GH distance and partitions

For a correspondence $R \subset X \times Y$,

$$\operatorname{dis} R = \sup\Big\{|d_X(x,x') - d_Y(y,y')| : (x,y),(x',y') \in R\Big\}, \qquad d_{\mathrm{GH}}(X,Y) = \tfrac{1}{2}\inf_R \operatorname{dis} R.$$

Here $\mathcal{D}_m(X) = \Big\{\{X_i\}_{i=1}^m : X = \bigsqcup_{i=1}^m X_i,\ X_i \neq \varnothing\Big\}$ is the set of partitions into exactly m nonempty parts. For $D = \{X_i\}_{i=1}^m \in \mathcal{D}_m(X)$, put $\alpha(D) = \min_{i\neq j} d(X_i, X_j)$ and $\operatorname{diam} D = \max_i \operatorname{diam} X_i$. Then

$$2d_{\mathrm{GH}}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\Big\{\operatorname{diam} D, \lambda - \alpha(D), \operatorname{diam} X - \lambda\Big\}.$$

1. Let X have three points with distances $0 < a \le b \le c \le a + b$. Compute $d_{\mathrm{GH}}(\lambda\Delta_2, X)$ for every $\lambda > 0$, and draw its graph as a function of λ.

2. For a metric space X, a cardinal number $n \le \#X$ and real λ such that $\lambda \ge 2\operatorname{diam} X$, prove the equality

$$2d_{\mathrm{GH}}(\lambda\Delta_n, X) = \begin{cases} \inf\limits_{D\in\mathcal{D}_n(X)} \big(\lambda - \alpha(D)\big) = \lambda - \sup\limits_{D\in\mathcal{D}_n} \alpha(D), & \text{for } n \ge 2, \\ \operatorname{diam} X & \text{for } n = 1. \end{cases}$$

3. Let $X = A \sqcup B$ satisfy $\operatorname{diam} A \le p$, $\operatorname{diam} B \le p$, and $d(A,B) = q > p$. Compute $d_{\mathrm{GH}}(\lambda\Delta_2, X)$ in terms of p, q, $\operatorname{diam} X$, provided that the displayed partition is optimal. Give a simple condition guaranteeing its optimality, and identify all values of λ at which the active term in the partition formula changes.

4. Prove that if two metrics spaces are *indistinguishable*, i. e., Gromov–Hausdorff distances from them to each simplex are the same, then they have equal diameters.

5. For a finite metric space X, determine the smallest threshold λ_0 such that

$$2d_{\mathrm{GH}}(\lambda\Delta_{k+1}, X) = \lambda - \sigma_k(X)$$

holds for every $\lambda \ge \lambda_0$. Express the answer using the partition formula, and compare it with the sufficient bound $2\operatorname{diam} X$.

MST and MST spectrum

If $\#X = n$, arrange the edge lengths of an MST in nonincreasing order: $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$. The lecture gives

$$\sigma_k(X) = \max_{D\in\mathcal{D}_{k+1}(X)} \alpha(D).$$

Deleting the k longest MST edges produces a maximizing partition.
Here $\mathcal{D}_m(X) = \Big\{\{X_i\}_{i=1}^m : X = \bigsqcup_{i=1}^m X_i,\ X_i \neq \varnothing\Big\}$ is the set of partitions into exactly m nonempty parts. For $D = \{X_i\}_{i=1}^m \in \mathcal{D}_m(X)$, put $\alpha(D) = \min_{i\neq j} d(X_i, X_j)$ and $\operatorname{diam} D = \max_i \operatorname{diam} X_i$.

1. Let M be a finite metric space. Partition M into nonempty subsets M_1 and M_2, and let $v_i \in M_i$ were chosen in such a way that $|v_1v_2| = |M_1M_2|$. Prove that there exists a minimum spanning tree $T \in \mathrm{MST}(M)$ such that v_1v_2 is an edge of T.

2. Given n distinct points in the plane. It is allowed to partition them into two nonempty groups and compute the minimum distance between two points belonging to different groups. What is the largest possible number of distinct values that can be obtained in this way?

3. Let X be a finite ultrametric space with distinct nonzero distances $0 < r_1 < \dots < r_s$. Put $N_0 = \#X$, and let N_j be the number of classes of the relation $x \sim_j y \iff d(x,y) \le r_j$. Prove that r_j occurs in the MST spectrum with multiplicity $N_{j-1} - N_j$. Hence determine the least number of parts of diameter strictly smaller than $\operatorname{diam} X$.

4. Construct, for each $n \ge 3$, an n-point metric space satisfying $\sigma_1(X) < \operatorname{diam} X$. How large can the ratio $\operatorname{diam} X/\sigma_1(X)$ be? Prove the optimal bound.

5. For $r \ge 0$, let G_r be the graph on a finite metric space X whose edges join pairs at distance at most r. Prove that the number of components of G_r determines the multiplicity of every value in the MST spectrum. Use this description to recover the whole spectrum without constructing an MST.

Borsuk partitions and graph metrics

A Borsuk partition has all part-diameters strictly smaller than $\operatorname{diam} X$. For $0 < a < b \leq 2a$, encode a graph G by the metric

$$d_G(u,v) = \begin{cases} b, & uv \in E(G), \\ a, & uv \notin E(G),\ u \neq v. \end{cases}$$

Then partitions into sets of diameter $< b$ are precisely proper colourings of G.

1. Let $G = C_{2r+1}$ (a cycle with $2r+1$ vertices) and let X_G be its (a,b)-metric space. Compute
$$\min\bigl\{m : 2d_{\mathrm{GH}}(a\Delta_m, X_G) < b\bigr\}$$
and exhibit an optimal partition. Repeat for complete multipartite graphs.
2. Fix $0 < \theta < 1$. Derive from the simplex–partition formula a useful necessary and sufficient condition for a partition $X = X_1 \sqcup \cdots \sqcup X_m$ with $\operatorname{diam} X_i \leq \theta \operatorname{diam} X$. Determine a range of λ for which the criterion has the cleanest form.
3. Let X_G be the (a,b)-metric space of a graph G. Prove that the least number of parts of diameter $< b$ equals $\chi(G)$. Reformulate this identity entirely in terms of distances from X_G to simplexes.
4. Apply the graph-metric construction to the d-dimensional cube graph Q_d, whose vertex set is $\{0,1\}^d$ and whose edges join binary vectors that differ in exactly one coordinate. Find the least number of Borsuk parts of X_{Q_d}, exhibit an optimal partition, and determine the corresponding GH inequality against a simplex.
5. Let G and H be graphs on the same vertex set which differ by one edge. Compare the least numbers of Borsuk parts of X_G and X_H. Construct examples showing that both equality and a strict increase can occur, and translate the comparison into GH-distance statements.

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Ultrametrization

For a chain $L = (x_0, \ldots, x_N)$ let $|L|_u = \max_i d(x_{i-1}, x_i)$ and define

$$d_u(x,y) = \inf_{L : x \rightsquigarrow y} |L|_u.$$

After identifying pairs at d_u-distance zero, one obtains the ultrametric space $\mathcal{U}(X)$.

1. Show that d_u is *the largest subdominant ultrametric*, i. e., it is the largest ultrametric that is less than or equal to the initial metric d.
2. Give an example of a metric space X that is not path-connected, for which $\mathcal{U}(X) = \Delta_1$.
3. Prove that a closure of the class of all path-connected metric spaces with respect to GH-convergence is the class of all metric spaces, which are sent to Δ_1 by ultrametrization.
4. Prove that for every finite X, every nonzero distance of $\mathcal{U}(X)$ is an MST edge length. Determine precisely which MST edge lengths actually occur as distances in $\mathcal{U}(X)$.
5. For each $n \geq 3$, construct an n-point metric space for which $\operatorname{diam}\mathcal{U}(X)/\operatorname{diam} X$ is as small as possible. Determine the infimum of this ratio when n is fixed.

Lower bound on d_{GH} via ultrametrization

For a chain $L = (x_0, \dots, x_N)$ let $|L|_u = \max_i d(x_{i-1}, x_i)$ and define

$$d_u(x, y) = \inf_{L: x \rightsquigarrow y} |L|_u.$$

After identifying pairs at d_u-distance zero, one obtains the ultrametric space $\mathcal{U}(X)$. Ultrametrization is nonexpanding:

$$d_{\mathrm{GH}}\big(\mathcal{U}(X), \mathcal{U}(Y)\big) \le d_{\mathrm{GH}}(X, Y).$$

Thus any computable difference between the ultrametric quotients gives a lower bound for the original GH distance.

1. Let X and Y be finite metric spaces whose ultrametrizations are simplexes $a\Delta_m$ and $b\Delta_n$. Compute $d_{\mathrm{GH}}\big(\mathcal{U}(X), \mathcal{U}(Y)\big)$ and hence obtain an explicit lower bound for $d_{\mathrm{GH}}(X, Y)$. Determine when this lower bound is sharp for X and Y themselves simplexes.
2. Construct finite metric spaces X_N, Y_N such that $d_{\mathrm{GH}}(\mathcal{U}(X_N), \mathcal{U}(Y_N)) = 0$ but $d_{\mathrm{GH}}(X_N, Y_N)$ is bounded below by a positive constant independent of N. Explain precisely which geometric information ultrametrization loses in your example.
3. Characterize equality in

   $$d_{\mathrm{GH}}\big(\mathcal{U}(X), \mathcal{U}(Y)\big) \le d_{\mathrm{GH}}(X, Y)$$

   for pairs of finite simplexes and for pairs of finite ultrametric spaces. Give an example where the inequality is strict.
4. Show that a map[1] $X \mapsto \mathbb{R} \times_{\ell_1} X$ is isometric for ultrametric spaces X.
5. Denote by $\mathbb{R}^n_\infty$ the space $\mathbb{R}^n$ endowed with ℓ_∞-norm. Let $\mathbb{Z}^n_\infty$ be the integer lattice in $\mathbb{R}^n_\infty$. Prove that $d_H(\mathbb{R}^n_\infty, \mathbb{Z}^n_\infty) = d_{GH}(\mathbb{R}^n_\infty, \mathbb{Z}^n_\infty)$.

[1] Here, $\mathbb{R} \times_{\ell_1} X$ is the Cartesian product equipped with the Manhattan distance.

Chebyshev radius and Jung constant

For a bounded nonempty set $A \subset Z$, define its Chebyshev radius in Z by

$$\mathrm{rad}_Z(A) = \inf_{z \in Z} \sup_{a \in A} d(z, a), \qquad J(Z) = \sup_{\operatorname{diam} A > 0} \frac{\mathrm{rad}_Z(A)}{\operatorname{diam} A}.$$

Always $\frac{1}{2} \le J(Z) \le 1$. For normed spaces,

$$J(\mathbb{R}^n) = \sqrt{\frac{n}{2(n+1)}}, \qquad J(\ell^n_\infty) = \frac{1}{2}.$$

Thus every bounded $A \subset Z$ lies in a ball of radius $J(Z) \operatorname{diam} A$, and $|\mathrm{rad}_Z(A) - \mathrm{rad}_Z(B)| \le d_H^Z(A, B)$.

1. Prove the universal estimates $\frac{1}{2} \le J(Z) \le 1$ for every metric space containing at least two points. Show that the Jung constant is unchanged under rescaling and isometry. Determine when either universal bound is attained.
2. Compute $J(\mathbb{R})$ and prove directly that $J(\ell^n_\infty) = \frac{1}{2}$. More generally, prove that for the ℓ_∞ product metric,

   $$J(X \times_\infty Y) = \max\{J(X), J(Y)\}.$$
3. Let $v_0, \dots, v_n$ be the vertices of a regular Euclidean n-simplex of side length 1. Compute the radius of its smallest enclosing ball. Use this example for the lower bound in Jung's theorem $J(\mathbb{R}^n) = \sqrt{n/(2n+2)}$, and prove the matching upper bound.
4. Compare the ambient and intrinsic choices of centres. Compute the Jung constant of the equilateral finite metric space $\lambda\Delta_m$ when centres must lie in the space itself, and compare it with the Chebyshev radius of a regular simplex embedded in $\mathbb{R}^{m-1}$. Explain the discrepancy.
5. If $d_H^Z(A, B) \le \varepsilon$, prove $|\operatorname{diam} A - \operatorname{diam} B| \le 2\varepsilon$ and $|\mathrm{rad}_Z(A) - \mathrm{rad}_Z(B)| \le \varepsilon$. Apply this to a finite ε-net in a compact set, and determine which constants are sharp.