Estimation of the GH-distance to a normed space via Jung's constant

Lecture 4 of the minicourse "Gromov–Hausdorff distance: applications and lower bounds"

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Lecture outline

1. Statement of the problem: estimation of $d_{GH}(V, X)$ in terms of $d_H(V, X)$.
2. Jung's constant $J(V)$, relative Jung constant $J_s(V)$ and Gulevich number $G(V)$.
3. Weak estimate: $d_{GH}(V, X) \geq \frac{1}{2J_s(V)} d_H(V, X)$ for any finite-dimensional normed V.
4. Intersection property.
5. Stronger estimate: $d_{GH}(V, X) \geq \frac{1}{2J(V)} d_H(V, X)$ for spaces with the intersection property.
6. Corollaries: exact equality for the sup-norm, cylindrical norms and $\dim V = 2$.

Problem statement

Conventions

Let V be a finite-dimensional normed space, $\dim V = n \geq 1$, with norm $\|\cdot\|$; put $|xy| := \|x - y\|$. Let $X \subseteq V$ be a non-empty subset with finite Hausdorff distance $d_H(V, X)$.

Reminder

$$d_H(V, X) = \inf\big\{r > 0 :\ V \subseteq U_r(X),\ X \subseteq U_r(V)\big\},$$

$$U_r(A) = \big\{v \in V : \exists a \in A,\ |va| < r\big\}.$$

Problem. Estimate $d_{GH}(V, X)$ from below using $d_H(V, X)$ and geometric characteristics of V. Always $d_{GH} \leq d_H$, so d_H gives only an upper bound; we need the lower one.

Circumsphere radius and relative radius

Definition

For a non-empty bounded $A \subseteq V$ the *circumscribed ball radius* is

$$R(A) := \inf_{v \in V} \sup_{a \in A} \|v - a\|.$$

For a non-empty $Y \subseteq V$ *relative circumscribed ball radius* of the set A relative to Y is

$$R_Y(A) := \inf_{y \in Y} \sup_{a \in A} \|y - a\|.$$

In particular, $R_V(A) = R(A)$.

The difference is significant: in $R(A)$ the center of the ball is sought in all of V, but in $R_Y(A)$ only in a subset of Y; it is this difference that distinguishes Jung's constant $J(V)$ from the relative Jung constant $J_s(V)$.

Jung's constant and relative Jung constant

Definition (Jung's constant)

$$J(V) = \sup_{\substack{A \subseteq V \\ \operatorname{diam} A > 0}} \frac{R(A)}{\operatorname{diam} A}.$$

Definition (relative Jung constant)

$$J_s(V) := \sup_{\substack{A \subseteq V \\ \operatorname{diam} A > 0}} \frac{R_{\operatorname{conv} A}(A)}{\operatorname{diam} A},$$

where $\operatorname{conv} A$ is the convex hull of A in V.

In Gulevich's work, the quantity $J_s(V)$ was denoted by $J(V)$ and called the Jung constant; in subsequent works the distinction between $J(V)$ and $J_s(V)$ was established. We follow the latest terminology.

Measure of non-convexity, Gulevich number and Gulevich's theorem

Definitions

For a non-empty bounded $A \subseteq V$ the *measure of non-convexity* is $\lambda(A) := \sup_{c \in \operatorname{conv} A} \inf_{a \in A} \|c - a\|$, and the *Gulevich number* is $G(V) := \sup_{\operatorname{diam} A > 0} \lambda(A) / \operatorname{diam} A$.

Fact (Gulevich's theorem)

For any Banach space V: $G(V) = J_s(V)$.

Meaning: the Gulevich number measures how far the points of a convex hull can be from the set itself.

Values and estimates of constants

Fact (estimates of Jung's constant)

For any normed space V

$$\frac{1}{2} \le J_s(V) \le 1, \qquad \frac{1}{2} \le J(V) \le 1.$$

If V is Hilbertian, then $J(V) = J_s(V) = \sqrt{n/\big(2(n+1)\big)}$; for $V = \mathbb{R}^n$ with norm $\|\cdot\|_\infty$:

$$J(V) = \frac{1}{2}, \qquad J_s(V) = \frac{n-1}{n} \quad (n \ge 2).$$

Fact (equality criterion)

The equality $J(V) = J_s(V)$ holds if and only if V is Hilbert or $\dim V \le 2$.

Weak estimate

Theorem (weak estimate)

Let V be a finite-dimensional normed space, and $X \subseteq V$ be a non-empty subset. Then

$$d_{GH}(V, X) \geq \frac{1}{2J_s(V)}\, d_H(V, X).$$

The estimate is valid for *any* finite-dimensional normed space and is based on the relative Jung constant $J_s(V)$.

The proof requires three auxiliary lemmas: on triangulation, on the continuous extension of a mapping with finite distortion, and Webster's lemma.

Triangulation Lemma

Lemma

Let V be an n-dimensional normed space. For any $\varepsilon > 0$ there is a triangulation Σ of the space V such that the diameter of each simplex is less than ε.

Idea of proof. Let us choose a basis $e_1, \ldots, e_n$ with $\|e_i\| < \varepsilon/n$ (in an arbitrary basis it is enough to replace f_i with $\lambda_i f_i$ for sufficiently small λ_i) and consider the cubic lattice generated by this basis; the diameter of each cubic cell is then less than ε. Then we sort through the cells in increasing order of dimension: each d-dimensional polyhedron P is divided into simplices $\operatorname{conv}(b_P \cup T)$, where b_P is the center of P, and T is the simplices of the partition of its faces. Partitions on common faces of neighboring cells coincide simply by construction, since the partition of a face is determined only by the face itself.

Continuous extension lemma

Lemma

Let V be a finite-dimensional normed space and $f\colon V \to X \subset V$ be an arbitrary mapping (not necessarily continuous) with $\operatorname{dis} f < C < \infty$. Then for any $\varepsilon > 0$ there is a continuous $f^*\colon V \to V$ such that:

1. $f^*(y) = f(y)$ for each vertex y of the triangulation Σ from the triangulation lemma;
2. $\operatorname{dis} f^* \leq 3C + 4\varepsilon$;
3. for any simplex $\Delta \in \Sigma$ we have $\operatorname{diam} f^*(\Delta) < C + \varepsilon$.

The extension is built linearly on each simplex: the values at the vertices are given, and inside the simplex the mapping is a convex combination of these values. The small diameter of the simplices ($< \varepsilon$) controls distortion, and the property of preserving values at the vertices allows "recognition" of the X image in the triangulation.

Webster's Lemma

Lemma (Webster)

Let $f\colon V \to W$ be a continuous map between finite-dimensional normed spaces, $\dim V \geq \dim W$ and $\operatorname{dis} f < \infty$. Then $\dim V = \dim W$ and f is surjective.

The lemma shows that a finite distortion mapping between spaces of the same (or greater) dimension must be "on": the dimension cannot fall, and the image cannot be a proper subset.

Proof of weak estimate: step 1

If $d_{GH}(V,X)=\infty$, the inequality is satisfied automatically; let $d_{GH}(V,X)<\infty$.

1. Let us choose C with $2d_{GH}(V,X)<C$; by the distance formula through correspondences there is $R\in\mathcal{R}(V,X)$ with $\operatorname{dis} R<C$.
2. Let us choose a mapping $f\colon V\to X$ such that $\big(v,f(v)\big)\in R$ for all $v\in V$. Then $\operatorname{dis} f\le\operatorname{dis} R<C$.
3. By the continuous extension lemma (with small ε), there is a continuous $f^*\colon V\to V$ with $\operatorname{dis} f^*<\infty$, coinciding with f at the triangulation vertices and having $\operatorname{diam} f^*(\Delta)<C+\varepsilon$ for each simplex $\Delta\in\Sigma$.
4. By Webster's lemma (applied to $f^*\colon V\to V$), the mapping f^* is surjective, which means

$$V=\bigcup_{\Delta\in\Sigma} f^*(\Delta).$$

Proof of weak estimate: step 2

Let $v \in V$; by surjectivity $v \in f^*(\Delta)$ for some simplex Δ. Let us denote

$$A := f(\text{vertices } \Delta) \subset X.$$

Since f^* is linear on Δ, the point v lies in $\operatorname{conv} A$, and by definition of the non-convexity measure

$$\inf_{a \in A} \|v - a\| \le \lambda(A) \overset{\text{Gulevich}}{\le} J_s(V) \cdot \operatorname{diam} A.$$

Moreover, $A \subset f^*(\Delta)$, therefore $\operatorname{diam} A \le \operatorname{diam} f^*(\Delta) < C + \varepsilon$. This means there is $a \in X$ with $\|v - a\| \le J_s(V)(C + \varepsilon)$.

Proof of weak estimate: ending

Since $v \in V$ is arbitrary,

$$\sup_{v\in V} |vX| \le J_s(V)(C+\varepsilon),$$

and according to the standard maximum formula for the Hausdorff distance

$$d_H(V,X) = \max\bigl\{\sup_{v\in V}|vX|,\ \sup_{x\in X}|xV|\bigr\} = \sup_{v\in V}|vX| \le J_s(V)(C+\varepsilon)$$

(the second supremum is equal to zero, since $X \subseteq V$). Letting $\varepsilon \to 0$, we obtain $d_H(V,X) \le J_s(V)C$. The number C is arbitrary provided $C > 2d_{GH}(V,X)$, therefore

$$d_H(V,X) \le 2J_s(V)\, d_{GH}(V,X), \qquad \text{that is} \qquad d_{GH}(V,X) \ge \frac{1}{2J_s(V)} d_H(V,X).$$

Intersection property

Definition

A finite-dimensional normed space V satisfies the *intersection property* if for any $\rho > 0$ there is a $t > 0$ such that for any $t' > t$ and any finite family of open balls $\{U_{r_i}(x_i)\}_{i=1}^m$ with $r_i \le \rho$ the intersection

$$\big(V \setminus B_{t'}(0)\big) \cap \left(\bigcap_{i=1}^m U_{r_i}(x_i)\right)$$

is either empty or contractible (in particular, connected).

Here $B_{t'}(0) = \{v : \|v\| \le t'\}$ is a closed ball, $U_{r_i}(x_i) = \{v : \|v - x_i\| < r_i\}$ is an open ball.

Who has the intersection property

Fact 1 (Euclidean spaces)

Every finite-dimensional Euclidean space has the intersection property.

Fact 2 (sup-norm)

Every finite-dimensional $V = \mathbb{R}^n$ with norm $\|\cdot\|_\infty$ has the intersection property.

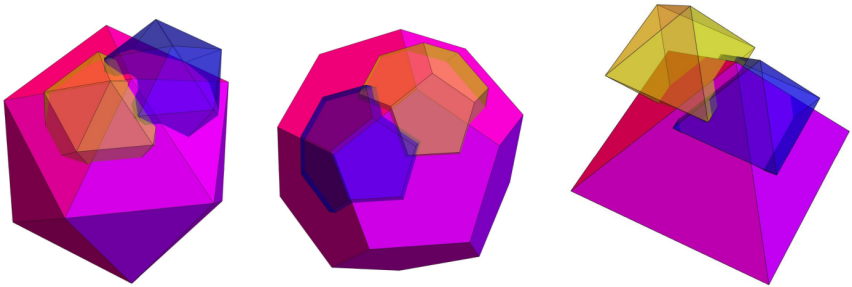
Fact 3 (cylindrical norms)

For a unit ball B^2 of an arbitrary norm in $\mathbb{R}^2$ and a cube $K^{n-2} \subset \mathbb{R}^{n-2}$, the cylindrical norm with unit ball $B^n = B^2 \times K^{n-2}$ has the intersection property.

Fact 4

Some finite-dimensional normed spaces lack the intersection property: in $\mathbb{R}^3$ the icosahedron, dodecahedron and octahedron are counterexamples.

Unit balls: icosahedron, dodecahedron, octahedron



The unit balls of these three norms do not have the intersection property (figure from Adams et al., 2026).

Strengthened estimate

Theorem (strengthened estimate)

Let V be a finite-dimensional normed space with $\dim V \geq 1$ and possessing the intersection property. Suppose that $X \subseteq V$ is a non-empty subset with finite $d_H(V, X)$. Then

$$d_{GH}(V, X) \geq \frac{1}{2J(V)}\, d_H(V, X),$$

where $J(V)$ is the Jung constant of the space V.

Comparison with the weak estimate: $J_s(V) \geq J(V)$, therefore $\frac{1}{2J(V)} \geq \frac{1}{2J_s(V)}$ — the estimate has been strengthened, but not for all normed spaces, but only for those with the intersection property.

Proof. By contradiction, using the Čech and Vietoris–Rips complexes, the nerve lemma and the homology of the sphere S^n_R.

Proof: steps 1–3

Step 1 (selection of parameters). Assume the opposite: $d_{GH}(V,X) < \frac{1}{2J(V)} d_H(V,X)$. Then there exist $r, \varepsilon > 0$ with

$$2d_{GH}(V,X) < r < r+\varepsilon < \frac{1}{J(V)} d_H(V,X), \qquad (r+\varepsilon)J(V) < 2r.$$

Step 2 (correspondence and mappings). From $2d_{GH}(V,X) < r$ it follows that there exists $C \in \mathcal{R}(V,X)$ with $\operatorname{dis} C < r$; choose $h\colon V \to X$ and $g\colon X \to V$ with $\big(v, h(v)\big) \in C$ and $\big(g(x), x\big) \in C$.

Step 3 (lemma on the closeness of complexes). By the lemma on the closeness of Vietoris–Rips complexes, the maps h and g induce simplicial maps

$$\mathrm{VR}(V,\varepsilon) \xrightarrow{\bar{h}} \mathrm{VR}(X, r+\varepsilon) \xrightarrow{\iota} \mathrm{VR}(X, 2(r+\varepsilon)J(V)) \xrightarrow{\bar{g}} \mathrm{VR}(V, r+2(r+\varepsilon)J(V)),$$

and the composition $\bar{g} \circ \iota \circ \bar{h}$ is close to the inclusion $\iota_V\colon \mathrm{VR}(V,\varepsilon) \hookrightarrow \mathrm{VR}(V, r+2(r+\varepsilon)J(V))$.

Proof: step 4

Step 4 (transition to Čech complexes). By the lemma on the connection between the Čech and Vietoris–Rips complexes

$$\check{C}_{\varepsilon/2}(V) \overset{\beta'}{\hookrightarrow} \mathrm{VR}(V,\varepsilon) \subseteq \check{C}_{J(V)\varepsilon}(V), \qquad \mathrm{VR}(V, r+2(r+\varepsilon)J(V)) \overset{\beta}{\hookrightarrow} \check{C}_{r+2(r+\varepsilon)J(V)}(V),$$

. We get a diagram

$$\begin{array}{ccccccc} \mathrm{VR}(V,\varepsilon) & \overset{\bar{h}}{\to} & \mathrm{VR}(X, r+\varepsilon) & \overset{\iota}{\to} & \mathrm{VR}(X, 2(r+\varepsilon)J(V)) & \overset{\bar{g}}{\to} & \mathrm{VR}(V, r+2(r+\varepsilon)J(V)) \\ \uparrow \beta' & & \downarrow \iota' & & \uparrow \iota'' & & \downarrow \beta \\ \check{C}_{\varepsilon/2}(V) & & & & \check{C}_{(r+\varepsilon)J(V)}(X;V) & & \check{C}_{r+2(r+\varepsilon)J(V)}(V) \end{array}$$

The composition $\beta \circ \bar{g} \circ \iota \circ \bar{h} \circ \beta'$ is close to the inclusion $\kappa_0 \colon \check{C}_{\varepsilon/2}(V) \hookrightarrow \check{C}_{r+2(r+\varepsilon)J(V)}(V)$.

Proof: step 5, point selection

Step 5 (intersection property). The first and third families of balls (of radii $\varepsilon/2$ and ρ, centered at the points of V) cover V: each point $v \in V$ lies in the balls of radii $\varepsilon/2$ and ρ centered at v. The middle family $\{U_{(r+\varepsilon)J(V)}(x)\}_{x\in X}$ (centered at the points of X) does not cover V. Indeed, $X \subseteq V$, hence $d_H(V,X) = \sup_{v\in V} d(v,X)$, and from the inequality $(r+\varepsilon)J(V) < d_H(V,X)$ it follows that there is a point

$$y \in V \setminus \bigcup_{x\in X} U_{(r+\varepsilon)J(V)}(x).$$

By moving the origin of coordinates we will achieve that $y = 0$, that is

$$0 \notin \bigcup_{x\in X} U_{(r+\varepsilon)J(V)}(x).$$

Let us apply the intersection property with $\rho = r + 2(r+\varepsilon)J(V)$: there is $t > 0$ such that for any $t' > t$ the intersection of the set $V \setminus B_{t'}(0)$ with any finite intersection of balls of radius $\leq \rho$ is either empty or contractible.

Proof: step 5, external sets

Let us add to the three families of balls from step 4 (centered at the points of V, of X and of V again) the external sets

$$U_1 := V \setminus B_{t+9r}(0), \qquad U_2 := V \setminus B_{t+6r}(0), \qquad U_3 := V \setminus B_t(0).$$

Here $t + 9r > t + 6r > t$, and the radii of all balls of the three families are at most ρ: since $J(V) \geq \frac{1}{2}$,

$$\varepsilon/2 < \varepsilon + 2r \leq \rho, \qquad (r + \varepsilon)J(V) \leq 2(r + \varepsilon)J(V) \leq \rho.$$

Therefore every non-empty intersection of the form $U_i \cap \bigcap_j U_{r_j}(x_j)$ is contractible.

At the same time, the sets U_i themselves are *not* contractible: each of them is the complement of a closed ball in V. So the completed coverings are *not* good in V, and the nerve lemma is not applicable here. That is why we pass to the sphere S^n_R (step 6).

Proof: step 6, stereographic projection

Step 6 (completion of the complexes to the sphere). Consider the stereographic projection $p\colon S^n_R \setminus \{N\} \to V$, where V is realized as the coordinate subspace $\{x^{n+1} = 0\}$ in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, where $x^1, \dots, x^{n+1}$ are Cartesian coordinates, S^n_R is the sphere of radius R centered at the origin, $N := (0, \dots, 0, R)$ is the north pole, and R is chosen so large that the inverse images of all balls participating in the construction are smaller than a hemisphere. To each family of balls the sets $V_i := p^{-1}(U_i) \cup \{N\}$ and the preimages $p^{-1}\big(U_\rho(x)\big)$ are added.

- All the necessary properties (nerves of good coverings, the Čech and Vietoris–Rips inclusions, the closeness of the composition) are verified directly.
- Completed coverings of the sphere S^n_R are good, which means that their nerves are homotopy equivalent to unions of the corresponding sets.
- Therefore, the *left* and *right* completed complexes are homotopy equivalent to S^n_R, and the *middle* one is homotopy equivalent to a proper open subset $U \subset S^n_R$; moreover, U does not have to be connected, nor does it have to be homeomorphic to a sphere without a point.

Proof: step 7, homology contradiction

Now we pass to homology, with coefficients in $\mathbb{Z}_2$:

- $H_n(\text{left}) = H_n(\text{right}) = \mathbb{Z}_2 \neq 0$: both complexes are homotopy equivalent to S^n_R;
- $H_n(\text{middle}) = 0$: the middle complex is homotopy equivalent to a proper open subset of S^n_R, and for a proper open subset of a closed connected n-manifold this group vanishes.

So in the sequence

$$H_n(\text{left}) \xrightarrow{\bar{h}_*} H_n(\text{middle}) \xrightarrow{\bar{g}_*} H_n(\text{right})$$

the middle group is zero, hence $\bar{g}_* \circ \bar{h}_* = 0$.
On the other hand, the composition $\bar{g} \circ \bar{h}$ is close to the inclusion $\check{C}_{\varepsilon/2}(S^n_R) \hookrightarrow \check{C}_{r+2(r+\varepsilon)J(V)}(S^n_R)$; both these complexes are homotopy equivalent to S^n_R, so this inclusion induces an *isomorphism* on the nth homology group: $\bar{g}_* \circ \bar{h}_* \neq 0$. Contradiction.
Thus the original assumption is false and $d_{GH}(V, X) \geq \frac{1}{2J(V)} d_H(V, X)$.

Corollary: exact equality for the sup-norm

Corollary

Let $V = \mathbb{R}^n$ with norm $\|\cdot\|_\infty$ (sup-norm), and let $X \subseteq V$ be a non-empty subset with finite $d_H(V, X)$. Then

$$d_{GH}(V, X) = d_H(V, X).$$

Proof.

1. *Upper bound*: by definition $d_{GH}(V, X) \leq d_H(V, X)$.
2. *Lower bound*: a space with sup-norm has the intersection property, and its Jung constant is equal to $J(V) = \frac{1}{2}$; by the strengthened estimate

$$d_{GH}(V, X) \geq \frac{1}{2 \cdot \frac{1}{2}} d_H(V, X) = d_H(V, X).$$

Combining the estimates, we obtain equality.

Corollary: cylindrical norms and dimension 2

Corollary

Let V be a finite-dimensional normed space with cylindrical norm (with the unit ball $B^2 \times K^{n-2}$) or $\dim V = 2$. Then for any $X \subseteq V$ with finite $d_H(V, X)$

$$d_{GH}(V, X) \geq \frac{1}{2J(V)} d_H(V, X).$$

Proof. Cylindrical norms have the intersection property (Fact 3); if $\dim V = 2$, any norm has the intersection property. It remains to apply the strengthened estimate.

Lecture summary

1. Definitions: circumscribed ball radius $R(A)$ and relative radius $R_Y(A)$, Jung's constant $J(V)$, relative Jung constant $J_s(V)$, Gulevich number $G(V)$, non-convexity measure $\lambda(A)$.
2. Gulevich's theorem: $G(V) = J_s(V)$ for Banach spaces, where $G(V) = \sup \frac{\lambda(A)}{\operatorname{diam} A}$; estimates $\frac{1}{2} \le J(V) \le 1$, $\frac{1}{2} \le J_s(V) \le 1$; values for Hilbert space and sup-norm; criterion $J(V) = J_s(V)$.
3. Auxiliary lemmas: triangulation of a finite-dimensional normed space, continuous extension of a mapping with finite distortion, Webster's lemma.
4. Weak estimate: $d_{GH}(V, X) \ge \frac{1}{2J_s(V)} d_H(V, X)$ holds for any finite-dimensional normed V.
5. Intersection property: definition, examples (Euclidean spaces, sup-norm, cylindrical norms) and counterexamples (icosahedron, dodecahedron, octahedron in $\mathbb{R}^3$).
6. Stronger estimate: $d_{GH}(V, X) \ge \frac{1}{2J(V)} d_H(V, X)$ for spaces with the intersection property; in the proof, completion of Čech complexes to the sphere S^n_R and a contradiction in $H_n(\cdot; \mathbb{Z}_2)$.
7. Corollaries: $d_{GH}(V, X) = d_H(V, X)$ for the sup-norm; estimate for cylindrical norms and $\dim V = 2$.