1 Solutions — Seminar sheet (Peking, 2026): partitions, MST, Borsuk partitions, ultrametrization, Jung constant

This file solves all 30 problems on the seminar sheet `Peking_seminars_2026 (5).pdf`. The sheet has six blocks of five problems each. The "MST and MST spectrum" box is printed twice but has only one list of problems.

Notation. $|xy|$ is the distance. $\lambda\Delta_m$ is the m-point space with all nonzero distances equal to λ, and Δ_1 is a point. $\mathcal{D}_m(X)$, $\alpha(D)$ and $\operatorname{diam} D$ are as on the sheet. By convention $\alpha(\{X\}) = +\infty$. The **partition formula** is the formula from the sheet, for $m \le \#X$:

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\}. \tag{PF}$$

Two facts are used throughout: - $d_{GH}(X, \Delta_1) = \frac{1}{2}\operatorname{diam} X$. - $d_{GH}(\cdot,\cdot)$ satisfies the triangle inequality.

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2 A. GH distance and partitions

2.1 A1. Three points

Let $X = \{x, y, z\}$ with $|xy| = a$, $|yz| = b$ and $|xz| = c$, where $0 < a \le b \le c \le a + b$. Then $\operatorname{diam} X = c$. The three 2-partitions are:

D	$\operatorname{diam} D$	$\alpha(D)$
$\{x, y\} \mid \{z\}$	a	$\min(b, c) = b$
$\{y, z\} \mid \{x\}$	b	$\min(a, c) = a$
$\{x, z\} \mid \{y\}$	c	$\min(a, b) = a$

The first row is the best in every entry: $a \le b \le c$ and $\lambda - b \le \lambda - a$. So by (PF)

$$d_{GH}(\lambda\Delta_2, X) = \tfrac{1}{2}\max\{a,\ \lambda - b,\ c - \lambda\} = \begin{cases} \dfrac{c-\lambda}{2}, & 0 < \lambda \le c - a, \\ \dfrac{a}{2}, & c - a \le \lambda \le a + b, \\ \dfrac{\lambda - b}{2}, & \lambda \ge a + b. \end{cases}$$

The middle interval is nonempty because $c - a \le b \le a + b$.

Graph. It is a "bathtub". It starts at $c/2$ as $\lambda \to 0^+$, which matches $d_{GH}(\Delta_1, X) = c/2$. It decreases with slope $-\frac{1}{2}$ down to the level $a/2$ at $\lambda = c - a$. It stays flat at $a/2$ up to $\lambda = a + b$. After that it increases with slope $\frac{1}{2}$ along the line $(\lambda - b)/2$. The minimum value is $a/2$, half the shortest side.

For an equilateral triangle ($a = b = c$) the first piece is empty, and the graph is flat at $a/2$ on $(0, 2a]$.

The large-λ behaviour $\frac{1}{2}(\lambda - b) = \frac{1}{2}(\lambda - \sigma_1(X))$ agrees with A2 and A5.

2.2 A2. Large λ

Let $\lambda \geq 2 \operatorname{diam} X$. In particular $\operatorname{diam} X < \infty$.
Case $n = 1$. $\mathcal{D}_1 = \{\{X\}\}$ and $\lambda - \alpha = -\infty$. So

$$2d_{GH} = \max\{\operatorname{diam} X,\ \operatorname{diam} X - \lambda\} = \operatorname{diam} X.$$

This is also immediate from $d_{GH}(\cdot, \Delta_1) = \frac{1}{2} \operatorname{diam}$.
Case $n \geq 2$. Every $D \in \mathcal{D}_n(X)$ has $\alpha(D) \leq \operatorname{diam} X$, because α is a distance between points of X. Hence

$$\lambda - \alpha(D) \ \geq\ \lambda - \operatorname{diam} X \ \geq\ \operatorname{diam} X \ \geq\ \operatorname{diam} D.$$

Also $\lambda - \operatorname{diam} X \geq \operatorname{diam} X - \lambda$, because $\lambda \geq \operatorname{diam} X$. So the maximum in (PF) is always $\lambda - \alpha(D)$, and

$$2d_{GH}(\lambda\Delta_n, X) = \inf_{D \in \mathcal{D}_n} (\lambda - \alpha(D)) = \lambda - \sup_{D \in \mathcal{D}_n} \alpha(D).$$

The hypothesis $n \leq \#X$ guarantees $\mathcal{D}_n \neq \varnothing$.

2.3 A3. Two clusters

Put $p := \max(\operatorname{diam} A, \operatorname{diam} B)$ and $\Delta := \operatorname{diam} X$. If p is only an upper bound for the two diameters, replace it by this maximum. Assume $q = d(A, B) > p$.
For $D_0 = \{A, B\}$ we have $\operatorname{diam} D_0 = p$ and $\alpha(D_0) = q$. Assuming D_0 is optimal,

$$2d_{GH}(\lambda\Delta_2, X) = \max\{p,\ \lambda - q,\ \Delta - \lambda\} = \begin{cases} \Delta - \lambda, & \lambda \leq \Delta - p, \\ p, & \Delta - p \leq \lambda \leq p + q, \\ \lambda - q, & \lambda \geq p + q. \end{cases}$$

The pieces fit together because $q \leq \Delta \leq 2p + q$. For $a, a' \in A$ and $b, b' \in B$:

$$|ab| \leq |aa'| + |a'b'| + |b'b| \leq 2p + |a'b'|,$$

and $|a'b'|$ can be taken arbitrarily close to q. Hence $0 < \Delta - p \leq p + q$. The active term changes exactly at

$$\lambda = \Delta - p \qquad \text{and} \qquad \lambda = p + q.$$

The crossing point $\lambda = (\Delta + q)/2$ of the two outer lines lies inside the plateau, so it is not a change point.
Optimality condition. The condition $q \geq p$ suffices, so D_0 is automatically optimal under the hypothesis $q > p$. Let $D = \{X_1, X_2\} \neq D_0$. Then some part, say X_1, meets both A and B, so $\operatorname{diam} D \geq q$. The other part X_2 meets A or B, say A. Since X_1 also meets A, we get $\alpha(D) \leq \operatorname{diam} A \leq p$. Therefore

$$\max\{\operatorname{diam} D, \lambda - \alpha(D), \Delta - \lambda\} \ \geq\ \max\{q,\ \lambda - p,\ \Delta - \lambda\} \ \geq\ \max\{p, \lambda - q, \Delta - \lambda\}$$

for every λ. So D_0 is optimal.

2.4 A4. Indistinguishable spaces have equal diameters

Take the one-point simplex: $\operatorname{diam} X = 2d_{GH}(X, \Delta_1) = 2d_{GH}(Y, \Delta_1) = \operatorname{diam} Y$. This includes the value $+\infty$.
If one wants to use only simplexes $\lambda\Delta_n$ with a fixed $n \geq 2$, note that $d_{GH}(\lambda\Delta_n, \Delta_1) = \lambda/2$. The triangle inequality then gives

$$|d_{GH}(\lambda\Delta_n, X) - \tfrac{1}{2}\operatorname{diam} X| \leq \lambda/2, \qquad \text{so} \qquad \operatorname{diam} X = 2 \lim_{\lambda \to 0^+} d_{GH}(\lambda\Delta_n, X).$$

The same limit for Y gives $\operatorname{diam} X = \operatorname{diam} Y$.

2.5 A5. The exact threshold

Let X be finite and $k + 1 \leq \#X$. Write $\sigma_k = \max_{D \in \mathcal{D}_{k+1}} \alpha(D)$, and define

$$\delta_k := \min\{\operatorname{diam} D : \ D \in \mathcal{D}_{k+1}(X), \ \alpha(D) = \sigma_k\}.$$

For every D we have $\lambda - \alpha(D) \geq \lambda - \sigma_k$, so $2d_{GH}(\lambda\Delta_{k+1}, X) \geq \lambda - \sigma_k$ always. Equality holds iff some D has all three terms $\leq \lambda - \sigma_k$. That means $\alpha(D) = \sigma_k$, $\operatorname{diam} D \leq \lambda - \sigma_k$ and $\operatorname{diam} X - \lambda \leq \lambda - \sigma_k$. The set of such λ is the ray $[\lambda_0, \infty)$, with

$$\boxed{\lambda_0 = \max\left\{\sigma_k + \delta_k, \ \tfrac{1}{2}(\operatorname{diam} X + \sigma_k)\right\}}.$$

Which partitions maximise α. $\alpha(D) \geq \sigma_k$ iff every component of the graph $G_{<\sigma_k}$ lies inside one part. The edges of $G_{<\sigma_k}$ join pairs at distance $< \sigma_k$. So δ_k is the least possible largest part-diameter when the components of $G_{<\sigma_k}$ are grouped into $k + 1$ parts. One such grouping is obtained by deleting the k longest edges of an MST.
Comparison with $2\operatorname{diam} X$. We have $\sigma_k \leq \operatorname{diam} X$ and $\delta_k \leq \operatorname{diam} X$. Hence

$$\lambda_0 \leq \operatorname{diam} X + \sigma_k \leq 2\operatorname{diam} X,$$

and usually λ_0 is much smaller.
Examples: - Two points at distance 1, $k = 1$: $\lambda_0 = 1$, while $2\operatorname{diam} X = 2$. - The three points of A1, $k = 1$: $\lambda_0 = a + b$, which matches A1. - $k + 1 = \#X$: $\delta_k = 0$ and $\sigma_k = \varepsilon(X)$ (the smallest distance), so $\lambda_0 = \frac{1}{2}(\operatorname{diam} X + \varepsilon(X))$. Both terms in the max can be the active one. - $X = \Delta_3$, $k = 1$: every 2-partition has a 2-point part, so $\sigma_1 = \delta_1 = 1$ and $\lambda_0 = 2 = 2\operatorname{diam} X$. So the bound $2\operatorname{diam} X$ is attained.

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3 B. MST and MST spectrum

Throughout, an MST is a minimum spanning tree of the complete graph on M (or X) with edge weights $|uv|$.

3.1 B1. Cut property

Let T be any MST. If $v_1v_2 \in T$, we are done. Otherwise $T + v_1v_2$ contains a unique cycle. The cycle goes from M_1 to M_2 along v_1v_2 and must come back, so it contains another edge f joining M_1 to M_2. Then $|f| \geq |M_1M_2| = |v_1v_2|$. The graph $T' = T - f + v_1v_2$ is a spanning tree with $w(T') \leq w(T)$. So T' is an MST, and it contains v_1v_2.

3.2 B2. How many values?

Answer: $n-1$ for $n \geq 2$, and 0 for $n = 1$.

The values are exactly the MST edge lengths. Fix an MST T. - Let (M_1, M_2) be a partition, and let v_1v_2 realise $|M_1M_2|$. By B1, v_1v_2 is an edge of some MST T'. All MSTs have the same multiset of edge lengths: this follows from B5, or by repeated exchange. So $|M_1M_2|$ is an MST edge length. - Conversely, let $e \in T$. Removing e splits T into two components M_1, M_2, so $|M_1M_2| \leq |e|$. Suppose some pair $u \in M_1$, $v \in M_2$ had $|uv| < |e|$. Then $T - e + uv$ would be a lighter spanning tree, a contradiction. So $|M_1M_2| = |e|$.

So there are at most $n-1$ distinct values. Exactly $n-1$ is attained by collinear points $0, 1, 3, 6, \ldots, \binom{n}{2}$, whose consecutive gaps are $1, 2, \ldots, n-1$. The MST is the path, and its $n-1$ edge lengths are distinct.

3.3 B3. Ultrametric spaces

In an ultrametric space, $x \sim_j y \iff d(x,y) \leq r_j$ is an equivalence relation. Transitivity is the strong triangle inequality.

Lemma (proved in B5). For any finite metric space and any MST T, the number of edges of T of length $\leq r$ equals $\#X - c(r)$. Here $c(r)$ is the number of components of G_r.

In an ultrametric space the components of G_{r_j} are exactly the $\sim_j$-classes: G_{r_j} is already a disjoint union of cliques. So T has $N_0 - N_j$ edges of length $\leq r_j$, and $N_0 - N_{j-1}$ edges of length $\leq r_{j-1}$. Subtracting, r_j has multiplicity $N_{j-1} - N_j$ in the spectrum.

Least number of parts of diameter $< \operatorname{diam} X = r_s$. A part of diameter $< r_s$ has diameter $\leq r_{s-1}$, so it lies in a single $\sim_{s-1}$-class. Conversely, each $\sim_{s-1}$-class has diameter $\leq r_{s-1}$. Hence the minimum is

$$N_{s-1} = 1 + (\text{multiplicity of } \operatorname{diam} X \text{ in } \sigma(X)),$$

using $N_s = 1$.

3.4 B4. $\sigma_1 < \operatorname{diam} X$, and the optimal ratio

Take $X = \{0, 1, \ldots, n-1\} \subset \mathbb{R}$. Its MST is the path with all edges of length 1, so $\sigma_1 = 1 < n - 1 = \operatorname{diam} X$.

Optimal bound: $\operatorname{diam} X / \sigma_1(X) \leq n - 1$, with equality for the example above. Any two points are joined by the path between them in an MST. That path has at most $n-1$ edges, each of length $\leq \sigma_1$. By the triangle inequality, $|xy| \leq (n-1)\sigma_1$. (For $n = 2$ the ratio is always 1, which is why $n \geq 3$ is needed for strict inequality.)

3.5 B5. Spectrum from the components of G_r

Claim. For every MST T and every $r \geq 0$, the forest $T_{\leq r}$ of edges of T of length $\leq r$ has the same components as G_r.

$T_{\leq r} \subseteq G_r$, so each component of $T_{\leq r}$ lies inside a component of G_r. Suppose $|uv| \leq r$ but u, v lie in different components of $T_{\leq r}$. Then the T-path from u to v contains an edge f with $|f| > r$. Now $T - f + uv$ is a spanning tree that is strictly lighter, a contradiction.

So T has exactly $\#X - c(r)$ edges of length $\leq r$. Consequently: - The multiplicity of a value $t > 0$ in $\sigma(X)$ is $c(t^-) - c(t)$, where $c(t^-) = \lim_{r \uparrow t} c(r)$ is the number of components of $G_{<t}$. - The spectrum consists of the jump points of the right-continuous step function c, each counted with its jump size. Equivalently, $\sigma_k(X) = \min\{r \geq 0 : \ c(r) \leq k\}$.

This does not depend on the choice of T, which also proves that all MSTs have the same spectrum. It also agrees with $\sigma_k = \max_{\mathcal{D}_{k+1}} \alpha$: there is a $(k+1)$-partition with $\alpha \geq r$ iff $G_{<r}$ has at least $k+1$ components.

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4 C. Borsuk partitions and graph metrics

Since $b \le 2a$, every triangle inequality $d(u,w) \le d(u,v) + d(v,w)$ holds: the right-hand side is $\ge 2a \ge b$. A set has diameter $< b$ iff it contains no edge, i.e. it is independent.

Key lemma. Let $1 \le m \le |V|$ and $0 < \lambda < a + b$. In (PF) for X_G: - $\alpha(D) \in \{a, b\}$, so $\lambda - \alpha(D) \le \lambda - a < b$. - $\operatorname{diam} X_G - \lambda < b$. - $\operatorname{diam} D \in \{0, a, b\}$. Hence

$$2d_{GH}(\lambda\Delta_m, X_G) < b \iff \text{some } D \in \mathcal{D}_m \text{ has all parts independent} \iff m \ge \chi(G).$$

The last step uses that a proper colouring with χ colours can be refined to exactly m nonempty classes for any $\chi \le m \le |V|$. If $m < \chi(G)$ and G has an edge, then every D has $\operatorname{diam} D = b$, so $2d_{GH}(\lambda\Delta_m, X_G) = b$ exactly.

4.1 C1. Odd cycles and complete multipartite graphs

By the key lemma with $\lambda = a$, $\min\{m : 2d_{GH}(a\Delta_m, X_G) < b\} = \chi(G)$.

Odd cycle C_{2r+1}, vertices $v_0, \dots, v_{2r}$. An odd cycle is not bipartite, so $\chi = 3$. An optimal partition is

$$\{v_0, v_2, \dots, v_{2r-2}\}, \quad \{v_1, v_3, \dots, v_{2r-1}\}, \quad \{v_{2r}\}.$$

- For $r \ge 2$ it gives $2d_{GH}(a\Delta_3, X_G) = \max\{a, b-a\} < b$. Every 3-partition has a part with at least 2 points, so $\operatorname{diam} D \ge a$, and the value cannot be lower. - For $r = 1$ (triangle) $X_G = b\Delta_3$, and the value is $b - a$. - For $m = 2$ the value is exactly b.

Complete multipartite $K_{n_1,\dots,n_t}$, with $t \ge 2$. Two vertices in different parts are adjacent, so every independent set lies in one part, and $\chi = t$. The t parts form the optimal partition. Here $\alpha = b$, so

$$2d_{GH}(a\Delta_t, X_G) = \max\{\operatorname{diam} D,\ b - a\} = \begin{cases} \max\{a, b-a\} & \text{if some } n_i \ge 2, \\ b - a & \text{if } G = K_t. \end{cases}$$

4.2 C2. A partition criterion with parameter θ

Write $d = \operatorname{diam} X$. The max in (PF) splits off the constant term:

$$2d_{GH}(\lambda\Delta_m, X) = \max\left\{d - \lambda,\ \inf_D \max\{\operatorname{diam} D, \lambda - \alpha(D)\}\right\}.$$

For finite X the infimum is a minimum. So, for every $\lambda \ge (1-\theta)d$,

$$2d_{GH}(\lambda\Delta_m, X) \le \theta d \iff \exists D \in \mathcal{D}_m :\ \operatorname{diam} D \le \theta d \ \text{ and } \ \alpha(D) \ge \lambda - \theta d.$$

For $\lambda < (1-\theta)d$ the left side always fails, because $2d_{GH} \ge d - \lambda$.

Cleanest range. If $\theta \ge \frac{1}{2}$ and

$$(1-\theta)\operatorname{diam} X \ \le\ \lambda \ \le\ \theta \operatorname{diam} X,$$

the separation condition is automatic, since $\alpha \ge 0 \ge \lambda - \theta d$. Then:

$$X \text{ splits into } m \text{ parts of diameter} \le \theta \operatorname{diam} X \iff 2d_{GH}(\lambda\Delta_m, X) \le \theta \operatorname{diam} X.$$

The most symmetric choice is $\lambda = d/2$. For infinite X (where the infimum need not be attained), "$\Rightarrow$" still holds. The implication "$\Leftarrow$" then only gives, for every $\varepsilon > 0$, a partition into m parts of diameter $\leq \theta d + \varepsilon$.

For $\theta < \frac{1}{2}$ the range is empty. Then the criterion necessarily also involves the separation $\alpha(D) \geq \lambda - \theta d \geq (1 - 2\theta)d > 0$, so the simplex distance cannot detect small-diameter partitions alone.

4.3 C3. $\chi(G)$ in terms of distances to simplexes

Assume G has at least one edge. Parts of diameter $< b$ are the independent sets, so the least number of such parts is $\chi(G)$. By the key lemma, for every $\lambda \in (0, a + b)$:

$$\boxed{\chi(G) = \min\{m \geq 1 :\ d_{GH}(\lambda\Delta_m, X_G) < b/2\}}.$$

Moreover $d_{GH}(\lambda\Delta_m, X_G) = b/2$ for all $m < \chi(G)$. For $\lambda = a$ and $m \geq \chi(G)$ it is at most $\frac{1}{2}\max\{a, b - a\}$.

4.4 C4. The cube graph Q_d, $d \geq 1$

Q_d is bipartite: colour a vector by the parity of its number of 1's. It has edges, so $\chi(Q_d) = 2$ and $\operatorname{diam} X_{Q_d} = b$.

- The least number of Borsuk parts is 2.
- An optimal partition is even-weight vectors versus odd-weight vectors.
- The GH inequality: $2d_{GH}(\lambda\Delta_2, X_{Q_d}) < b$ for every $\lambda \in (0, a + b)$, while $2d_{GH}(\lambda\Delta_1, X_{Q_d}) = b$.

Exact values for $\lambda = a$: - For $d \geq 2$: $2d_{GH}(a\Delta_2, X_{Q_d}) = \max\{a, b - a\}$. Every 2-partition of at least 3 points has a part with two points, so $\operatorname{diam} D \geq a$. - For $d = 1$: $X_{Q_1} = b\Delta_2$ and the value is $b - a$.

4.5 C5. Adding or removing one edge

Let $H = G + e$. Then

$$\chi(G) \leq \chi(H) \leq \chi(G) + 1.$$

A proper colouring of H is a proper colouring of G. Conversely, recolouring one endpoint of e with a new colour turns a colouring of G into one of H. If G has at least one edge, the Borsuk numbers of X_G and X_H are $\chi(G)$ and $\chi(H)$.

Caveat: if G has no edges, then $X_G = a\Delta_{|V|}$ has Borsuk number $|V|$, whereas X_{G+e} has Borsuk number 2.

Examples. - Equality: the path $P_4 = u{-}v{-}w{-}x$ plus the edge ux gives C_4. Both have $\chi = 2$. - Strict increase: the path $P_3 = u{-}v{-}w$ plus the edge uw gives a triangle, so χ goes from 2 to 3.

GH translation. Fix $\lambda \in (0, a + b)$. By C3:

$$d_{GH}(\lambda\Delta_m, X_H) < \tfrac{b}{2} \Rightarrow d_{GH}(\lambda\Delta_m, X_G) < \tfrac{b}{2}, \qquad d_{GH}(\lambda\Delta_m, X_G) < \tfrac{b}{2} \Rightarrow d_{GH}(\lambda\Delta_{m+1}, X_H) < \tfrac{b}{2}.$$

Also, the identity correspondence has distortion $b - a$, so $d_{GH}(X_G, X_H) \leq \frac{b-a}{2}$. Hence

$$|d_{GH}(\lambda\Delta_m, X_G) - d_{GH}(\lambda\Delta_m, X_H)| \leq \tfrac{b-a}{2}$$

for all m and λ.

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5 D. Ultrametrization

5.1 D1. d_u is the largest subdominant ultrametric

- $d_u(x,x) = 0$ and d_u is symmetric: use the trivial chain, and reverse chains.
- $d_u \le d$: use the one-step chain.
- Concatenating chains gives $d_u(x,z) \le \max\{d_u(x,y), d_u(y,z)\}$.

So d_u is a pseudo-ultrametric below d. Now let ρ be any ultrametric with $\rho \le d$. For any chain L from x to y, the strong triangle inequality applied repeatedly gives

$$\rho(x,y) \le \max_i \rho(x_{i-1}, x_i) \le \max_i d(x_{i-1}, x_i) = |L|_u.$$

Taking the infimum over chains, $\rho \le d_u$.

5.2 D2. Not path-connected, but $\mathcal{U}(X) = \Delta_1$

Examples: $X = \mathbb{Q} \subset \mathbb{R}$, or $X = [0,1] \setminus \{\frac{1}{2}\}$. For every $\varepsilon > 0$, any two points are joined by an ε-chain, so $d_u \equiv 0$ and $\mathcal{U}(X)$ is a point. Neither space is path-connected.

5.3 D3. The GH closure of path-connected spaces is $\{X : \mathcal{U}(X) = \Delta_1\}$

Here "closure" means X such that for every $\varepsilon > 0$ there is a path-connected Y with $d_{GH}(X,Y) < \varepsilon$.

Nonexpansion (used below). Let R be a correspondence with $\operatorname{dis} R \le r$, and let $(x,y),(x',y') \in R$. Take a chain $x = x_0, \dots, x_N = x'$ with steps $\le s$. Choose y_i with $(x_i, y_i) \in R$, $y_0 = y$ and $y_N = y'$. Then the steps $|y_{i-1}y_i|$ are $\le s + r$. Hence $d_u^Y(y,y') \le d_u^X(x,x') + r$, and symmetrically. So R induces a correspondence between $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ of distortion $\le r$, and

$$d_{GH}(\mathcal{U}X, \mathcal{U}Y) \le d_{GH}(X,Y).$$

($\subseteq$). A path-connected Y has $d_u \equiv 0$. A path $\gamma : [0,1] \to Y$ is uniformly continuous, so a fine subdivision gives an ε-chain from $\gamma(0)$ to $\gamma(1)$. Thus $\mathcal{U}(Y) = \Delta_1$. If $d_{GH}(X, Y_n) \to 0$ with Y_n path-connected, then

$$\tfrac{1}{2} \operatorname{diam} \mathcal{U}(X) = d_{GH}(\mathcal{U}X, \Delta_1) \le d_{GH}(X, Y_n) \to 0.$$

So $\mathcal{U}(X)$ is a point.

($\supseteq$). Suppose $d_u \equiv 0$ on X. Embed X isometrically into the Banach space $\ell^\infty(X)$ by the Kuratowski map $x \mapsto d(x,\cdot) - d(x_0,\cdot)$. For $\varepsilon > 0$ let

$$Y_\varepsilon = X \cup \bigcup \{[x,y] : \ x,y \in X,\ d(x,y) < \varepsilon\}$$

(straight segments), with the metric induced from ℓ^∞. Then: - Every point of $[x,y]$ is within $d(x,y)/2 < \varepsilon$ of X. So $d_H(X, Y_\varepsilon) \le \varepsilon$, and hence $d_{GH}(X, Y_\varepsilon) \le \varepsilon$. - Y_ε is path-connected: any two points of X are joined by an ε-chain, and therefore by a polygonal path inside Y_ε. Each segment point is joined to its endpoints.

So $Y_\varepsilon \to X$ as $\varepsilon \to 0$.

5.4 D4. Distances of $\mathcal{U}(X)$ are MST edge lengths

Let X be finite and T an MST. Let $x \ne y$, let P be the T-path from x to y, and let e be its longest edge.

Removing e from T gives a partition with x and y on different sides. By the argument of B2, every pair across this partition has distance $\geq |e|$. Every chain from x to y crosses the partition, so it has $|L|_u \geq |e|$. The path P itself achieves $|e|$. Hence

$$d_u(x,y) = \max\{|f| : \ f \in P\},$$

which is an MST edge length.

In particular $d_u(x,y) > 0$ for $x \neq y$, so $\mathcal{U}(X) = (X, d_u)$ has $\#X$ points.

Which MST lengths occur: all of them. For an edge $e = uv \in T$, the T-path is e itself, so $d_u(u,v) = |e|$. Thus the set of nonzero distances of $\mathcal{U}(X)$ is exactly the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$.

More is true: by B5 applied to d_u (the graphs G_r have the same components for d and d_u), $\mathcal{U}(X)$ has the same MST spectrum as X, with multiplicities.

5.5 D5. Minimising $\operatorname{diam}\mathcal{U}(X)/\operatorname{diam} X$

By D4, $\operatorname{diam}\mathcal{U}(X) = \max_{x,y} d_u(x,y) = \sigma_1(X)$. By B4, $\sigma_1 \geq \operatorname{diam} X/(n-1)$. So

$$\frac{\operatorname{diam}\mathcal{U}(X)}{\operatorname{diam} X} \ \geq \ \frac{1}{n-1}.$$

Equality holds for $X = \{0, 1, \ldots, n-1\} \subset \mathbb{R}$, where $\mathcal{U}(X) = 1 \cdot \Delta_n$. The infimum is $1/(n-1)$, and it is attained.

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6 E. Lower bound on d_{GH} via ultrametrization

6.1 E1. Ultrametrizations that are simplexes

For finite X, $\mathcal{U}(X)$ has $\#X$ points (D4). So $m = \#X$ and $n = \#Y$. Apply (PF) to $b\Delta_n$ with $m \leq n$, $n \geq 2$. There $\alpha(D) = b$, and $\operatorname{diam} D = b$ if $m < n$, 0 if $m = n$. This gives

$$2d_{GH}(a\Delta_m, b\Delta_n) = \begin{cases} |a-b|, & m = n, \\ \max\{b,\ a-b\}, & m < n, \\ \max\{a,\ b-a\}, & m > n. \end{cases}$$

For $m = 1$, read $a\Delta_1$ as a point; the formula gives b, which is correct. By nonexpansion (D3), $d_{GH}(X,Y)$ is at least half of the right-hand side.

Sharpness for simplexes. A simplex is ultrametric, and $\mathcal{U}(Z) = Z$ for ultrametric Z: by D1, d itself is an ultrametric $\leq d$, so $d \leq d_u \leq d$. Hence for $X = a\Delta_m$ and $Y = b\Delta_n$ the lower bound is always an equality.

6.2 E2. Equal ultrametrizations, far-apart spaces

Take

$$X_N = \{0, \tfrac{1}{N}, \tfrac{2}{N}, \ldots, 1\} \subset \mathbb{R}, \qquad Y_N = \tfrac{1}{N}\Delta_{N+1}.$$

Both have $\mathcal{U} = \frac{1}{N}\Delta_{N+1}$, so $d_{GH}(\mathcal{U}X_N, \mathcal{U}Y_N) = 0$. By (PF) with singletons, $2d_{GH}(X, \lambda\Delta_{\#X}) = \max\{\lambda - \varepsilon(X), \operatorname{diam} X - \lambda\}$. So

$$d_{GH}(X_N, Y_N) = \tfrac{1}{2}\Big(1 - \tfrac{1}{N}\Big) \ \geq \ \tfrac{1}{4} \qquad (N \geq 2).$$

Note that $X_N \to [0,1]$ while $Y_N \to \Delta_1$.

What is lost. d_u remembers only the bottleneck (the largest step needed to travel between points). It forgets how many steps are needed, i.e. the large-scale geometry obtained by adding up steps along chains. In particular it forgets the diameter: a long chain of short steps and a cluster of points at the same mutual distance look the same.

6.3 E3. When $d_{GH}(\mathcal{U}X,\mathcal{U}Y) = d_{GH}(X,Y)$

For pairs of finite simplexes, and more generally for pairs of (finite) ultrametric spaces, equality **always** holds, because $\mathcal{U}$ is the identity on ultrametric spaces (E1). Equality can fail as soon as one space is not ultrametric. For example, with $X = \{0,1,2\} \subset \mathbb{R}$ and $Y = \Delta_3$:

$$d_{GH}(\mathcal{U}X,\mathcal{U}Y) = d_{GH}(\Delta_3,\Delta_3) = 0 < \tfrac{1}{2} = d_{GH}(X,Y).$$

Here $2d_{GH}(X,\Delta_3) = \max\{1-1,\, 2-1\}$ by E2's formula.

6.4 E4. $X \mapsto \mathbb{R} \times_{\ell_1} X$ preserves d_{GH} on ultrametric spaces

$\leq$ (for all spaces). Let R be a correspondence between X and Y. Then $\{((t,x),(t,y)) : (x,y) \in R\}$ is a correspondence between the products with the same distortion, because the $|t-s|$ terms cancel. So $d_{GH}(\mathbb{R} \times_1 X, \mathbb{R} \times_1 Y) \leq d_{GH}(X,Y)$.

$\geq$. Compute $\mathcal{U}(\mathbb{R} \times_1 X)$ for ultrametric X. Each chain step has length $|\Delta t| + d(x_i,x_{i+1}) \geq d(x_i,x_{i+1})$. So

$$d_u((t,x),(s,x')) \geq d_u^X(x,x') = d(x,x').$$

Conversely, move along $\mathbb{R}$ with tiny steps at fixed x, then make one jump $(s,x) \to (s,x')$ of length $d(x,x')$. Hence $d_u((t,x),(s,x')) = d(x,x')$, and $\mathcal{U}(\mathbb{R} \times_1 X) = X$. By nonexpansion,

$$d_{GH}(X,Y) = d_{GH}(\mathcal{U}(\mathbb{R} \times_1 X),\mathcal{U}(\mathbb{R} \times_1 Y)) \leq d_{GH}(\mathbb{R} \times_1 X, \mathbb{R} \times_1 Y).$$

All of this uses the correspondence form $d_{GH} = \frac{1}{2}\inf \operatorname{dis} R$, which is valid for arbitrary metric spaces (values in $[0,\infty]$).

6.5 E5. $d_H(\mathbb{R}^n_\infty,\mathbb{Z}^n_\infty) = d_{GH}(\mathbb{R}^n_\infty,\mathbb{Z}^n_\infty) = \frac{1}{2}$

Hausdorff distance. Rounding each coordinate gives a lattice point within ℓ_∞-distance $\frac{1}{2}$. The point $(\frac{1}{2},\dots,\frac{1}{2})$ is at distance exactly $\frac{1}{2}$ from $\mathbb{Z}^n$. So $d_H = \frac{1}{2}$, and hence $d_{GH} \leq \frac{1}{2}$.

Lower bound. - $\mathbb{R}^n$ is path-connected, so $\mathcal{U}(\mathbb{R}^n_\infty) = \Delta_1$. - In $\mathbb{Z}^n_\infty$, distinct points are at distance ≥ 1, and unit steps connect everything. So $d_u \equiv 1$ off the diagonal, and $\mathcal{U}(\mathbb{Z}^n_\infty) = 1 \cdot \Delta_{\aleph_0}$.

By nonexpansion,

$$d_{GH}(\mathbb{R}^n_\infty,\mathbb{Z}^n_\infty) \geq d_{GH}(\Delta_1,\Delta_{\aleph_0}) = \tfrac{1}{2}\operatorname{diam}\Delta_{\aleph_0} = \tfrac{1}{2}.$$

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7 F. Chebyshev radius and Jung constant

7.1 F1. Universal bounds, invariance, extremal cases

Let A be bounded with $\operatorname{diam} A > 0$. - **Lower bound.** For every $z \in Z$ and $a,a' \in A$, $|aa'| \leq |za| + |za'| \leq 2\sup_A |z\cdot|$. So $\operatorname{rad}_Z(A) \geq \frac{1}{2}\operatorname{diam} A$. Since Z has two points, $J(Z) \geq \frac{1}{2}$. - **Upper bound.** Choose the centre $a_0 \in A$: $\sup_a |a_0 a| \leq \operatorname{diam} A$. So rad $\leq$ diam and $J(Z) \leq 1$.

Invariance. Rescaling by $\lambda > 0$ multiplies both rad and diam by λ. An isometry $Z \to Z'$ carries bounded sets, centres, radii and diameters to each other. So J is unchanged.

When the bounds are attained. - $J(Z) = \frac{1}{2}$ iff $\operatorname{rad}_Z(A) = \frac{1}{2}\operatorname{diam} A$ for every bounded A, i.e. every set of diameter D fits in balls of radius $D/2+\varepsilon$ for all $\varepsilon > 0$. Examples: $\mathbb{R}$, ℓ_∞^n (F2), and more generally hyperconvex (injective) spaces. - $J(Z) = 1$ iff there are bounded sets for which no centre does noticeably better than a point of the set: $\sup_A \operatorname{rad} / \operatorname{diam} = 1$. Example: every ultrametric space with at least 2 points. If $\sup_a |za| < \operatorname{diam} A$ for some z, the strong triangle inequality gives $|aa'| \le \max(|za|, |za'|) < \operatorname{diam} A$ for all $a, a' \in A$, which is impossible. So rad = diam for every A. This covers $\lambda\Delta_m$, $m \ge 2$, and also the two-point space.

7.2 F2. $J(\mathbb{R})$, $J(\ell_\infty^n)$, and ℓ_∞-products

$\mathbb{R}$. $A \subset [\inf A, \sup A]$, and the midpoint is within $\frac{1}{2}\operatorname{diam} A$ of every point. So $J(\mathbb{R}) = \frac{1}{2}$.

ℓ_∞^n. Take the centre with coordinates $z_i = \frac{1}{2}(\inf_A a_i + \sup_A a_i)$. Each coordinate projection is 1-Lipschitz, so $\sup_A a_i - \inf_A a_i \le \operatorname{diam} A$. Hence $\|z-a\|_\infty \le \frac{1}{2}\operatorname{diam} A$, and $J(\ell_\infty^n) = \frac{1}{2}$.

Products. Assume X and Y each have at least 2 points. For $A \subset X \times_\infty Y$ with projections A_X, A_Y:

$$\operatorname{diam} A = \max(\operatorname{diam} A_X, \operatorname{diam} A_Y), \qquad \operatorname{rad}(A) = \max(\operatorname{rad}_X A_X, \operatorname{rad}_Y A_Y).$$

For "$\le$" in the second identity, take a pair of almost-optimal centres; "$\ge$" follows by projecting. Hence

$$\operatorname{rad}(A) \le \max(J(X)\operatorname{diam} A_X, J(Y)\operatorname{diam} A_Y) \le \max(J(X), J(Y))\operatorname{diam} A.$$

Conversely, $A = A_X \times \{y_0\}$ has rad $= \operatorname{rad}_X A_X$ and diam $= \operatorname{diam} A_X$. So $J(X \times_\infty Y) \ge J(X)$, and similarly $\ge J(Y)$. This gives $J(X \times_\infty Y) = \max\{J(X), J(Y)\}$. By induction, $J(\ell_\infty^n) = J(\mathbb{R}) = \frac{1}{2}$ again.

7.3 F3. Regular simplex and Jung's theorem

Realise the side-1 regular simplex as $v_i = e_i/\sqrt{2} \in \mathbb{R}^{n+1}$, inside the hyperplane $\sum x_i = 1/\sqrt{2}$. Its centroid is $c = \frac{1}{n+1}\sum v_i$, and

$$|v_i - c|^2 = \tfrac{1}{2}\Big[\big(1 - \tfrac{1}{n+1}\big)^2 + \tfrac{n}{(n+1)^2}\Big] = \frac{n}{2(n+1)}.$$

For any z, $\sum_i |z - v_i|^2 = \sum_i |c - v_i|^2 + (n+1)|z-c|^2$. So $\max_i |z - v_i| \ge \sqrt{n/(2(n+1))}$, with equality only at $z = c$. The smallest enclosing ball is the circumball, of radius

$$R_n = \sqrt{\frac{n}{2(n+1)}}.$$

Hence $J(\mathbb{R}^n) \ge \sqrt{n/(2n+2)}$.

Upper bound (Jung).

Step 1: at most $n+1$ points. Let S be a set of at most $n+1$ points with diameter D. Let $B(c, r)$ be its smallest enclosing ball, which exists by compactness. Then c lies in the convex hull of the contact points p_i (those with $|p_i - c| = r$). Otherwise a separating direction would let us move c and shrink the ball. Put $c = 0$ and write $0 = \sum w_i p_i$ as a convex combination of $k \le n+1$ contact points. For each i,

$$\sum_j w_j |p_i - p_j|^2 = \sum_j w_j (2r^2 - 2\langle p_i, p_j\rangle) = 2r^2.$$

Average over i with weights w_i, and use $|p_i - p_j| \le D$ for $i \ne j$:

$$2r^2 \le D^2\Big(1 - \sum w_i^2\Big) \le D^2\Big(1 - \frac{1}{k}\Big) \le D^2 \frac{n}{n+1}.$$

So $r \le D\sqrt{n/(2(n+1))}$.

Step 2: arbitrary bounded sets. Let A be bounded with diameter D, and put $r_* = D\sqrt{n/(2n+2)}$. The closed balls $B(a, r_*)$, $a \in A$, are compact and convex. Any $n+1$ of them intersect, by Step 1. By Helly's theorem together with compactness (the finite intersection property), all of them intersect. A common point z satisfies $|z - a| \le r_*$ for all $a \in A$. Hence $\mathrm{rad}(A) \le r_*$, and

$$J(\mathbb{R}^n) = \sqrt{n/(2n+2)}.$$

7.4 F4. Intrinsic versus ambient centres

In $\lambda\Delta_m$ with $m \ge 2$, every A with $\operatorname{diam} A > 0$ has at least two points, so $\operatorname{diam} A = \lambda$. Every centre $z \in \lambda\Delta_m$ is at distance λ from some point of $A \setminus \{z\}$. Hence $\mathrm{rad} = \lambda$ and $J(\lambda\Delta_m) = 1$, the maximal value.

The same m points, embedded as a regular simplex of side λ in $\mathbb{R}^{m-1}$, have Chebyshev radius $\lambda\sqrt{(m-1)/(2m)}$ by F3. This equals $\lambda/2$ for $m = 2$ and increases to $\lambda/\sqrt{2}$ as $m \to \infty$; it is always $< \lambda$.

Explanation. $\mathrm{rad}_Z(A)$ depends on the ambient space Z, not just on A. If $Z \subset W$ isometrically, then $\mathrm{rad}_W(A) \le \mathrm{rad}_Z(A)$, because there are more candidate centres. $\lambda\Delta_m$ has no midpoints and no "interior", so the only available centres are the points themselves. Euclidean space supplies the centroid. Thus the Jung constant measures how well the space itself can fill in centres; it is not an invariant of finite configurations.

7.5 F5. Stability of diameter and radius

Suppose $d_H^Z(A, B) \le \varepsilon$, and let $\eta > 0$. - **Diameter.** For $a, a' \in A$ choose $b, b' \in B$ with $|ab|, |a'b'| \le \varepsilon + \eta$. Then $|aa'| \le \operatorname{diam} B + 2\varepsilon + 2\eta$. Symmetrically, $|\operatorname{diam} A - \operatorname{diam} B| \le 2\varepsilon$. - **Radius.** For every z, $\sup_A |za| \le \sup_B |zb| + \varepsilon + \eta$. Taking the infimum over z, $\operatorname{rad} A \le \operatorname{rad} B + \varepsilon$, and symmetrically.

Application. Let $S \subset K$ be a finite ε-net of a compact set K, so $d_H(S, K) \le \varepsilon$. Since $S \subset K$, the diameter and radius are monotone, and

$$\operatorname{diam} K - 2\varepsilon \le \operatorname{diam} S \le \operatorname{diam} K, \qquad \operatorname{rad}_Z K - \varepsilon \le \operatorname{rad}_Z S \le \operatorname{rad}_Z K.$$

So both quantities can be computed to accuracy 2ε and ε respectively from finite data.

Both constants are sharp. In $Z = \mathbb{R}$ take $K = [-\varepsilon, \varepsilon]$ and $S = \{0\}$. Then S is an ε-net, $d_H = \varepsilon$, $\operatorname{diam} K - \operatorname{diam} S = 2\varepsilon$, and $\operatorname{rad} K - \operatorname{rad} S = \varepsilon$.