Chapter 1

Applications of GH-distance: MST-spectrum, Borsuk problem and combinatorial invariants

In this chapter we will show how the Gromov–Hausdorff distance to simplexes allows us to calculate various combinatorial and geometric characteristics of finite metric spaces:

(1) MST-spectrum (lengths of the edges of the minimum spanning tree),

(2) existence of Borsuk partition,

(3) chromatic number of the graph,

(4) the clique cover number of the graph,

(5) the minimum dimension of a rectangular (Manhattan) space into which a given finite metric space is isometrically embedded.

1.1 MST-spectrum and GH-distances to simplexes

Let Δ_k denote the k-point simplex (all distances are equal to 1, k is a natural number). For a metric space X, let λX denote the space obtained by multiplying all distances by λ.

Let us recall the notion of a minimum spanning tree.

Definition 1.1. Let G be a finite connected weighted graph. A ***spanning tree*** of G is a subgraph of G which is a tree and contains all vertices of G; the ***weight*** of a spanning tree is the sum of the weights of its edges. A ***minimum spanning tree*** (MST) is a spanning tree of the least weight.

Let us recall the definition of MST-spectrum.

Definition 1.2. Let X be a finite metric space with $\#X = n$. MST-***spectrum*** $\sigma(X) = (\sigma_1, \ldots, \sigma_{n-1})$ is the set of edge lengths of an arbitrary minimum spanning tree (MST) on X, arranged in non-increasing order.

Remark 1.3. A minimum spanning tree exists for any finite connected weighted graph. In the case of a metric space X, the graph is taken to be a complete graph on the set X with edge weights equal to the distances between the vertices. The multiset of edge lengths of an MST does not depend on the choice of the tree (see [46, Proposition 2.1]).

We need the following lemma, which expresses the MST-spectrum in terms of space partitions.

Lemma 1.4 ([46, Theorem 2.1]). *Let X be a finite metric space with $\#X = n$, and $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$ be its* MST*-spectrum. Then for each $k = 1, \dots, n-1$:*

$$\sigma_k = \max_{D \in \mathcal{D}_{k+1}(X)} \alpha(D),$$

where:

- $\mathcal{D}_{k+1}(X)$: *the set of all partitions of X into $k+1$ non-empty subsets*;
- $\alpha(D) = \min_{1 \le i < j \le k+1} |X_i X_j|$, *where* $|X_i X_j| = \inf\{|xy| : x \in X_i, y \in X_j\}$.

Proof. Step 1. *Relationship between splits and cuts.* Consider an arbitrary minimal spanning tree T on X. Removing from T its k edges with the largest weights (i.e. edges whose length is equal to $\sigma_1, \dots, \sigma_k$) splits T into $k+1$ connected components. Let us denote by D_T the partition of X into sets of vertices of these components.

Step 2. *Estimate $\alpha(D_T) \ge \sigma_k$.* Let $e_1, \dots, e_k$ be the k largest edges of T, and $T' = T \setminus \{e_1, \dots, e_k\}$. Then the components T' give the partition D_T. Any edge $e \in T$ connecting two different components of D_T does not belong to T', which means $e \in \{e_1, \dots, e_k\}$, therefore $|e| \ge \sigma_k$ (since σ_k is the smallest among the removed ones). Therefore $\alpha(D_T) \ge \sigma_k$.

Step 3. *Estimate $\alpha(D) \le \sigma_k$ for any partition D.* Let $D = \{X_1, \dots, X_{k+1}\} \in \mathcal{D}_{k+1}(X)$ be an arbitrary partition. Consider the graph G_D on $k+1$ vertices (corresponding to the elements of the partition), in which two vertices i, j are connected by all edges from MST, which in X connect X_i and X_j. Since T is connected, G_D is connected and contains at least k edges. Let us choose k such edges with the least weights. Then among them there is an edge of length $\le \sigma_k$ (otherwise there would be more than k edges of length $> \sigma_k$ in T, which contradicts the definition of σ_k as the kth largest edge).

Therefore, there is a pair of distinct parts X_i, X_j with $|X_i X_j| \le \sigma_k$. Thus, $\alpha(D) \le \sigma_k$.

Step 4. *Conclusion.* From steps 2 and 3 we obtain that $\sigma_k = \max_{D \in \mathcal{D}_{k+1}(X)} \alpha(D)$. □

Remark 1.5. Lemma 1.4 is a standard result in the theory of minimum spanning trees: the k-th longest edge of MST is equal to the maximum, over all partitions of the set of vertices into $k+1$ parts, the length of the shortest edge between the parts. This is a cut characterization of Kruskal's algorithm; see, for example, [5, Section 4.7, Claim (4.26)].

Now we are ready to establish the connection between the MST-spectrum and GH-distances to simplexes.

Theorem 1.6 ([46, Corollary 3.2]). *Let X be a finite metric space with $\#X = n$, and $\lambda \ge 2 \operatorname{diam} X$. Then*

$$\sigma_k(X) = \lambda - 2 d_{GH}(\lambda \Delta_{k+1}, X)$$

for each $k = 1, \dots, n-1$.

Proof. Step 1. *Application of the* MST*-spectrum lemma.* By Lemma 1.4,

$$\sigma_k = \max_{D \in \mathcal{D}_{k+1}(X)} \alpha(D).$$

Step 2. *Application of the formula for the distance to the simplex.* According to the formula for the distance to a simplex with a smaller or equal number of points (see also [30, Corollary 2.5]), for $m = k+1 \le n$:

$$2 d_{GH}(\lambda \Delta_m, X) = \inf_{D \in \mathcal{D}_m(X)} \max\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\}.$$

Step 3. *Simplification with* $\lambda \geq 2\operatorname{diam} X$. Since $\operatorname{diam} D \leq \operatorname{diam} X$ for any partition D, and $\lambda \geq 2\operatorname{diam} X$, then

$$\lambda - \alpha(D) \geq \lambda - \operatorname{diam} X \geq \operatorname{diam} X \geq \operatorname{diam} D.$$

Moreover, $\operatorname{diam} X - \lambda \leq -\operatorname{diam} X \leq 0$, so $\max\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\} = \lambda - \alpha(D)$.

Therefore,

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \big(\lambda - \alpha(D)\big) = \lambda - \sup_{D\in\mathcal{D}_m(X)} \alpha(D).$$

Step 4. *Substitution* $m = k+1$. With $m = k+1$ we get:

$$2d_{GH}(\lambda\Delta_{k+1}, X) = \lambda - \max_{D\in\mathcal{D}_{k+1}(X)} \alpha(D) = \lambda - \sigma_k.$$

Therefore,

$$\sigma_k = \lambda - 2d_{GH}(\lambda\Delta_{k+1}, X).$$

□

Remark 1.7. The condition $\lambda \geq 2\operatorname{diam} X$ is essential: it guarantees that $\lambda - \alpha(D) \geq \operatorname{diam} D$ for all partitions D, which allows us to simplify the maximum. If $\lambda < 2\operatorname{diam} X$, the formula becomes more complex and requires taking into account $\operatorname{diam} D$.

Example 1.8. Let X be a three-point space with distances $a \leq b \leq c$, where $c \leq a + b$ (triangle inequality), and $\lambda \geq 2c$. Then $\operatorname{diam} X = c$, and the MST-spectrum consists of two numbers: $\sigma_1 = b$ (the largest edge of MST) and $\sigma_2 = a$ (the smallest edge of MST). It is known [31, Example 2.12] that for two three-point spaces X_1 and X_2 with distances $a_1 \leq b_1 \leq c_1$ and $a_2 \leq b_2 \leq c_2$ it holds

$$2d_{GH}(X_1, X_2) = \max\{|a_1 - a_2|, |b_1 - b_2|, |c_1 - c_2|\},$$

where the distances can be zero. Thus, considering $\lambda\Delta_2$ as a three-point space with distances $0 \leq \lambda \leq \lambda$, we get

$$2d_{GH}(\lambda\Delta_2, X) = \max\{a, \lambda - b, \lambda - c\} = \lambda - b,$$
$$2d_{GH}(\lambda\Delta_3, X) = \max\{\lambda - a, \lambda - b, \lambda - c\} = \lambda - a.$$

On the other hand, by Theorem 1.6 we have

$$\sigma_1 = \lambda - 2d_{GH}(\lambda\Delta_2, X) = b, \qquad \sigma_2 = \lambda - 2d_{GH}(\lambda\Delta_3, X) = a,$$

which is consistent with direct calculation.

1.2 Generalized Borsuk problem

The classic Borsuk problem goes back to Karol Borsuk's question: into how many parts should an arbitrary bounded subset of Euclidean space be divided in order to obtain pieces of strictly smaller diameters. In 1933, Borsuk put forward a conjecture: any bounded subset of $\mathbb{R}^n$ can be partitioned into at most $n+1$ subsets of smaller diameter [10]. The conjecture was proved by Hadwiger for convex bodies with a smooth boundary [11] and refuted in the general case in 1993 by Kahn and Kalai [12]; the current state of the issue is presented, for example, in the review by Raigorodskii [14]. On the other hand, Lyusternik and Shnirelman [13], and

a little later independently Borsuk [10] showed that the standard sphere and the standard ball in $\mathbb{R}^n$, $n \geq 2$, cannot be divided into $m \leq n$ subsets of smaller diameter.

Since the 1993 refutation, the main line of work has been to reduce the dimensionality, to which there are already counterexamples. The Kahn–Kalai construction uses the Frankl–Wilson theorem: a family of subsets of a finite set is encoded by points in Euclidean space such that the diameter is realized exactly on pairs with a prescribed intersection; then the estimate for the independent set in the corresponding diameter graph gives a lower estimate for the Borsuk number $f(n)$. Their original example works in dimension 1325 and in all dimensions $n \geq 2015$, and also gives the exponential bound $f(n) \geq (1.2)^{\sqrt{n}}$ [12, 21]. Nilli (Alon) simplified the argument and reduced the dimension to 946 [24]. Further refinements of the same combinatorial scheme were given by 903 (Grey-Weissbach) [17], 561 (Raigorodskii) [26] and 560 (Weissbach) [29]. The next jump is associated not with $\{0,1\}$-codes, but with spherical codes of the Lich lattice: Hinrichs obtained the dimension 323 [18], Pikhurko — 321 [25], Hinrichs and Richter — 298 [19]. In 2013, Bondarenko answered Larman's question about two-point sets: the Euclidean representation of a strongly regular graph $G_2(4)$ gives a two-point set of 416 points in $\mathbb{R}^{65}$, which cannot be partitioned into fewer than 84 parts of smaller diameter [16]. Jenrich and Brouwer cut from it a two-point subset of 352 points in $\mathbb{R}^{64}$, which cannot be divided into less than 71 part [20]. Until 2026, this remained the smallest published counterexample dimension. The best asymptotic lower bound for the Borsuk number is still due to Raigorodsky: $f(n) \geq \Big(\big(2/\sqrt{3}\big)^{\sqrt{2}} + o(1)\Big)\sqrt{n} = \big(1.2255\ldots + o(1)\big)\sqrt{n}$ [27].

In 2025-2026, further work on the same problem appeared on arXiv. Bikeev and Raigorodskii refined the upper bounds for partitions into parts of diameter less than a given one $b \in (0,1]$ [15]. Musin reformulated the two-point case in terms of graphs [23]. Tolmachev and Voronov lowered the upper bound for the Borsuk number in $\mathbb{R}^4$ from 9 (Lassac) to 8 [28]. The smallest known dimension of the counterexample is now 63: Ji, starting from the Euclidean representation $G_2(4)$ and the 320-point Jenrich–Brauer kernel, constructed a 321-point set in $\mathbb{R}^{63}$, which cannot be divided into 64 parts of smaller diameter, so that $b(63) \geq 65$ [22]. Thus, the range $4 \leq n \leq 62$ remains open.

Following [33], we consider a generalization of this problem to an arbitrary bounded metric space X and partitions of any cardinality $m \leq \#X$ (not necessarily finite).

Recall that Δ_m is a simplex of cardinality m (all non-zero distances are equal to 1), and $\mathcal{D}_m(X)$ is the family of all partitions of the set X into m non-empty subsets. For $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ we set

$$\operatorname{diam} D = \sup_{i\in I} \operatorname{diam} X_i, \qquad \alpha(D) = \inf_{i\neq j} |X_i X_j|.$$

Definition 1.9. Let X be a bounded metric space and m be a cardinal number, $m \leq \#X$. The partition $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ is called ***Borsuk partition*** (into parts of strictly smaller diameter) if there exists $\varepsilon > 0$ such that $\operatorname{diam} X_i \leq \operatorname{diam} X - \varepsilon$ for all $i \in I$. In other words, $\operatorname{diam} D < \operatorname{diam} X$.

Generalized Borsuk problem: can a given bounded metric space X be divided into a given, possibly infinite, number m of subsets, each of which has a strictly smaller diameter than X?

Theorem 1.10 ([48, Theorem 8.7])**.** *Let X be an arbitrary bounded metric space and m be a cardinal number, $m \leq \#X$. Let us choose an arbitrary number $0 < \lambda < \operatorname{diam} X$. Then X admits partitioning into m subsets of strictly smaller diameter if and only if*

$$2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X.$$

Proof. By the formula for the distance to a simplex of arbitrary cardinality (see [30], as well as [33, Theorem 1.3]) for $m \leq \#X$

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\bigl\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\bigr\}.$$

In particular, $2d_{GH}(\lambda\Delta_m, X) \leq \operatorname{diam} X$. If $0 < \lambda < \operatorname{diam} X$, then $\operatorname{diam} X - \lambda < \operatorname{diam} X$ and $\lambda - \alpha(D) < \operatorname{diam} X$, thus the equality $2d_{GH}(\lambda\Delta_m, X) = \operatorname{diam} X$ is achieved if and only if $\operatorname{diam} D = \operatorname{diam} X$ for any $D \in \mathcal{D}_m(X)$. This means that there is no partition of X into m parts of strictly smaller diameter.

Therefore, $2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X$ if and only if there exists $D \in \mathcal{D}_m(X)$ with $\operatorname{diam} D < \operatorname{diam} X$, i.e. a Borsuk partition of cardinality m. □

1.3 Calculation of the chromatic number, the clique cover number and the Manhattan dimension via GH distances

In this section we will show that the Gromov–Hausdorff distance to simplices allows us to calculate three classical combinatorial and geometric characteristics of finite metric spaces:

(1) chromatic number of the graph,

(2) the clique cover number of the graph,

(3) the minimum dimension of a rectangular (Manhattan) space into which a given finite metric space is isometrically embedded.

1.3.1 Metric spaces with two distances

We will need a special class of metric spaces.

Definition 1.11. Let $0 < a < b$. A finite metric space (X, d) is called a (a, b)-***space*** if for any distinct $x, y \in X$ $d(x, y) \in \{a, b\}$ holds.

For each (a, b)-space X we define the graph G_X as follows:

- set of vertices $V(G_X) := X$;
- two vertices $x, y \in X$ are connected by an edge in G_X if and only if $d(x, y) = b$.

We similarly define the graph H_X by placing an edge between x and y if and only if $d(x, y) = a$.

Remark 1.12. The graph H_X is complementary to the graph G_X: $H_X = \overline{G_X}$.

1.3.2 The clique cover number

Definition 1.13. A subset C of the vertices of a (simple) graph $G = (V, E)$ is called a ***clique*** if any two distinct vertices from C are joined by an edge (equivalently, the subgraph induced by the set C is complete). By a clique we mean exactly a set of vertices, not a subgraph. In particular, a one-point set is also a clique.

Definition 1.14. ***The clique cover number*** of a graph $G = (V, E)$ is the smallest natural number k for which there exists a family of cliques $C_1, \dots, C_k$ of G such that $V = \bigcup_{i=1}^{k} C_i$. Notation: $\theta(G)$.

Let $0 < a < b \le 2a$. Using the graph G, we construct a (a, b)-space $Y_G = (V, d_G^a)$, in which

$$d_G^a(u, v) := \begin{cases} a, & \text{if } \{u, v\} \in E, \\ b, & \text{if } \{u, v\} \notin E,\ u \neq v. \end{cases}$$

In other words, the edges of G correspond to a smaller distance a. If G is complete, all nonzero distances are equal to a; otherwise Y_G has distance b and $\operatorname{diam} Y_G = b$.

Theorem 1.15 ([48, Corollary 8.9])**.** *For any finite graph $G = (V, E)$ and any $m \ge 1$*

$$\theta(G) \le m \iff 2d_{GH}(a\Delta_m, Y_G) < b.$$

In particular,

$$\theta(G) = \min\bigl\{m \ge 1 :\ 2d_{GH}(a\Delta_m, Y_G) < b\bigr\}.$$

Proof. Let $n = \#V$ be the number of vertices of G.

First, a key observation. Suppose G is not complete, so that Y_G contains a pair of points at distance b; then $\operatorname{diam} Y_G = b$. A subset $W \subseteq V$ is a clique of G if and only if W contains no pair of non-adjacent vertices; since the nonzero distances in Y_G take only the values a and b, this is equivalent to $\operatorname{diam} W < b = \operatorname{diam} Y_G$. In other words, partitions of the vertex set V into cliques are exactly Borsuk partitions of Y_G.

For $m \le n$, a covering of V by at most m cliques can be turned into a partition of V into exactly m cliques, so the inequality $\theta(G) \le m$ is equivalent to the existence of a partition of V into m cliques. Applying Theorem 1.10 to $X = Y_G$ with $\lambda = a$ (recall that $0 < a < b = \operatorname{diam} Y_G$), we find: a Borsuk partition into m parts exists if and only if $2d_{GH}(a\Delta_m, Y_G) < b$. Hence

$$\theta(G) \le m \iff 2d_{GH}(a\Delta_m, Y_G) < b, \qquad m \le n.$$

If $m \ge n$, then $\theta(G) \le n \le m$, while any surjection $\pi\colon a\Delta_m \to Y_G$ gives a correspondence of distortion at most a; hence $2d_{GH}(a\Delta_m, Y_G) \le a < b$, and the equivalence holds for $m \ge n$ as well.

It remains to consider the case of a complete graph G. Then all nonzero distances in Y_G equal a, so $Y_G = a\Delta_n$ and $\theta(G) = 1$. A similar correspondence between $a\Delta_m$ and $a\Delta_n$ yields $2d_{GH}(a\Delta_m, a\Delta_n) \le a < b$ for all $m \ge 1$. Thus here too the inequality $\theta(G) \le m$ is equivalent to $2d_{GH}(a\Delta_m, Y_G) < b$.

This proves the equivalence for all $m \ge 1$. The smallest $m \ge 1$ for which the inequality holds equals $\theta(G)$: for $m < \theta(G)$ neither side holds, while for $m = \theta(G)$ both do. □

1.3.3 Chromatic number

Definition 1.16. ***The chromatic number*** of a graph $G = (V, E)$ is the smallest natural number k for which there exists a coloring $\gamma : V \to \{1, \dots, k\}$ such that for any edge $\{u, v\} \in E$, $\gamma(u) \neq \gamma(v)$ holds. Notation: $\chi(G)$.

A proper coloring of G is a clique cover of the complementary graph, so $\chi(G) = \theta(\overline{G})$.

Let again $0 < a < b \le 2a$. Using the graph G, we construct a (a, b)-space $X_G = (V, d_G^b)$, in which

$$d_G^b(u, v) := \begin{cases} b, & \text{if } \{u, v\} \in E, \\ a, & \text{if } \{u, v\} \notin E,\ u \neq v. \end{cases}$$

This is the same space as $Y_{\overline{G}}$ in Theorem 1.15: the edges of G correspond to the larger distance b. If G is empty, all nonzero distances are equal to a; otherwise $\operatorname{diam} X_G = b$.

Theorem 1.17 ([48, Corollary 8.10]). *For any finite graph $G = (V, E)$ and any $m \geq 1$*

$$\chi(G) \leq m \iff 2d_{GH}(a\Delta_m, X_G) < b.$$

In particular,

$$\chi(G) = \min\bigl\{m \geq 1 : \ 2d_{GH}(a\Delta_m, X_G) < b\bigr\}.$$

Proof. By duality $\chi(G) = \theta(\overline{G})$ and $X_G = Y_{\overline{G}}$. It remains to apply Theorem 1.15 to $\overline{G}$. □

1.3.4 Manhattan dimension

Now consider the problem of an isometric embedding of a finite metric space into a rectangular (Manhattan) space $\mathbb{R}^m_1$ with metric ℓ_1:

$$\|x - y\|_1 = \sum_{i=1}^{m} |x_i - y_i|.$$

A pseudometric space (X, d) is said to be ***m*-embeddable** into a rectangular space if there is an isometric embedding $f\colon X \to \mathbb{R}^m_1$, that is, $d(x, y) = \bigl\|f(x) - f(y)\bigr\|_1$ for all $x, y \in X$.

Definition 1.18. The ***Manhattan dimension*** (or ℓ_1-dimension) of a finite metric space (X, d) is the smallest natural number $m \geq 1$ such that there exists an isometric embedding $f\colon X \to \mathbb{R}^m_1$. Notation: $\dim_{\ell_1}(X)$.

To connect this quantity with cuts, hypergraphs and the Gromov–Hausdorff distance, we fix all the constructions used.

Definition 1.19 (Cut and cut pseudometrics). Let X be a finite set. ***A cut*** of a set X is an unordered pair $c = \{S, S'\}$, where $S \subset X$ is a proper non-empty subset and $S' = X \setminus S$.

The cut $c = \{S, S'\}$ is associated with the ***cut pseudometric*** δ_c on X:

$$\delta_c(x, y) = \begin{cases} 1, & \text{if } \#\bigl(S \cap \{x, y\}\bigr) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\delta_c(x, y) = 1$ if and only if x and y lie on opposite sides of the cut, and $\delta_c(x, y) = 0$ otherwise (in particular, $\delta_c(x, x) = 0$).

If $\mathcal{C}$ is a family of cuts of the set X and $\lambda\colon \mathcal{C} \to \mathbb{R}$ is a mapping with $\lambda_c := \lambda(c) > 0$ for all $c \in \mathcal{C}$, then the function

$$d_{\mathcal{C},\lambda} = \sum_{c \in \mathcal{C}} \lambda_c \delta_c$$

is called a ***cut metric*** (more precisely, a cut pseudometric) corresponding to the family $\mathcal{C}$. The set of all such functions forms a ***cone of cuts*** $\mathrm{CUT}_{\#X}$.

Known fact 1.20 (Assouad criterion, [43]). A finite metric space (X, d) is isometrically embedded in some rectangular space (equivalently: in some ℓ_1-space) if and only if $d \in \mathrm{CUT}_{\#X}$, that is, when there exists a family of cuts $\mathcal{C}$ of the set X and positive coefficients $\lambda_c > 0$ such that

$$d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c.$$

If such a representation exists, it is natural to ask what is the smallest feasible dimension of rectangular space. The answer is given in terms of the coloring of a special hypergraph constructed from the family of cuts $\mathcal{C}$.

Definition 1.21 (Hypergraph)**.** ***Hypergraph*** is a pair $H = (V, E)$, where V is a set of vertices, and E is a family of non-empty subsets of V, called ***hyperedges*** (or simply edges). Unlike a simple graph, a hyperedge can contain any number of vertices, at least 1.

A coloring of the vertices of a hypergraph H in m colors is called ***proper*** if no hyperedge is monochromatic, that is, in each $e \in E$ there are at least two different colors. The smallest such $m \geq 1$ is called the ***chromatic number*** of the hypergraph and is denoted by $\chi(H)$. A hypergraph is m***-colorable*** if and only if $\chi(H) \leq m$.

Definition 1.22 (Incompatible cuts and asteroidal triples)**.** Let $\mathcal{C}$ be a family of cuts of a finite set X.

Two cuts $\{A, A'\}$ and $\{B, B'\}$ are said to be ***incompatible*** if all four intersections

$$A \cap B, \qquad A \cap B', \qquad A' \cap B, \qquad A' \cap B'$$

are not empty.

Three cuts $\{A, A'\}$, $\{B, B'\}$ and $\{C, C'\}$ form an ***asteroidal triple*** if one can select one side from each cut — the sets $\widetilde{A} \in \{A, A'\}$, $\widetilde{B} \in \{B, B'\}$ and $\widetilde{C} \in \{C, C'\}$ — so that $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are pairwise disjoint.

Definition 1.23 (Nested hypergraph)**.** Let $\mathcal{C}$ be a family of cuts of a finite set X. ***Nested hypergraph*** $\Gamma(\mathcal{C})$ is a hypergraph with a set of vertices $\mathcal{C}$, the hyperedges of which are:

- all two-element sets $\{c, c'\} \subset \mathcal{C}$ consisting of incompatible cuts;
- all three-element sets $\{c, c', c''\} \subset \mathcal{C}$ forming an asteroidal triple.

Thus, $\Gamma(\mathcal{C})$ is a hypergraph with edges of sizes 2 and 3.

Note that for the asteroidal triple

$$\Big\{\big\{\{a,b\},\{c\}\big\}, \big\{\{b,c\},\{a\}\big\}, \big\{\{c,a\},\{b\}\big\}\Big\},$$

none of the pairs $\big\{\{a,b\},\{c\}\big\}$, $\big\{\{b,c\},\{a\}\big\}$, $\big\{\{c,a\},\{b\}\big\}\Big\}$ is an edge of $\Gamma(\mathcal{C})$. Therefore, hyperedges of size 3 cannot simply be forgotten and one cannot pass to the underlying simple graph.

The following criterion is due to Bandelt, Cepoy and Laurent.

Theorem 1.24 (Bandelt–Chepoi–Laurent, [42])**.** *Let (X, d) be a finite metric space and*

$$d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c, \qquad \lambda_c > 0,$$

where $\mathcal{C}$ is a family of cuts of the set X. Then (X, d) is isometrically embedded in an m-dimensional rectangular space $\mathbb{R}_1^m$ if and only if the nested hypergraph $\Gamma(\mathcal{C})$ is m-colorable. In particular,

$$\dim_{\ell_1}(X) = \chi(\Gamma(\mathcal{C})).$$

To reduce the coloring of the hypergraph $\Gamma(\mathcal{C})$ to the coloring of simple graphs (and thereby apply Theorem 1.17), we replace each hyperedge of size 3 with one ordinary edge.

Definition 1.25 (Family of simple graphs $\mathcal{G}(\mathcal{C})$)**.** Let $\mathcal{C}$ be a family of cuts, $\Gamma(\mathcal{C})$ be the corresponding nested hypergraph, and let $\{T_1, \dots, T_k\}$ be all asteroidal triples in $\Gamma(\mathcal{C})$ (hyperedges of size 3). Let E_2 denote the set of all hyperedges of size 2 (that is, all pairs of incompatible cuts).

For each triple $T_i = \{c_1, c_2, c_3\}$ we choose one of three possible edges $\{c_1, c_2\}$, $\{c_2, c_3\}$ or $\{c_1, c_3\}$. The resulting set of choices defines a simple graph G with a set of vertices $\mathcal{C}$ and a set of edges

$$E(G) = E_2 \cup \{\, e_i : \ i = 1, \dots, k \,\},$$

where e_i is the edge selected for T_i. If the same edge is selected for multiple triplets, it is included in $E(G)$ once (there are no multiple edges). The family of all simple graphs obtained by all possible such choices is denoted by $\mathcal{G}(\mathcal{C})$. Its power does not exceed 3^k.

If there are no asteroidal triples, then $\Gamma(\mathcal{C})$ is itself a simple graph, and $\mathcal{G}(\mathcal{C}) = \big\{\Gamma(\mathcal{C})\big\}$.

Lemma 1.26 ([45, Lemma 3.2])**.** *Let $\mathcal{C}$ be the family of cuts of a finite set X, $\Gamma(\mathcal{C})$ the corresponding nested hypergraph, and $\mathcal{G}(\mathcal{C})$ the family of simple graphs from Definition 1.25. Then $\Gamma(\mathcal{C})$ is m-colorable if and only if at least one graph from $\mathcal{G}(\mathcal{C})$ is m-colorable. Therefore,*

$$\chi(\Gamma(\mathcal{C})) = \min_{G \in \mathcal{G}(\mathcal{C})} \chi(G).$$

Proof. Let ξ be a proper m-coloring of $\Gamma(\mathcal{C})$. Then each asteroidal triple contains at least two vertices of different colors. For each such triple, we add an edge connecting some pair of different-colored vertices and delete the triple itself. The resulting simple graph G belongs to $\mathcal{G}(\mathcal{C})$, and the same coloring of ξ is correct for G.

Conversely, let $G \in \mathcal{G}(\mathcal{C})$ and ξ be a proper coloring of G. Then ξ is also correct for $\Gamma(\mathcal{C})$: every hyperedge of size 2 is an edge of G, and every asteroidal triple contains at least one edge of G and is therefore not monochromatic. □

Corollary 1.27 ([45, Corollary 3.3])**.** *Let (X, d) be a finite metric space and $d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c$ with $\lambda_c > 0$. Then (X, d) is embeddable in $\mathbb{R}_1^m$ if and only if the family $\mathcal{G}(\mathcal{C})$ contains an m-colorable simple graph, and*

$$\dim_{\ell_1}(X) = \min_{G \in \mathcal{G}(\mathcal{C})} \chi(G).$$

Let us fix the numbers $0 < a < b \le 2a$. For each simple graph $G \in \mathcal{G}(\mathcal{C})$ we construct a (a, b)-space $C_G = (\mathcal{C}, d_G^b)$ on the set of vertices $\mathcal{C}$ (that is, on the set of cuts) according to the rule of Theorem 1.17: the distance between adjacent vertices is b, between non-adjacent vertices is a. This is the same space as X_G in Theorem 1.17, only the support is the family of cuts $\mathcal{C}$.

Theorem 1.28 ([45, Theorem 3.4])**.** *Let (X, d) be a finite metric space,*

$$d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c, \qquad \lambda_c > 0,$$

and let $0 < a < b \le 2a$. Then

$$\dim_{\ell_1}(X) = \min_{G \in \mathcal{G}(\mathcal{C})} \chi(G) = \min_{G \in \mathcal{G}(\mathcal{C})} \min\big\{m \ge 1 : \ 2d_{GH}(a\Delta_m, C_G) < b\big\}.$$

In other words, $\dim_{\ell_1}(X) = m$ if and only if m is the smallest natural number for which

$$\min_{G \in \mathcal{G}(\mathcal{C})} 2d_{GH}(a\Delta_m, C_G) < b.$$

Proof. The equality $\dim_{\ell_1}(X) = \min_{G\in\mathcal{G}(\mathcal{C})}\chi(G)$ is Corollary 1.27, that is, a combination of Theorem 1.24 and Lemma 1.26. For each $G \in \mathcal{G}(\mathcal{C})$ Theorem 1.17 gives

$$\chi(G) = \min\bigl\{m \geq 1 :\ 2d_{GH}(a\Delta_m, C_G) < b\bigr\},$$

whence follows the second formula.

The set of such m is non-empty: for $m \geq \#\mathcal{C}$, any correspondence that must embed $\mathcal{C}$ in $a\Delta_m$ has distortion less than b, and therefore $2d_{GH}(a\Delta_m, C_G) < b$ for any G. In addition, always $2d_{GH}(a\Delta_m, C_G) \leq b$.

If the smallest such m is equal to 1, then $2d_{GH}(a\Delta_1, C_G) = \operatorname{diam} C_G < b$ is impossible in the presence of distance b, then all distances in C_G are equal to a, the graph G is empty, the nested hypergraph is also empty and is colored in one color. Then $\dim_{\ell_1}(X) = 1$.

If $m \geq 2$, then there exists $G_0 \in \mathcal{G}(\mathcal{C})$ with $2d_{GH}(a\Delta_m, C_{G_0}) < b$, and for all smaller k the inequality does not hold for any G. By Theorem 1.17 this exactly means $\chi(G_0) = m$ and $\chi(G) \geq m$ for all $G \in \mathcal{G}(\mathcal{C})$, that is, $\dim_{\ell_1}(X) = m$. □

Remark 1.29. Checking whether d belongs to the cut cone and constructing the family $\mathcal{G}(\mathcal{C})$ are difficult in the general case: the problem of ℓ_1-embedding of a finite metric space is NP-complete. For special classes (trees, hypercubes, ultrametrics), the cut decomposition and nested hypergraph are described explicitly.

Remark 1.30. In the notation of Theorem 1.17 the space C_G coincides with X_G, and the family $\mathcal{G}(\mathcal{C})$ plays the role of $\mathcal{G}(X)$. We write $\mathcal{G}(\mathcal{C})$ to emphasize that the vertices of the graphs are cuts from the fixed representation $d = \sum \lambda_c \delta_c$, and not points in the original space X. If the representation d is fixed, we also set $\mathcal{G}(X) := \mathcal{G}(\mathcal{C})$.

1.3.5 Example: calculating the chromatic number for a loop

Let us illustrate Theorem 1.17 using a simple example.

Example 1.31. Consider the graph $G = C_5$, a cycle of length 5. It is known that $\chi(C_5) = 3$ (see [33, Theorem 1.3]).

Let us construct the (a, b)-space X_{C_5}: the vertices of the cycle, the distance b between neighboring ones, a between non-neighboring ones. Let us take $a = 1$, $b = 2$ (the condition $b \leq 2a$ is satisfied).

According to Theorem 1.17:

$$\chi(C_5) = \min\bigl\{m \geq 1 \ \bigm|\ 2d_{GH}(\Delta_m, X_{C_5}) < 2\bigr\}.$$

Let us check:

- For $m = 1$: $2d_{GH}(a\Delta_1, X_{C_5}) = \operatorname{diam} X_{C_5} = 2$, so the condition < 2 is not satisfied.
- For $m = 2$: Δ_2 is a two-point simplex with distance 1. By the formula for (a, b)-spaces (see [30, Corollary 2.5]), $2d_{GH}(\Delta_2, X_{C_5}) = 2$ (since X_{C_5} does not allow splitting into 2 parts of smaller diameter). Consequently, the condition < 2 is not satisfied.
- For $m = 3$: X_{C_5} can be divided into 3 independent sets (for example, $\{1, 3\}$, $\{2, 4\}$, $\{5\}$). Then $\operatorname{diam} D = 1 < 2$, which means $2d_{GH}(\Delta_3, X_{C_5}) < 2$.

Therefore, $\chi(C_5) = 3$, which is consistent with the classical result.

Chapter Summary

This chapter proves:

(1) expression of MST-spectrum through partitions (Lemma 1.4);

(2) formula for MST-spectrum in terms of GH-distance to simplices (Theorem 1.6);

(3) criterion for the existence of a Borsuk partition through the GH-distance (Theorem 1.10);

(4) formulas for the chromatic number and the clique cover number through the GH-distance (Theorems 1.17, 1.15);

(5) formula for the Manhattan dimension through the GH-distance (Theorem 1.28).

The constructions of a cut, a hypergraph, a nested hypergraph $\Gamma(\mathcal{C})$, and a family of simple graphs $\mathcal{G}(\mathcal{C})$ are given in the text. Accepted as a known fact: invariance of the MST-spectrum ([30]), duality of clique cover and the chromatic number, Assouad criterion ([43]), Bandelt–Chepoi–Laurent theorem ([42]).

Chapter 2

Ultrametrization and its applications in GH-geometry

In this chapter, we study an important method for estimating the Gromov–Hausdorff distance — ***ultrametrization***. This method allows us to replace these metric spaces by canonical ultrametric spaces, the mutual distance between which bounds the original GH-distance from below.

The material in this chapter is based on the work of [36], [37] and [39].

2.1 Ultrametric spaces

Let us recall the definition of ultrametrics.

Definition 2.1. A metric space (X, d) is said to be ***ultrametric*** if its distance function satisfies the ***strong triangle inequality***:

$$d(x, z) \leq \max\big\{d(x, y), d(y, z)\big\} \quad \forall x, y, z \in X.$$

An important property follows from this inequality: in any triangle, two large distances are equal, and the third does not exceed them. This property is called the ***isosceles property*** of ultrametric triangles.

Lemma 2.2. *The ultrametric space X has the following properties*:

(1) **Polygon inequality**: *for any finite sequence of points $x_0, x_1, \ldots, x_n \in X$*:

$$d(x_0, x_n) \leq \max_{1 \leq i \leq n} d(x_{i-1}, x_i).$$

(2) **Isosceles**: *for any $x, y, z \in X$ among three distances $d(x, y)$, $d(y, z)$, $d(z, x)$ two are equal, and the third does not exceed them.*

(3) **Property of balls**: *every ball of non-zero radius is both open and closed*; *if two balls intersect, then one is contained in the other.*

(4) **Diameter of component**: *if $d(x, y) = \operatorname{diam} X$ for some $x, y \in X$, then for any $z \in X$ we have $d(x, z) = \operatorname{diam} X$ or $d(y, z) = \operatorname{diam} X$.*

(5) **Center of the ball**: *each point of the ball is its center.*

2.1.1 Examples of ultrametric spaces

Example 2.3. (1) ***Discrete metric***: $d(x,y) = 1$ for $x \neq y$ is ultrametric.

(2) ***Trees with maximum edge metric***: let T be a tree with weight $\omega(e) > 0$ assigned to each edge e. For vertices v, w we define

$$d(v,w) = \max\{\omega(e) : e \text{ lies on the path from } v \text{ to } w\}.$$

Then d is an ultrametric. Conversely, any finite ultrametric space can be represented in this form (see [35]).

(3) ***p-adic numbers***: for a prime p, every nonzero $x \in \mathbb{Q}$ can be written uniquely as $x = p^k \frac{m}{n}$, where $k \in \mathbb{Z}$ and the integers m, n are not divisible by p; put

$$|x|_p = p^{-k}, \qquad |0|_p = 0.$$

Then $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the metric $|x-y|_p$, and since $|x+y|_p \leq \max\{|x|_p, |y|_p\}$, this metric is an ultrametric.

(4) ***Function spaces***: if X is a set, and $f : X \to (0,\infty)$ is a function, then

$$d(x,y) = \max\{f(x), f(y)\} \quad (x \neq y)$$

is ultrametric.

2.2 Definition of ultrametrization

Definition 2.4. Let (X,d) be a metric space. ***Polyline*** in X is a finite sequence of points $x_0, x_1, \ldots, x_n \in X$. The ***ultralength*** of the polyline $L = (x_0, \ldots, x_n)$ is

$$|L|_u := \max_{1 \leq i \leq n} d(x_{i-1}, x_i).$$

The ***ultradistance*** between points $x, y \in X$ is

$$d_u(x,y) := \inf\{|L|_u : L \text{ is a polyline in } X, \text{ connecting } x \text{ and } y\}.$$

Proposition 2.5 ([37, Definition 2.12])**.** *The function d_u is a pseudometric on X and satisfies the strong triangle inequality*:

$$d_u(x,z) \leq \max\{d_u(x,y), d_u(y,z)\}.$$

Proof. The symmetry of d_u is obvious. The triangle inequality in the usual sense follows from the concatenation of polylines: if L_{xy} connects x to y, and L_{yz} connects y to z, then their concatenation $L_{xy} + L_{yz}$ connects x to z, and

$$|L_{xy} + L_{yz}|_u = \max\{|L_{xy}|_u, |L_{yz}|_u\}.$$

Therefore,

$$d_u(x,z) \leq \inf_{L_{xy}, L_{yz}} \max\{|L_{xy}|_u, |L_{yz}|_u\} = \max\{d_u(x,y), d_u(y,z)\}.$$

Positive definiteness may be violated: $d_u(x,y) = 0$ is possible for $x \neq y$ (for example, if the space is connected). □

Definition 2.6. An ***ultrametrization*** of a metric space X is a quotient space

$$\mathcal{U}(X) := X/\sim,$$

where $x \sim y \iff d_u(x,y) = 0$, with the metric induced by d_u.

Remark 2.7. The space $\mathcal{U}(X)$ is ultrametric. If X is already ultrametric, then $\mathcal{U}(X) = X$ (up to isometry), since $d_u = d$.

Example 2.8. (1) ***Connected space***: if X is connected and each pair of points is connected by a curve of finite length, then $\mathcal{U}(X) = \Delta_1$ (one-point space). Indeed, for any x, y and any $\varepsilon > 0$, we can connect them by a polyline with ultralength $< \varepsilon$.

(2) ***Discrete set***: if X is discrete with $d(x,y) \geq \delta > 0$ for all $x \neq y$, then $\mathcal{U}(X) = X$ as sets, but the metric on $\mathcal{U}(X)$ is, generally speaking, different from the metric on X.

(3) ***Circle*** S^1: $\mathcal{U}(S^1) = \Delta_1$, since S^1 is connected.

(4) For the set P_n ***vertices of a regular*** n***-gon*** inscribed in the unit circle, we have $\mathcal{U}(P_n) = (2\pi/n)\Delta_n$ (the metric on P_n is induced from the intrinsic metric of the circle).

2.3 Main theorem about ultrametrization

The following result is key for applications.

Theorem 2.9 ([39, Theorem H])**.** *For any non-empty metric spaces X and Y,*

$$d_{GH}(X,Y) \geq d_{GH}\big(\mathcal{U}(X), \mathcal{U}(Y)\big).$$

Proof. If $d_{GH}(X,Y) = \infty$, the inequality is trivial. Let $d := d_{GH}(X,Y) < \infty$.

Let us choose an arbitrary $\varepsilon > 0$. By the characterization of the Gromov–Hausdorff distance in terms of correspondences, there is a correspondence $R \in \mathcal{R}(X,Y)$ with $\operatorname{dis} R < 2d + \varepsilon$.

For a point $x \in X$ we denote by $[x]$ its class in $\mathcal{U}(X)$, and for $y \in Y$ by $[y]$ its class in $\mathcal{U}(Y)$. Let us put

$$R_u := \Big\{ \big([x],[y]\big) : (x,y) \in R \Big\} \subset \mathcal{U}(X) \times \mathcal{U}(Y).$$

The projections of R onto X and onto Y are surjective, so both projections of R_u are also surjective, that is, R_u is a correspondence between $\mathcal{U}(X)$ and $\mathcal{U}(Y)$.

Let us evaluate its distortion. For all $x, x' \in X$ we have $d_{\mathcal{U}(X)}\big([x],[x']\big) = d_u(x,x')$, and similarly in Y. Note that $d_u(x,x') \leq d(x,x') < \infty$: the polyline (x,x') consists of one link, so its ultralength is equal to $d(x,x')$. This means that all the quantities under consideration are finite numbers, and it is enough to check that for any $(x,y),(x',y') \in R$ two inequalities are satisfied

$$d_u(y,y') \leq d_u(x,x') + \operatorname{dis} R, \qquad d_u(x,x') \leq d_u(y,y') + \operatorname{dis} R;$$

together they are equivalent to the estimate $\big|d_u(x,x') - d_u(y,y')\big| \leq \operatorname{dis} R$ and, therefore, the estimate $\operatorname{dis} R_u \leq \operatorname{dis} R$.

Let us prove the first of these two inequalities; the second is obtained by applying the same reasoning to the correspondence $R^{-1} \in \mathcal{R}(Y,X)$, whose distortion is equal to $\operatorname{dis} R$.

So, let us fix $(x,y),(x',y') \in R$. By definition of d_u for any $\delta > 0$ there is a polyline $L = (x_0 = x, x_1, \dots, x_n = x')$ in X such that $|L|_u < d_u(x,x') + \delta$; if $x = x'$, we take the polyline (x,x), so that in all cases we can assume that $n \geq 1$.

For $i = 1, \dots, n-1$, choose a point y_i such that $(x_i, y_i) \in R$ (this is possible, since R is a correspondence), and put $y_0 := y$, $y_n := y'$. Both choices are consistent with R: $(x_0, y_0) = (x,y) \in R$ and $(x_n, y_n) = (x', y') \in R$; therefore, $(x_i, y_i) \in R$ for all $i = 0, \dots, n$.

By the definition of distortion applied to the pairs $(x_i, y_i), (x_{i+1}, y_{i+1}) \in R$,

$$\big||x_i x_{i+1}|_X - |y_i y_{i+1}|_Y\big| \leq \operatorname{dis} R, \qquad i = 0, \dots, n-1,$$

that is, $|y_i y_{i+1}|_Y \leq |x_i x_{i+1}|_X + \operatorname{dis} R$ for each i. The sequence $L' := (y_0, y_1, \dots, y_n)$ is a polyline in Y connecting y and y', so

$$\begin{aligned} d_u(y,y') \leq |L'|_u &= \max_{0 \leq i \leq n-1} |y_i y_{i+1}|_Y \\ &\leq \max_{0 \leq i \leq n-1} |x_i x_{i+1}|_X + \operatorname{dis} R = |L|_u + \operatorname{dis} R \\ &< d_u(x,x') + \delta + \operatorname{dis} R. \end{aligned}$$

Since $\delta > 0$ is arbitrary, we obtain $d_u(y,y') \leq d_u(x,x') + \operatorname{dis} R$, as required.

Thus, $\operatorname{dis} R_u \leq \operatorname{dis} R < 2d + \varepsilon$. Passing to the infimum over R and then to the limit $\varepsilon \to 0$, we obtain

$$d_{GH}\big(\mathcal{U}(X), \mathcal{U}(Y)\big) \leq d.$$

□

Corollary 2.10. *If $d_{GH}(X,Y) = 0$, then $d_{GH}\big(\mathcal{U}(X), \mathcal{U}(Y)\big) = 0$.*

Corollary 2.11. *For any metric space X:*

$$\operatorname{diam} \mathcal{U}(X) \leq \operatorname{diam} X.$$

If X is connected and each pair of points is connected by a curve of finite length, then $\operatorname{diam} \mathcal{U}(X) = 0$.

Corollary 2.12. *Let a metric space X be such that any two of its points are connected by a curve of finite length. Then for any metric space Y:*

$$d_{GH}(X,Y) \geq \frac{1}{2} \operatorname{diam} \mathcal{U}(Y).$$

In particular, this is true for any normed space X and any subset $Y \subseteq X$.

Proof. By Corollary 2.11 we have $\mathcal{U}(X) = \Delta_1$. The distance from the one-point space to a metric space Y is equal to half its diameter: $d_{GH}(\Delta_1, Y) = \frac{1}{2} \operatorname{diam} Y$. By Main Theorem 2.9

$$d_{GH}(X,Y) \geq d_{GH}\big(\mathcal{U}(X), \mathcal{U}(Y)\big) = d_{GH}\big(\Delta_1, \mathcal{U}(Y)\big) = \frac{1}{2} \operatorname{diam} \mathcal{U}(Y).$$

A normed space is connected, and any two of its points are connected by a segment, which is a curve of finite length, so the case of a normed space is a special case of the proved statement. □

2.4 Application to the calculation of GH distances

2.4.1 Distance between a circle and a regular polygon

Theorem 2.13 ([39, Proposition C.2]). *Let S^1 be a standard circle with intrinsic metric, and $P_n \subset S^1$ be the set of vertices of a regular n-gon. Then*

$$d_{GH}(P_n, S^1) = \frac{\pi}{n}.$$

Proof. Upper bound. The Hausdorff distance between P_n and S^1 is π/n, so $d_{GH}(P_n, S^1) \le \pi/n$.

Lower bound. Since S^1 is connected, $\mathcal{U}(S^1) = \Delta_1$. The space P_n with intrinsic metric (induced from S^1) is ultrametric with $\mathcal{U}(P_n) = P_n$, where all nonzero distances are equal to $2\pi/n$. Thus, $\mathcal{U}(P_n) = (2\pi/n)\Delta_n$.

According to Theorem 2.9 and an elementary lower bound for the Gromov–Hausdorff distance between simplices:

$$d_{GH}(P_n, S^1) \ge d_{GH}\big(\mathcal{U}(P_n), \mathcal{U}(S^1)\big) = d_{GH}\big(\frac{2\pi}{n}\Delta_n, \Delta_1\big) = \frac{1}{2} \cdot \frac{2\pi}{n} = \frac{\pi}{n}.$$

Combining the estimates, we obtain equality. □

2.4.2 Distance between regular polygons

Theorem 2.14 ([38, Theorem 3.1]). *Let P_m and P_n be the sets of vertices of regular m- and n-gons inscribed in one circle ($m \le n$). Then*

$$d_{GH}(P_m, P_n) \ge \frac{\pi}{m} - \frac{\pi}{n}.$$

If n is divisible by m, then

$$d_{GH}(P_m, P_n) = \frac{\pi}{m} - \frac{\pi}{n}.$$

Proof. Lower bound. If $m = n$, then $P_m = P_n$ and the inequality is trivial, so we further assume that $m < n$. As above, $\mathcal{U}(P_k) = (2\pi/k)\Delta_k$, and by Theorem 2.9

$$d_{GH}(P_m, P_n) \ge d_{GH}\left(\frac{2\pi}{m}\Delta_m, \frac{2\pi}{n}\Delta_n\right).$$

Let us put $\lambda = 2\pi/m$ and $\mu = 2\pi/n$; since $m < n$, we have $\mu < \lambda$. Let us apply the theorem on the distance to a simplex with a large number of points to the simplex $\mu\Delta_n$ and the space $\lambda\Delta_m$ (see [30]):

$$2d_{GH}(\lambda\Delta_m, \mu\Delta_n) = \max\{\mu, \lambda - \mu\} \ge \lambda - \mu = \frac{2\pi}{m} - \frac{2\pi}{n}.$$

Therefore,

$$d_{GH}(P_m, P_n) \ge \frac{\pi}{m} - \frac{\pi}{n}.$$

Upper bound for $n = pm$. Let us construct the correspondence $R \in \mathcal{R}(P_m, P_n)$:

$$R = \bigcup_{i=0}^{m-1} \big\{(u_i, v_{pi+k}) : k = 0, \dots, p-1\big\},$$

where u_i are the vertices of the m-gon, v_j are the vertices of the n-gon. A direct check shows that $\operatorname{dis} R = 2\pi/m - 2\pi/n$. Therefore,

$$d_{GH}(P_m, P_n) \le \frac{\pi}{m} - \frac{\pi}{n}.$$

Combining the estimates, we obtain equality. □

2.5 Ultrametrization and metric trees

For metric trees, ultrametrization has a simple geometric interpretation.

Definition 2.15. ***Metric tree*** is a connected graph without cycles, each edge of which is a segment with a given length, and the distance between points is equal to the length of the only path connecting them.

Since every tree is path connected and the lengths of all paths are finite, we get

Proposition 2.16. *Let T be a metric tree. Then $\mathcal{U}(T) = \Delta_1$ (one-point space).*

Corollary 2.17. *For any metric tree T and any metric space X:*

$$d_{GH}(T, X) \geq \frac{1}{2} \operatorname{diam} \mathcal{U}(X).$$

This is a special case of Corollary 2.12, because every metric tree is path connected and the lengths of all paths in it are finite. Thus we obtain a simple lower bound for the distance from a tree to an arbitrary space; the following example shows that this estimate is sharp.

Example 2.18. Let T be the union of three unit segments $[o, a]$, $[o, b]$ and $[o, c]$ emanating from one point o (in other words, ***tripod***), endowed with an intrinsic metric. As X we take the set of ends of these segments: $X = \{a, b, c\}$, and the metric on X is induced from T. Then

$$d_X(a, b) = d_X(b, c) = d_X(c, a) = 2,$$

since the only path from a to b passes through o and consists of two unit segments. Let us calculate $d_{GH}(T, X)$.

First we get a lower estimate. Each polyline in X connecting two different points consists of links of length 0 or 2, so its ultralength is 2; this means $d_u(x, y) = d_X(x, y)$ for all $x, y \in X$ and $\mathcal{U}(X) = X$. In particular, $\operatorname{diam} \mathcal{U}(X) = 2$, and Corollary 2.17 gives

$$d_{GH}(T, X) \geq \frac{1}{2} \operatorname{diam} \mathcal{U}(X) = 1.$$

On the other hand, X lies in T, so it is sufficient to estimate the Hausdorff distance between T and X, considering both sets to be subsets of the ambient space T. Each point $t \in T$ lies on one of the segments $[o, x]$, so $d_T(t, x) \leq 1$; in this case $d_T(o, X) = 1$, since all three ends are located at a distance of 1 from the center o. So this Hausdorff distance is equal to 1. Since in the definition of d_{GH} the infimum is taken over all possible embeddings in the ambient space, and T and X are already embedded in T, we obtain

$$d_{GH}(T, X) \leq 1.$$

So, $d_{GH}(T, X) = 1$: the estimate of Corollary 2.17 in this example becomes an equality.

2.6 Ultrametrization and Jung's constant

The connection between ultrametrization and Jung's constant allows us to obtain estimates of the GH distance to normed spaces. We will show that the ultrametrization diameter of a set is estimated from above through the Hausdorff distance to the ambient space and Jung's constant; combined with the lower bound of Corollary 2.12, this gives estimates of the GH distance to normed spaces. Let us first determine Jung's constant.

Definition 2.19. For a non-empty bounded subset A of the metric space Z, we define ***radius of the circumscribed ball*** (Chebyshev radius):

$$R_Z(A) := \inf_{z\in Z} \sup_{a\in A} |za|_Z.$$

Definition 2.20. The ***Jung constant*** of a metric space Z is the quantity

$$J(Z) := \sup_{\substack{A\subseteq Z \\ \operatorname{diam} A>0}} \frac{R_Z(A)}{\operatorname{diam} A}.$$

Remark 2.21. For any metric space Z containing at least two points, $1/2 \le J(Z) \le 1$ holds. The lower bound follows from the triangle inequality, the upper one from the fact that a ball of radius $\operatorname{diam} A$ with center at any point $a \in A$ contains the entire set A. In particular, Jung's constant is always positive.

Recall also that for a subset X of a metric space Z, the value $d_H(X, Z)$ is equal to the smallest radius for which the neighborhoods of points of X cover Z, i.e.

$$d_H(X, Z) = \sup_{z\in Z} |zX| = \inf\bigl\{r > 0 : Z \subseteq U_r(X)\bigr\}.$$

Thus, the condition $d_H(X, Z) < \infty$ means that X is a net in Z. If $d_H(X, Z) = \infty$, then the estimates below are trivial, so the main interest is in the case $d_H(X, Z) < \infty$. The estimates given in the following lemma are auxiliary: they relate the ultrametrization diameter to the Hausdorff distance to the ambient space, and we prove them here. An estimate for the Gromov–Hausdorff distance from below in terms of the Hausdorff distance and the Jung constant was obtained in the work [49, Theorems 3.1 and 6.1].

Lemma 2.22. *Let Z be a metric space, $X \subseteq Z$ and $d_H(X, Z) < \infty$. Then*:

(1) *if $J(Z) < 1$, then*

$$\operatorname{diam} \mathcal{U}(X) \le \frac{d_H(X, Z)}{1 - J(Z)};$$

(2) *if any two points $a, b \in Z$ have a midpoint in Z, i.e. such a point $m \in Z$ such that $|am|_Z = |mb|_Z = \frac{1}{2}|ab|_Z$ (this is true, for example, for a normed space, a metric tree or a sphere with an intrinsic metric), then*

$$\operatorname{diam} \mathcal{U}(X) \le 2\, d_H(X, Z).$$

Proof. Let $r := d_H(X, Z) < \infty$. Then for every $\delta > 0$ and every point $z \in Z$ there is a point $x \in X$ such that $|zx|_Z < r + \delta$; it is only this property of the Hausdorff distance that we will use below.

Point 1. Let $J := J(Z) < 1$. Let us first show that for any distinct $a, b \in X$ there is a point $x \in X$ such that

$$\max\bigl\{|ax|_Z, |xb|_Z\bigr\} \le J|ab|_Z + r + \delta.$$

By the definition of the circumscribed ball radius, for a two-point set $\{a, b\}$ there is a point $z \in Z$ such that

$$\max\bigl\{|az|_Z, |zb|_Z\bigr\} < R_Z(\{a, b\}) + \frac{\delta}{2},$$

and by definition of the Jung constant $R_Z(\{a, b\}) \le J|ab|_Z$. In addition, for a point z there is a point $x \in X$ such that $|zx|_Z < r + \delta/2$. Then

$$|ax|_Z \le |az|_Z + |zx|_Z < J|ab|_Z + r + \delta,$$

and in the same way $|xb|_Z < J|ab|_Z + r + \delta$, as required. For $a = b$ the statement is trivial: any point $x \in X$ with $|ax|_Z < r + \delta$ is suitable, and such a point exists by definition of r.

Now for arbitrary $x, y \in X$ we construct by induction the polylines L_k connecting x and y, the ultralength of which does not exceed D_k, where

$$D_0 := |xy|_Z, \qquad D_{k+1} := JD_k + r + \delta.$$

Let us set $L_0 := (x, y)$. If $L_k = (u_0, \dots, u_{2^k})$, then, replacing each link (u_i, u_{i+1}) with a pair of links (u_i, x_i), (x_i, u_{i+1}), where points $x_i \in X$ are chosen as proven for the pair u_i, u_{i+1}, we obtain a polyline L_{k+1}, all links of which do not exceed

$$J|u_i u_{i+1}|_Z + r + \delta \le JD_k + r + \delta = D_{k+1}.$$

Since $J < 1$, the sequence D_k converges to $(r + \delta)/(1 - J)$, so for all k we have $d_u(x, y) \le |L_k|_u \le D_k$ and, therefore,

$$d_u(x, y) \le \inf_k D_k \le \lim_{k\to\infty} D_k = \frac{r + \delta}{1 - J}.$$

Since $\delta > 0$ is arbitrary, we obtain $d_u(x, y) \le r/(1 - J)$, i.e. $\operatorname{diam} \mathcal{U}(X) \le d_H(X, Z)/(1 - J)$ — the first estimate is proven.

Point 2. Now let any two points $a, b \in Z$ have a midpoint in Z, i.e. such a point $m \in Z$ such that $|am|_Z = |mb|_Z = \frac{1}{2}|ab|_Z$. Then for any $a, b \in X$ there is a point $x \in X$ such that

$$\max\bigl\{|ax|_Z, |xb|_Z\bigr\} \le \frac{1}{2}|ab|_Z + r + \delta.$$

Indeed, let us take the midpoint m of the points a and b; since $\max\bigl\{|am|_Z, |mb|_Z\bigr\} = \frac{1}{2}|ab|_Z$, and by the definition of r there is a point $x \in X$ such that $|mx|_Z < r + \delta$, then

$$|ax|_Z \le |am|_Z + |mx|_Z < \frac{1}{2}|ab|_Z + r + \delta,$$

and in the same way $|xb|_Z < \frac{1}{2}|ab|_Z + r + \delta$.

Next, we repeat the reasoning of paragraph 1 verbatim, replacing the factor J with $1/2$. Namely, for arbitrary $x, y \in X$ we construct by induction the polylines L_k connecting x and y, the ultralength of which does not exceed D_k, where

$$D_0 := |xy|_Z, \qquad D_{k+1} := \frac{1}{2}D_k + r + \delta.$$

Here D_k converges to $2(r + \delta)$, so $d_u(x, y) \le \inf_k D_k \le 2(r + \delta)$, and since $\delta > 0$ is arbitrary, we obtain $d_u(x, y) \le 2r$, i.e. $\operatorname{diam} \mathcal{U}(X) \le 2d_H(X, Z)$ — the second estimate is proven. □

Note that the lower bound of Corollary 2.12 cannot be derived from the Hausdorff estimate: if $Y \subseteq X$, then $d_{GH}(X, Y) \le d_H(X, Y)$ always holds, so the Hausdorff distance gives an upper bound, not a lower one. That is why its proof is based on Main Theorem 2.9.

Example 2.23. Let us apply the first point of Lemma 2.22 to a uniform net in a segment. Let $Z = [0, 1]$ and $X = \bigl\{\frac{k}{n} : k = 0, \dots, n\bigr\} \subset Z$. For any $A \subset Z$ we have $R_Z(A) = \frac{1}{2}\operatorname{diam} A$, therefore $J(Z) = \frac{1}{2}$; in addition, $d_H(X, Z) = \frac{1}{2n}$. According to the first point of the lemma

$$\operatorname{diam} \mathcal{U}(X) \le \frac{d_H(X, Z)}{1 - J(Z)} = \frac{1}{n}.$$

The estimate here is sharp. Indeed, neighboring points of the set X are separated from each other by $\frac{1}{n}$, and any two distinct points of it are no less than $\frac{1}{n}$ apart. Therefore, in any polyline connecting different points $x, y \in X$, there is a link of length at least $\frac{1}{n}$, while a polyline passing through all intermediate nodes consists of links of length exactly $\frac{1}{n}$. This means $d_u(x, y) = \frac{1}{n}$ for any different $x, y \in X$ and

$$\operatorname{diam} \mathcal{U}(X) = \frac{1}{n} = 2\, d_H(X, Z).$$

The segment is connected, so $\mathcal{U}(Z) = \Delta_1$, and the distance from the one-point space to the metric space is equal to half its diameter; therefore, by Main Theorem 2.9

$$d_{GH}(Z, X) \geq d_{GH}\big(\Delta_1, \mathcal{U}(X)\big) = \frac{1}{2} \operatorname{diam} \mathcal{U}(X) = \frac{1}{2n}.$$

Since, in addition, $d_{GH}(Z, X) \leq d_H(Z, X) = \frac{1}{2n}$, we obtain $d_{GH}(Z, X) = \frac{1}{2n}$.

For $J(Z) > \frac{1}{2}$ the factor $1/\big(1 - J(Z)\big)$ is greater than 2: for example, for the Euclidean plane $J(\mathbb{R}^2) = 1/\sqrt{3}$ (Jung's theorem, see [50]), and the first point of the lemma gives for any net $X \subset \mathbb{R}^2$ estimate $\operatorname{diam} \mathcal{U}(X) \leq \frac{3+\sqrt{3}}{2}\, d_H(X, \mathbb{R}^2) \approx 2.37\, d_H(X, \mathbb{R}^2)$.

2.7 Connection with MST-spectrum

It turns out that ultrametrization is closely related to the MST-spectrum of finite metric spaces.

The results of this section draw on two sources. The first is a classic fact of cluster analysis: the ultrametric of a single linkage, coinciding with the ultradistance d_u, for a finite metric space is calculated from the minimum spanning tree of the complete graph as the maximum length of edges on the path between points (see [47]; the construction of ultrametrization itself goes back to the work of [36]). The second is the work [35, Theorem 3.1, Corollary 3.2], where it is shown that a finite metric space is ultrametric if and only if its distances are equal to the maximum lengths of the edges on the paths in MST, and that in this case all its distances belong to the MST-spectrum, and the diameter is equal to σ_1. Theorem 2.24 given below transfers these statements from X itself to its ultrametrization $\mathcal{U}(X)$, and Corollary 2.25 supplements them with an estimate of $\sigma_1 \leq \operatorname{diam} X$ and a description of the case of equality. The proofs are presented here for completeness; the property of cuts of the minimum spanning tree used in them is a standard fact of graph theory (step 1; see, for example, [5, Section 4.7, Claim (4.26)]).

Recall that here and below, $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$ denotes the set of edge lengths of a minimum spanning tree on X, arranged in non-increasing order, that is, the MST-spectrum of the space X (see Definition 1.2).

Theorem 2.24 ([35, Theorem 3.1, Corollary 3.2])**.** *Let X be a finite metric space, $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$ its* MST*-spectrum. Then $\mathcal{U}(X)$ is an ultrametric space in which all non-zero distances belong to the set $\{\sigma_1, \dots, \sigma_{n-1}\}$.*

More precisely, let T be the minimum spanning tree on X. Then for any σ_k from the MST*-spectrum, the partition of the space X into classes with respect to the relation $d_u(x, y) < \sigma_k$ coincides with the partition into sets of vertices of the components obtained from T by removing all edges of length $\geq \sigma_k$; and the least of the distances in $\mathcal{U}(X)$ between points from different components equals σ_k.*

Proof. Step 1. *Property of cuts of a minimum spanning tree.* Let e be an edge of tree T. Let us remove it and denote by A and B the sets of vertices of the two resulting subtrees. Then

$$d(e) = \min\big\{|ab| : a \in A,\ b \in B\big\}.$$

Indeed, the edge e connects A and B, so the minimum on the right side does not exceed $d(e)$. If there were $a \in A$ and $b \in B$ with $|ab| < d(e)$, then, replacing the edge e in T with the edge ab, we would obtain a spanning tree of strictly less weight, which contradicts the minimality of T.

Step 2. *Ultra-distance is expressed in terms of tree* T. For points $x, y \in X$ we denote by $P_T(x, y)$ the path in the tree T connecting x and y and set

$$b(x, y) = \max\bigl\{d(e) : e \in P_T(x, y)\bigr\}.$$

Let us show that $d_u(x, y) = b(x, y)$.

The inequality $d_u(x, y) \le b(x, y)$ is satisfied, since the sequence of vertices of the path $P_T(x, y)$ is a polyline in X connecting x and y, the ultralength of which is equal to $b(x, y)$.

Let us prove the reverse inequality. Let $e \in P_T(x, y)$ be an edge of length $b(x, y)$, and let A and B be the sets of vertices of subtrees into which T splits after removing e, with $x \in A$ and $y \in B$. By step 1 we have $|ab| \ge b(x, y)$ for any $a \in A$ and $b \in B$. Any polyline connecting x and y contains a step leading from A to B, so its ultralength is at least $b(x, y)$. Therefore, $d_u(x, y) \ge b(x, y)$.

Step 3. *Corollaries for* $\mathcal{U}(X)$. So, $d_u(x, y)$ is the maximum length of the edges of the tree T on the path between x and y. Therefore, all non-zero distances in $\mathcal{U}(X)$ are the lengths of the edges of the tree T, i.e. belong to the set $\{\sigma_1, \dots, \sigma_{n-1}\}$.

Moreover, $d_u(x, y) < \sigma_k$ if and only if all edges of the path $P_T(x, y)$ have length $< \sigma_k$, i.e. if and only if x and y lie in the same component of the tree obtained from T by removing all edges of length $\ge \sigma_k$. This means that the equivalence classes at level $< \sigma_k$ are exactly the sets of vertices of such components. Finally, if e is an edge of tree T of length σ_k, and x and y are its ends, then the path $P_T(x, y)$ consists of one edge, so $d_u(x, y) = \sigma_k$, with x and y lying in different components. Thus, the smallest distance between different components is σ_k. □

Corollary 2.25. *For a finite metric space* X *the following holds:*

$$\operatorname{diam} \mathcal{U}(X) = \sigma_1 \le \operatorname{diam} X,$$

where σ_1 *is the largest element of the* MST*-spectrum. Moreover,* $\sigma_1 = \operatorname{diam} X$ *if and only if the diameter* X *is reached on the edge of some minimum spanning tree.*

Proof. By Theorem 2.24, $d_u(x, y)$ is the maximum length of the edges of the tree T on the path $P_T(x, y)$. The path contains at least one edge, so $d_u(x, y) \le \max_{e \in T} d(e) = \sigma_1$ for any $x \ne y$, i.e. $\operatorname{diam} \mathcal{U}(X) \le \sigma_1$. On the other hand, if e is an edge of the tree T of length σ_1, and x and y are its ends, then the path $P_T(x, y)$ consists of one edge and $d_u(x, y) = \sigma_1$. This means $\operatorname{diam} \mathcal{U}(X) = \sigma_1$.

Further, each edge of the tree T connects two points in the space X, so $\sigma_1 \le \operatorname{diam} X$. Finally, $\sigma_1 = \operatorname{diam} X$ if and only if among the edges of tree T there is an edge of length $\operatorname{diam} X$, i.e. when the diameter of X is reached at the edge of the minimum spanning tree. □

Note that the inequality $\sigma_1 \le \operatorname{diam} X$ can be strict. Let us take $X = \{0, 1, 2\}$, the vertices of the path 0—1—2, which is a segment $[0, 2]$ of two unit segments, with the usual distance on a straight line. Then $\operatorname{diam} X = 2$, and the minimum spanning tree consists of edges $\{0, 1\}$ and $\{1, 2\}$ of length 1, so $\sigma_1 = 1 < 2 = \operatorname{diam} X$.

2.8 Chapter summary

This chapter proves:

(1) Definition of ultrametric space and its basic properties (Lemma 2.2).

(2) Definition of ultrametrization $\mathcal{U}(X)$ (Definition 2.6).

(3) Main theorem: $d_{GH}(X,Y) \geq d_{GH}\big(\mathcal{U}(X),\mathcal{U}(Y)\big)$ (Theorem 2.9).

(4) Application to the calculation of $d_{GH}(P_n, S^1)$ (Theorem 2.13).

(5) Application to the calculation of $d_{GH}(P_m, P_n)$ (Theorem 2.14).

(6) Determination of the radius of a circumscribed ball and Jung's constant (Definitions 2.19, 2.20).

(7) Relationship between ultrametrization and Jung's constant (Lemma 2.22).

(8) Relationship between ultrametrization and MST-spectrum (Theorem 2.24).

Accepted as a known fact: the theory of ultrametric spaces and their representation through trees ([35]); results on the distance between simplexes ([30]). Left outside the scope: generalization to infinite spaces and p-ultrametrization.