq r + 2(r+\varepsilon)J(V) = \rho, \qquad (r+\varepsilon)J(V) \leq 2(r+\varepsilon)J(V) \leq \rho.$$

Therefore, if we add the sets U_1, U_2, U_3 to the first, second and third families of balls from step 5, then a non-empty intersection of the form $U_i \cap \bigcap_j U_{r_j}(x_j)$ over any non-empty finite set of balls is contractible: this is exactly the intersection of $V \setminus B_{t'}(0)$ with finite intersection of balls of radius at most ρ, where t' is equal to $t + 9r$, $t + 6r$ or t.

Moreover, the sets U_i themselves are not contractible: each of them is the complement of a closed ball in V, and such a set is homotopy equivalent to a sphere of dimension $n-1$, where $n = \dim V$. This means that the completed families are not yet good, and the nerve lemma cannot be applied to them in the space V. Let us denote by K_1, K_2, K_3 the nerves of these three completed families; for example, K_1 is the nerve of the family $\{U_{\varepsilon/2}(v)\}_{v\in V} \cup \{U_1\}$.

Let us continue mapping h_0 and g_0 from step 4 to new vertices, putting

$$h_0(U_1) := U_2, \qquad g_0(U_2) := U_3;$$

on the remaining vertices they act as before: $U_{\varepsilon/2}(v) \mapsto U_{(r+\varepsilon)J(V)}(h(v))$ and $U_{(r+\varepsilon)J(V)}(x) \mapsto B(g(x);\rho)$.

Step 7. Properties of completed complexes and transition to the sphere. The completed complexes and the maps between them have all the properties necessary for the transition to homology: the maps h_0 and g_0 are simplicial, the composition $g_0 \circ h_0$ is close to the inclusion, and the completed coverings are good (Definition 3.14). All these properties are verified directly, see [49].

Let us now move from the space V to the sphere. As we saw in step 6, the completed families are not good in V itself: the set U_i is the complement of a closed ball and is not contractible. Let us set $n := \dim V$ and realize V as the coordinate subspace $\{x^{n+1} = 0\}$ in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, where $x^1, \dots, x^{n+1}$ are Cartesian coordinates. For $R > 0$ we denote by $S^n_R \subset \mathbb{R}^{n+1}$ the sphere of radius R centered at the origin and put $N := (0, \dots, 0, R)$. Consider the stereographic projection

$$p\colon S^n_R \setminus \{N\} \to V$$

from the north pole N onto V. We choose R so large that the inverse images of all balls participating in the construction are smaller than a hemisphere (their angular size tends to zero uniformly as $R \to \infty$).

Let us put

$$V_i := p^{-1}(U_i) \cup \{N\}, \qquad i = 1, 2, 3,$$

and for the ball $U_s(x) \subset V$ we denote by $\widetilde{U}_s(x) := p^{-1}\big(U_s(x)\big)$ its inverse image in S^n_R. Each of the sets V_i is the complement of a closed ball in S^n_R:

$$V_1 = S^n_R \setminus p^{-1}\big(B_{t+9r}(0)\big), \qquad V_2 = S^n_R \setminus p^{-1}\big(B_{t+6r}(0)\big), \qquad V_3 = S^n_R \setminus p^{-1}\big(B_t(0)\big),$$

therefore V_i is open in S^n_R and homeomorphic to an open ball in V, that is, contractible.

Consider three families of open subsets of the sphere:

$$\mathcal{F}_1 := \{\widetilde{U}_{\varepsilon/2}(v)\}_{v\in V} \cup \{V_1\}, \qquad \mathcal{F}_2 := \{\widetilde{U}_{(r+\varepsilon)J(V)}(x)\}_{x\in X} \cup \{V_2\},$$

$$\mathcal{F}_3 := \{\widetilde{U}_\rho(v)\}_{v\in V} \cup \{V_3\}.$$

Their nerves as abstract simplicial complexes coincide exactly with K_1, K_2, K_3.

Let us show that the families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ are good. A non-empty finite intersection of their sets has one of three types.

- The intersection of the inverse images of the balls is equal to $p^{-1}\bigl(\bigcap_j U_{r_j}(x_j)\bigr)$; the intersection of balls in V is convex, and the inverse image of a convex set under a homeomorphism is contractible.
- The intersection of V_i with the inverse images of the balls is equal to $p^{-1}\bigl(U_i \cap \bigcap_j U_{r_j}(x_j)\bigr)$; by the intersection property and by the choice of the number t (see step 6), the set $U_i \cap \bigcap_j U_{r_j}(x_j)$ is empty or contractible, and the inverse image of a non-empty contractible set is contractible.
- Finally, the intersection consisting of one set is V_i itself; it is contractible according to what was proved above.

So, each family $\mathcal{F}_i$ is a good open cover of its union, and the union of sets in the family is a metric space, that is, paracompact. By the nerve lemma (Fact 3.15), realizations of the complexes K_1, K_2, K_3 are homotopy equivalent to these unions.

The first and third families have the same structure: the balls $\widetilde{U}_{\varepsilon/2}(v)$ and $\widetilde{U}_\rho(v)$ with centers at all points $v \in V$ cover V, and the set V_i contains the point N, so

$$\bigcup \mathcal{F}_1 = S^n_R, \qquad \bigcup \mathcal{F}_3 = S^n_R, \qquad \text{that is} \qquad |K_1| \simeq S^n_R \simeq |K_3|.$$

The union of the middle family is its own open subset of the sphere

$$U := \bigcup \mathcal{F}_2 = \Bigl(\bigcup_{x\in X} \widetilde{U}_{(r+\varepsilon)J(V)}(x)\Bigr) \cup V_2 \subset S^n_R, \qquad U \neq S^n_R :$$

it is open and does not contain the point $p^{-1}(0)$, since $0 \notin U_2$ and $\|x\| > (r+\varepsilon)J(V)$ for all $x \in X$ (see step 5). So $|K_2| \simeq U$, where U is a proper open subset of S^n_R.

Finally, the mappings h_0 and g_0 are simplicial mappings between the same abstract complexes K_1, K_2, K_3. In the sphere they correspond to the mappings $p^{-1} \circ h \circ p$ and $p^{-1} \circ g \circ p$, continued according to the rules $h_0(U_1) = U_2$, $g_0(U_2) = U_3$, that is, $V_1 \mapsto V_2$ and $V_2 \mapsto V_3$.

Step 8. *Contradiction in homology.* Only now let us move on to homology with coefficients in $\mathbb{Z}_2$. As proven in step 7,

$$H_n(K_1;\mathbb{Z}_2) \cong H_n(S^n_R;\mathbb{Z}_2) = \mathbb{Z}_2, \qquad H_n(K_3;\mathbb{Z}_2) \cong H_n(S^n_R;\mathbb{Z}_2) = \mathbb{Z}_2,$$

$$H_n(K_2;\mathbb{Z}_2) \cong H_n(U;\mathbb{Z}_2) = 0.$$

Here $H_n(S^n_R;\mathbb{Z}_2) = \mathbb{Z}_2$, since the sphere S^n_R is a compact connected n-dimensional manifold without boundary (Fact 3.24), and $H_n(U;\mathbb{Z}_2) = 0$, since U is a proper open subset of a connected closed n-dimensional manifold S^n_R (Corollary 3.25).

The simplicial maps $h_0\colon K_1 \to K_2$ and $g_0\colon K_2 \to K_3$ induce a sequence of homomorphisms

$$H_n(S^n_R;\mathbb{Z}_2) \xrightarrow{(h_0)_*} H_n(U;\mathbb{Z}_2) = 0 \xrightarrow{(g_0)_*} H_n(S^n_R;\mathbb{Z}_2).$$

The middle group is zero, so

$$(g_0)_* \circ (h_0)_* = 0.$$

On the other hand, in step 7 the composition $g_0 \circ h_0$ is close to the inclusion $\kappa\colon K_1 \to K_3$, which takes the vertex U_1 to U_3. Close simplicial maps induce identical homology homomorphisms (Fact 3.20), therefore

$$(g_0)_* \circ (h_0)_* = \kappa_*.$$

It remains to note that κ_* is an isomorphism. Indeed, every set of the family $\mathcal{F}_1$ lies in the corresponding set of the family $\mathcal{F}_3$: $\widetilde{U}_{\varepsilon/2}(v) \subseteq \widetilde{U}_\rho(v)$, since $\varepsilon/2 < \rho$, and $V_1 \subseteq V_3$, since $U_1 \subseteq U_3$. This means that κ takes one good covering of the sphere to another, subordinate to it, and the homotopy equivalences $|K_1| \to S^n_R$ and $|K_3| \to S^n_R$ from the nerve lemma can be chosen consistently: the witness points in non-empty intersections of nested sets are taken to be the same. Therefore κ is a homotopy equivalence, and $\kappa_*\colon H_n(S^n_R;\mathbb{Z}_2) \to H_n(S^n_R;\mathbb{Z}_2)$ is an isomorphism.

So,

$$0 = (g_0)_* \circ (h_0)_* = \kappa_*,$$

but κ_* is an isomorphism of the group $\mathbb{Z}_2 \neq 0$. The resulting contradiction shows that the original assumption is incorrect, and

$$d_{GH}(V,X) \geq \frac{1}{2J(V)} d_H(V,X).$$

□

4.6 Corollaries: equality for the sup-norm

Corollary 4.21 ([49, Corollary 6.2])**.** *Let $V = \mathbb{R}^n$ with norm $\|\cdot\|_\infty$ (sup-norm), and let $X \subseteq V$ be a non-empty subset with finite $d_H(V,X)$. Then*

$$d_{GH}(V,X) = d_H(V,X).$$

Proof. Step 1. *Top estimate.* By definition, for the Gromov–Hausdorff distance, we have

$$d_{GH}(V,X) \leq d_H(V,X).$$

Step 2. *Lower estimate.* By Fact 4.17, the space V with sup-norm satisfies the intersection property. In Fact 4.8, $J(V) = 1/2$ for sup-norm. Applying Theorem 4.20, we obtain:

$$d_{GH}(V,X) \geq \frac{1}{2 \cdot (1/2)} d_H(V,X) = d_H(V,X).$$

Combining the upper and lower bounds, we obtain the equality $d_{GH}(V,X) = d_H(V,X)$. □

Corollary 4.22 ([49, Corollary 6.2])**.** *Let V be a finite-dimensional normed space with cylindrical norm (Fact 4.18) or $\dim V = 2$. Then for any $X \subseteq V$ with finite $d_H(V,X)$:*

$$d_{GH}(V,X) \geq \frac{1}{2J(V)} d_H(V,X).$$

Proof. By Fact 4.18 (for cylindrical norms) and by Fact 4.16 (for $\dim V = 2$ the intersection property holds for any norm), the space V satisfies the intersection property. We apply Theorem 4.20. □

Chapter Summary

This chapter proves:

(1) definitions of the circumscribed ball radius $R(A)$, relative radius $R_Y(A)$, Jung constant $J(V)$ and relative Jung constant $J_s(V)$ (Definitions 2.19, 4.2, 2.20, 4.3);

(2) definition of the measure of nonconvexity $\lambda(A)$ and of the Gulevich number $G(V)$ (Definitions 4.5, 4.6);

(3) lemma on triangulation of a finite-dimensional normed space (Lemma 4.11);

(4) lemma on continuous extension of a mapping with finite distortion (Lemma 4.12);

(5) weak estimate $d_{GH}(V,X) \geq \frac{1}{2J_s(V)} d_H(V,X)$ for any finite-dimensional normed space V (Theorem 4.10);

(6) definition of the intersection property (Definition 4.14);

(7) strong estimate $d_{GH}(V,X) \geq \frac{1}{2J(V)} d_H(V,X)$ for spaces satisfying the intersection property (Theorem 4.20);

(8) equality $d_{GH}(V,X) = d_H(V,X)$ for sup-norm (Corollary 4.21);

(9) estimate for cylindrical norms and $\dim V = 2$ (Corollary 4.22).

Accepted as a known fact: Gulevich's theorem on the coincidence of the Gulevich number $G(V)$ with the relative Jung constant $J_s(V)$ for Banach spaces ([49, Theorem 2.6]); values of the Jung constant for Euclidean spaces and sup-norm ([50], [53], [52]); equality criterion $J(V) = J_s(V)$ ([54], [55]); intersection property for Euclidean spaces, sup-norm and cylindrical norms ([49, Lemmas 4.1, 4.2, 4.4]); the existence of norms that do not satisfy the intersection property ([49, Remark 4.5]); Webster's lemma ([56, Proposition 4.1]). Left outside the scope: generalization of the results to infinite-dimensional normed spaces.