Topological methods in GH-distance estimates

Lecture 3 of the minicourse "Gromov–Hausdorff distance: applications and lower bounds"

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Lecture outline

1. Simplicial complexes, chains and homology (reminder).
2. Čech complex and the nerve of a covering.
3. The Vietoris–Rips complex and its connection with the Čech complex.
4. Proximity of Vietoris–Rips complexes for spaces that are close in the GH sense.
5. Jung's constant and inclusion of complexes.
6. Higher homology of connected manifolds.

General idea: a simplicial complex is constructed from a metric space, the homology of which reflects its topology; spaces that are close in the GH sense have close complexes, which makes it possible to transfer topological invariants.

Simplicial complexes

Definition (abstract simplicial complex)

An *abstract simplicial complex* is a pair (V, Σ), where V is a set (its elements are called *vertices*), and $\Sigma \subset 2^V$ is a family of non-empty finite subsets (*simplices*), closed under the transition to non-empty subsets: if $\sigma \in \Sigma$ and $\tau \subset \sigma$, $\tau \neq \varnothing$, then $\tau \in \Sigma$. A simplex of $k+1$ vertices is *k-dimensional*; if $\tau \subset \sigma$, $\tau \neq \varnothing$, then τ is a *face* of the simplex σ.

- The word "abstract" emphasizes that a complex is a purely combinatorial object: it is given by a set of vertices V and a family Σ of its subsets, and no geometry is assumed in advance.
- A complex is *finite* if it has finitely many simplices, and *k-dimensional* if the dimensions of its simplices do not exceed k, and it has a k-dimensional simplex. All complexes in this lecture are finite.

Geometric realization of a complex

Definition (geometric realization)

Let Σ be an abstract simplicial complex on a vertex set V. Consider the vector space $\mathbb{R}^V$ of all functions $f\colon V \to \mathbb{R}$ with pointwise operations. Its *geometric realization* $|\Sigma|$ is the set of all f such that

$$f(v) \geq 0, \quad \sum_{v \in V} f(v) = 1, \quad \operatorname{supp} f := \{v \in V : f(v) > 0\} \in \Sigma$$

(only finitely many summands in the sum are non-zero).

In other words, a point of the realization is a convex combination of vertices of a simplex, written in its *barycentric coordinates* $f(v)$; the vertices themselves correspond to the functions f with a single unit value.

For a simplex $\sigma = \{v_0, \dots, v_k\}$ the set $|\sigma|$ consists of all f with $\operatorname{supp} f \subset \sigma$, and the mapping $f \mapsto \big(f(v_0), \dots, f(v_k)\big)$ is a bijection of $|\sigma|$ onto the standard k-simplex

$$\Delta^k = \Big\{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : t_i \geq 0,\ \sum_{i=0}^{k} t_i = 1\Big\}.$$

Since $|\tau| \subset |\sigma|$ for $\tau \subset \sigma$, we have $|\Sigma| = \bigcup_{\sigma \in \Sigma} |\sigma|$.

Topology on the realization

Definition (weak topology)

On $|\Sigma|$ we consider the *weak* (CW-)topology defined by the family of subspaces $|\sigma|$: a subset $A \subset |\Sigma|$ is closed if and only if every intersection $A \cap |\sigma|$ is closed in $|\sigma|$. Passing to complements: a subset $U \subset |\Sigma|$ is open if and only if every intersection $U \cap |\sigma|$ is open in $|\sigma|$.

- A mapping $g\colon |\Sigma| \to Z$ is continuous if and only if all its restrictions $g|_{|\sigma|}$ are continuous.
- A complex is *locally finite* if every simplex is contained in only finitely many simplices. For locally finite complexes the weak topology coincides with the topology induced from $\mathbb{R}^V$. All complexes in this lecture are finite, so for them both descriptions of the topology are equivalent.

Orientation of a simplex

Definition (orientation)

For a simplex $\sigma = \{v_0, \dots, v_k\}$, $k \geq 1$, *orientation* is the equivalence class of its orderings: $(v_0, \dots, v_k) \sim (v_{\pi(0)}, \dots, v_{\pi(k)})$ if the permutation π is even. The notation $[v_0, \dots, v_k]$ denotes a simplex with this orientation.

- A simplex of dimension $k \geq 1$ has exactly two orientations: geometrically, this is the choice of the direction of traversal ($[a, b]$ and $[b, a]$ are opposite).
- The zero-dimensional simplex $\{v\}$ has one orientation.

Chain group

Definition (group of k-dimensional chains)

Let Σ be a complex and G an Abelian group of coefficients ($G = \mathbb{Z}$ or $\mathbb{Z}/2$). The group $C_k(\Sigma; G)$ is defined by the generators $g[v_0, \dots, v_k]$, where $g \in G$, and $\{v_0, \dots, v_k\}$ are all k-simplices with all possible orientations and relations

$$(g_1 + g_2)[v_0, \dots, v_k] = g_1[v_0, \dots, v_k] + g_2[v_0, \dots, v_k],$$

$$g[v_{\pi(0)}, \dots, v_{\pi(k)}] = \operatorname{sgn}(\pi)\, g[v_0, \dots, v_k].$$

Here $\operatorname{sgn}(\pi) = \pm 1$ is an integer acting on an element of the group (G is a module over $\mathbb{Z}$).

In other words, $C_k(\Sigma; G)$ is the direct sum of copies of G, numbered k-simplices: two orientations of one simplex *do not* give two independent summands, and changing the orientation changes the sign of the coefficient.

A note about coefficients

Note

Each k-simplex corresponds to exactly one copy of G in the direct sum expansion: the choice of orientation is just a choice of notation for the summand. For $G = \mathbb{Z}/2$ we have $-g = g$, so changing the orientation does not change the elements of the group at all, and $C_k(\Sigma; \mathbb{Z}/2)$ is a vector space over $\mathbb{Z}/2$ with a basis of unoriented k-simplices.

The coefficients $\mathbb{Z}_2$ are convenient because they remove all questions of orientation; they are used in the last section of the lecture.

Boundary operator

Definition (boundary operator)

The homomorphism $\partial_k \colon C_k(\Sigma; G) \to C_{k-1}(\Sigma; G)$ is given on the generators by the formula

$$\partial_k\big(g[v_0, \dots, v_k]\big) = \sum_{i=0}^{k} (-1)^i g[v_0, \dots, \widehat{v_i}, \dots, v_k],$$

where $\widehat{v_i}$ means that the vertex v_i is omitted, and the factors $(-1)^i$ are integers acting on the elements of the group $C_{k-1}(\Sigma; G)$.

The formula indeed defines a homomorphism: it is additive in the coefficient g, and when the vertices of a simplex are permuted both sides are multiplied by the same integer $\operatorname{sgn}(\pi)$, so the value does not depend on the chosen orientation of the simplex.

Lemma: the boundary squared is zero

Lemma

For every k the composition of two consecutive boundary operators is zero:

$$\partial_{k-1} \circ \partial_k = 0.$$

Proof. It suffices to verify the equality on a generator. Applying the formula from the definition of the boundary twice, we see that every simplex $[v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_k]$, $i < j$, occurs in the double sum exactly twice: with the coefficient $(-1)^i(-1)^{j-1}$ (first the vertex v_i is omitted, then the vertex v_j) and with the coefficient $(-1)^j(-1)^i$ (first v_j, then v_i). These integers are opposite, so the contributions corresponding to the pair $i < j$ cancel out, and the whole double sum is zero.

If a complex has no simplices of dimension $k+1$, then $C_{k+1}(\Sigma; G) = 0$ and $\partial_{k+1} = 0$; for $k = 0$ we set $\partial_{-1} = 0$.

Homology group

Definition (homology)

k-dimensional homology group with coefficients in G: $H_k(\Sigma; G) = \ker \partial_k / \operatorname{im} \partial_{k+1}$. The elements of $\ker \partial_k$ are *cycles*, and the elements of $\operatorname{im} \partial_{k+1}$ are *boundaries*.

Why the quotient is defined: by the lemma, $\partial_k(\partial_{k+1} c) = (\partial_k \circ \partial_{k+1})c = 0$ for any chain $c \in C_{k+1}(\Sigma; G)$, hence $\operatorname{im} \partial_{k+1} \subset \ker \partial_k$. The group $C_k(\Sigma; G)$ is Abelian, so the subgroup $\operatorname{im} \partial_{k+1}$ is normal in $\ker \partial_k$.

Homotopic invariance of homology

Fact

Continuous $f\colon |\Sigma| \to |\Delta|$ induces $f_*\colon H_k(\Sigma; G) \to H_k(\Delta; G)$; if f and g are homotopic, then $f_* = g_*$.

This is the main transfer tool: spaces that are close in the GH sense give close complexes, and close simplicial maps give coinciding homomorphisms of homology.

Source: Hatcher, Chapter 2, Theorem 2.10.

Nerve of a covering

Definition (nerve)

Let $\{U_\alpha\}_{\alpha \in A}$ be a family of subsets of a topological space Y. The *nerve* $\mathcal{N}(\{U_\alpha\})$ is the abstract simplicial complex with vertex set A in which $\{\alpha_0, \ldots, \alpha_k\}$ is a simplex if and only if

$$\bigcap_{i=0}^{k} U_{\alpha_i} \neq \varnothing.$$

Thus, the nerve remembers only the pattern of intersections of the sets: each set U_α gives a vertex, and every non-empty intersection of $k+1$ sets gives a k-simplex.

Good coverings

Reminder (contractibility)

A topological space is *contractible* if it is homotopy equivalent to a point, that is, if its identity mapping is homotopic to a constant one.

Definition (good covering)

An open covering $\mathcal{U} = \{U_\alpha\}$ of a space Y is called *good* if every finite non-empty intersection $\bigcap_{i=1}^{k} U_{\alpha_i}$ is contractible.

A typical example: every open convex subset of $\mathbb{R}^n$ is contractible, so any covering of $\mathbb{R}^n$ by open convex sets is good.

Reminder: locally finite families and paracompactness

A family $\{W_\gamma\}_{\gamma\in\Gamma}$ of subsets of a space Y is called *locally finite* if every point $y \in Y$ has a neighbourhood intersecting only finitely many elements of this family.

A topological space Y is called *paracompact* if every open covering $\{V_\beta\}_{\beta\in B}$ of Y admits a locally finite open refinement: there exist a locally finite family of open sets $\{W_\gamma\}_{\gamma\in\Gamma}$ and a mapping $\varphi\colon \Gamma \to B$ such that $Y = \bigcup_{\gamma\in\Gamma} W_\gamma$ and $W_\gamma \subset V_{\varphi(\gamma)}$ for all $\gamma \in \Gamma$.

Nerve Lemma

Fact (nerve lemma; Hatcher, Corollary 4G.3)

If Y is paracompact and $\mathcal{U}$ is a good open covering of Y, then the nerve $\mathcal{N}(\mathcal{U})$ is homotopy equivalent to Y.

Every metric space is paracompact (Stone, Corollary 1 of his 1948 note), so the nerve lemma applies to any metric space. This means that the homology of a metric space can be calculated from the nerve of any good covering of it.

External Čech complex

Definition

Let Z be a metric space, $X \subset Z$, $r > 0$. *External Čech complex* $\check{C}_r(X; Z)$ is a complex with the set X of vertices in which $\{x_0, \dots, x_k\} \subset X$ is a simplex if and only if there is a point $z \in Z$ with

$$|zx_i|_Z < r \qquad \text{for all } i = 0, \dots, k.$$

In other words, $\check{C}_r(X; Z)$ is the nerve of the family of open balls $\{U_r(x)\}_{x \in X}$ in the ambient space Z.

Unlike the Vietoris–Rips complex, here the proximity of points is measured *in the ambient* space Z; this is why the complex is called *external*.

Vietoris–Rips complex

Definition

Vietoris–Rips complex $\mathrm{VR}(X, r)$ of a metric space X for $r > 0$ is a complex with vertices X in which $\{x_0, \dots, x_k\}$ is a simplex if and only if

$$\operatorname{diam}\{x_0, \dots, x_k\} < r.$$

This is a construction that uses only the intrinsic metric of X; the parameter r plays the role of "resolution": new simplices appear as r grows. The condition $\operatorname{diam}\{x_0, \dots, x_k\} < r$ means that all pairwise distances between these points are less than r.

Lemma

For any metric space Z, its subset $X \subset Z$, and any $r > 0$ we get

$$\mathrm{VR}(X, r) \xrightarrow{\iota_1} \check{C}_r(X; Z) \xrightarrow{\iota_2} \mathrm{VR}(X, 2r),$$

where ι_i are inclusions.

Proof. Let $\sigma = \{x_0, \ldots, x_k\} \in \mathrm{VR}(X, r)$, that is, $\operatorname{diam}\sigma < r$. Take any vertex $x_i \in \sigma$: then $|x_i x_j|_Z \le \operatorname{diam}\sigma < r$ for all j, which means $\sigma \in \check{C}_r(X; Z)$. This proves the inclusion ι_1.

Conversely, if $\sigma \in \check{C}_r(X; Z)$, then there is $z \in Z$ with $|z x_i|_Z < r$ for all i. By the triangle inequality

$$|x_i x_j|_Z \le |x_i z|_Z + |z x_j|_Z < 2r,$$

that is, $\operatorname{diam}\sigma < 2r$ and $\sigma \in \mathrm{VR}(X, 2r)$. This proves the inclusion ι_2.

Simplicial mappings

Definition (simplicial mapping)

A mapping $f\colon \Sigma \to \Delta$ of simplicial complexes is called *simplicial* if it maps the vertices of the complex Σ to the vertices of the complex Δ and takes every simplex of Σ to a simplex of Δ.

- A simplicial mapping induces a continuous mapping of geometric realizations $|f|\colon |\Sigma| \to |\Delta|$: it is linear on each simplex, $|f|\bigl(\sum_i t_i v_i\bigr) = \sum_i t_i f(v_i)$, and is glued from the mappings of the standard simplices.
- A simplicial mapping also induces a homomorphism of chain groups $f_*\colon C_k(\Sigma; G) \to C_k(\Delta; G)$, given on the generators by the formula $f_*\bigl(g[v_0, \ldots, v_k]\bigr) = g[f(v_0), \ldots, f(v_k)]$ (if f glues vertices of a simplex together, the value is considered to be the zero chain).

Close simplicial mappings

Definition

Simplicial mappings $f, g\colon \Sigma \to \Delta$ are called *close* (contiguous) if for any simplex $\sigma \in \Sigma$ the set $f(\sigma) \cup g(\sigma)$ is contained in some simplex of the complex Δ.

Fact (arXiv:2309.16648, Section 2)

Close simplicial mappings induce homotopic mappings of geometric realizations and, therefore, identical mappings in homology.

That is why in the next lemma it is enough to establish the *closeness* of the composition to the inclusion: the coincidence of the induced homomorphisms of homology automatically follows from it.

Key lemma: close spaces — close complexes

Lemma (arXiv:2607.18447, Lemma 5.2)

Let X, Y be compact metric spaces and $2d_{GH}(X,Y) < r$. Then there exists a correspondence $R \in \mathcal{R}(X,Y)$ with $\operatorname{dis} R < r$. For any mappings $h\colon Y \to X$ and $g\colon X \to Y$ such that

$$\big(h(y), y\big) \in R \ \text{ for all } y \in Y, \qquad \big(x, g(x)\big) \in R \ \text{ for all } x \in X,$$

and for any $\varepsilon > 0$, $\nu \geq 0$, these mappings induce simplicial mappings

$$\mathrm{VR}(Y,\varepsilon) \xrightarrow{h} \mathrm{VR}(X, r+\varepsilon) \xrightarrow{\iota_\nu} \mathrm{VR}(X, r+\varepsilon+\nu) \xrightarrow{g} \mathrm{VR}(Y, 2r+\varepsilon+\nu),$$

and the composition $g \circ \iota_\nu \circ h$ is close to the inclusion $\iota\colon \mathrm{VR}(Y,\varepsilon) \hookrightarrow \mathrm{VR}(Y, 2r+\varepsilon+\nu)$.

Here ι_ν is the inclusion of X-complexes (complexes grow with the parameter), and h, g denote also the induced simplicial mappings.

Idea for proving the key lemma

Step 1 (h simplicially). If $\sigma = \{y_0, \dots, y_k\} \in \mathrm{VR}(Y, \varepsilon)$ then $\operatorname{diam} \sigma < \varepsilon$ and

$$\big|h(y_i)h(y_j)\big|_X \le |y_i y_j|_Y + \operatorname{dis} R < \varepsilon + r,$$

therefore $h(\sigma) \in \mathrm{VR}(X, r + \varepsilon)$.

Step 2 (g simplicially). If $\tau \in \mathrm{VR}(X, r + \varepsilon + \nu)$ then $\operatorname{diam} \tau < r + \varepsilon + \nu$ and

$$\big|g(x_i)g(x_j)\big|_Y \le |x_i x_j|_X + \operatorname{dis} R < 2r + \varepsilon + \nu,$$

therefore $g(\tau) \in \mathrm{VR}(Y, 2r + \varepsilon + \nu)$.

Proof idea: proximity of composition

Step 3 (closeness of $g \circ h$ to inclusion). Let $\sigma = \{y_0, \dots, y_k\} \in \mathrm{VR}(Y, \varepsilon)$. Couples

$$\big(h(y_i), y_i\big), \qquad \big(h(y_j), g(h(y_j))\big)$$

lie in R (the second — since $h(y_j) \in X$). Applying the definition of distortion, we get

$$\big|y_i\,(g \circ h)(y_j)\big|_Y < \big|h(y_i)h(y_j)\big|_X + r.$$

Further, the pairs $\big(h(y_i), y_i\big)$, $\big(h(y_j), y_j\big) \in R$ give $\big|h(y_i)h(y_j)\big|_X < |y_i y_j|_Y + r < \varepsilon + r$. So,

$$\big|y_i\,(g \circ h)(y_j)\big|_Y < \varepsilon + 2r,$$

and in the same way $\big|g(h(y_i))g(h(y_j))\big|_Y < \varepsilon + 2r$. That is, all distances between points y_i, $(g \circ h)(y_j)$ and $g(h(y_i))$ are less than $2(r + \varepsilon) + \nu$, and this means proximity.

Reminder: Jung's constant

Definitions (lecture 2)

For a non-empty bounded $A \subset Z$

$$R_Z(A) = \inf_{z \in Z} \sup_{a \in A} |za|_Z, \qquad J(Z) = \sup_{\substack{A \subset Z \\ \operatorname{diam} A > 0}} \frac{R_Z(A)}{\operatorname{diam} A} \in \left[\frac{1}{2}, 1\right].$$

It is this constant that allows us to compare the intrinsic Vietoris–Rips complex with the *external* Čech complex: the radii differ by the factor $J(Z)$.

Connection of the Čech and Vietoris–Rips complexes through $J(Z)$

Lemma (arXiv:2607.18447, Lemma 5.3)

For any metric space Z, its non-empty subset X and positive numbers $s, r > 0$:

1. each simplex of the complex $\mathrm{VR}(X, s)$ belongs to $\check{C}_{J(Z)s}(X; Z)$;
2. each simplex of the complex $\check{C}_r(X; Z)$ belongs to $\mathrm{VR}(X, 2r)$.

Proof. (1) Let $\sigma = \{x_0, \dots, x_k\} \in \mathrm{VR}(X, s)$, that is, $\operatorname{diam}\sigma < s$. By the definition of Jung's constant, there is $z \in Z$ with

$$\sup_{x \in \sigma} |zx|_Z \leq J(Z) \operatorname{diam}\sigma < J(Z)s,$$

hence $\sigma \in \check{C}_{J(Z)s}(X; Z)$.

Item (2) is exactly the second inclusion of the lemma about the connection between Čech and Vietoris–Rips.

Corollary: chain of inclusions

Corollary (arXiv:2607.18447, Lemma 5.3)

For any metric space Z and its subset X there are inclusions

$$\mathrm{VR}(X,s) \xrightarrow{\iota} \check{C}_{J(Z)s}(X;Z) \xrightarrow{\iota'} \mathrm{VR}(X,2J(Z)s).$$

Proof. Directly from the previous lemma: the first inclusion is Item (1), the second inclusion is Item (2) for $r = J(Z)s$.

Reminder: manifolds

Definition (topological manifold)

A topological space M is called an *n-dimensional manifold* if every point of it has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, and M is Hausdorff and has a countable base.

- If every point of M has a neighbourhood homeomorphic to an open subset of the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$, then M is called a manifold *with boundary*; the points corresponding to $x_n = 0$ form its *boundary* ∂M. If $\partial M = \varnothing$, one speaks of a manifold *without boundary*.
- A manifold is called *closed* if it is compact and has no boundary.
- In the statement below a subset $U \subset M$ is called *proper* if $U \neq M$.

Higher homology of manifolds

Fact (Hatcher, Theorem 3.26 and Proposition 3.29)

Let M be a connected n-dimensional manifold without boundary. Then its n-dimensional homology with coefficients in $\mathbb{Z}_2$ is equal to

$$H_n(M;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & \text{if } M \text{ compact,} \\ 0, & \text{if } M \text{ is not compact.} \end{cases}$$

Corollary

Let M be a connected closed n-dimensional manifold, and $U \subset M$ be any proper open subset. Then $H_n(U;\mathbb{Z}_2) = 0$.

Corollary: proof

Proof of the corollary. Let us decompose U into the union of all its connected components. Each component C is open in M (as a component of an open subset of a manifold), and hence it is a connected n-dimensional manifold without boundary. The image of any singular simplex $\Delta^n \to U$ (a continuous mapping of the standard simplex) is connected and lies in one component, therefore

$$C_\bullet(U;\mathbb{Z}_2) = \bigoplus_C C_\bullet(C;\mathbb{Z}_2), \qquad \text{thus} \qquad H_n(U;\mathbb{Z}_2) = \bigoplus_C H_n(C;\mathbb{Z}_2).$$

Each component of C is *non-compact*: if a component C_i of C were compact, then it would be closed in the Hausdorff M and at the same time open, that is, clopen; in a connected M this would give $C_i = M$, which means $M \subset U$ is a contradiction with the properness of U.

It remains to apply the non-compact case of the fact: $H_n(C;\mathbb{Z}_2) = 0$, and the direct sum of zeros is equal to zero. For $U = \varnothing$ the statement is trivial.

Lecture summary

1. Simplicial complexes, chain groups, boundary operator, homology; homotopy invariance.
2. Covering nerve, good coverings, nerve lemma (the nerve of a good covering is homotopy equivalent to a space).
3. External Čech complex $\check{C}_r(X;Z)$ is a nerve of balls $U_r(x)$ in the ambient space Z.
4. Vietoris–Rips complex $\mathrm{VR}(X,r)$ and connection with Čech: $\mathrm{VR}(X,r) \subset \check{C}_r(X;Z) \subset \mathrm{VR}(X,2r)$.
5. Key lemma: for $2d_{GH}(X,Y) < r$ there are simplicial mappings $\mathrm{VR}(Y,\varepsilon) \to \mathrm{VR}(X,r+\varepsilon) \to \mathrm{VR}(X,r+\varepsilon+\nu) \to \mathrm{VR}(Y,2r+\varepsilon+\nu)$, the composition of which is close to the inclusion.
6. Relationship through Jung's constant: $\mathrm{VR}(X,s) \subset \check{C}_{J(Z)s}(X;Z) \subset \mathrm{VR}(X,2J(Z)s)$.
7. Higher homology: $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$ or 0 (compact / non-compact case), whence $H_n(U;\mathbb{Z}_2) = 0$ for a proper open U in a closed connected n-manifold.