# Applications of GH-distance: MST-spectrum, Borsuk problem and combinatorial invariants

Lecture 1 of the minicourse "Gromov–Hausdorff distance: applications and lower bounds"

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# Lecture outline

1. $\mathrm{MST}$-spectrum and GH-distances to simplexes.
2. Generalized Borsuk problem.
3. Combinatorial invariants through GH-distances: clique cover number, chromatic number, Manhattan dimension.

General idea: combinatorial characteristics of a finite metric space $X$ are expressed in terms of the distance $d_{GH}(\lambda\Delta_m, X)$ from $X$ to the simplex $\lambda\Delta_m$, all non-zero distances of which are equal to $\lambda$.

# Conventions and notations

- For a metric space $X$ and $x, y \in X$, let $|xy|$ denote the distance between $x$ and $y$.
- $\Delta_k$ is a $k$-point simplex: all non-zero distances are equal to 1; $\lambda X$ denotes the space obtained by multiplying all distances of the space $X$ by $\lambda$.
- $d_{GH}$ is the Gromov–Hausdorff distance; $d_H$ is the Hausdorff distance.
- $\mathcal{D}_m(X)$ is the set of all partitions of the space $X$ into $m$ non-empty subsets;
- For $D = \{X_i\} \in \mathcal{D}_m(X)$ we set $|X_iX_j| = \inf\big\{|xy| : x \in X_i,\ y \in X_j\big\}$, $\alpha(D) = \min_{i\neq j} |X_iX_j|$ and $\operatorname{diam} D = \max_i \operatorname{diam} X_i$.

Let us recall the basic formula that we will use: for $m \leq \#X$

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\big\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\big\}.$$

# MST-spectrum: definition

**Definition (minimum spanning tree)**

Let $X$ be a finite metric space, then *the minimum spanning tree on* $X$ is the shortest of all trees $G = (X, E)$ with a set of vertices $X$, where *the length* $|G|$ *of the tree* $G$ is the sum of the lengths $|e|$ of the edges $e \in E$, and *the length of the edge* $e = xy$ connecting the vertices $x \in X$ and $y \in X$, — the distance $|xy|$ in $X$ between these vertices.

**Definition (MST-spectrum)**

Let $\#X = n$. MST-*spectrum* $\sigma(X) = (\sigma_1, \ldots, \sigma_{n-1})$ is a set ordered in non-increasing order of edge lengths of an arbitrary minimum spanning tree on $X$ (this set does not depend on the choice of tree).

# Lemma: MST-spectrum through partitions

## Lemma (cut-characterization)

Let $X$ be a finite metric space, $\#X = n$ and $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$ be the MST spectrum. Then for each $k = 1, \dots, n-1$

$$\sigma_k = \max_{D \in \mathcal{D}_{k+1}(X)} \alpha(D),$$

where $\mathcal{D}_{k+1}(X)$ is the set of all partitions of $X$ into $k+1$ non-empty subsets.

## Note

This is a standard result of the theory of minimum spanning trees: the $k$-th longest edge of the MST is equal to the maximum length of the shortest edge between parts over all partitions of vertices into $k+1$ parts. This is a cut characterization of Kruskal's algorithm (Kleinberg–Tardos, Section 4.7, Claim (4.26)).

# Cut-characterization: proof outline

Let $T$ be the minimum spanning tree on $X$, and let $e_1, \dots, e_k$ be its $k$ largest edges, $|e_1| \geq \dots \geq |e_k|$.

1. Removing these edges splits $T$ into $k+1$ connected components; let $D_T$ be a partition of $X$ into sets of component vertices.
2. $\alpha(D_T) \geq \sigma_k$: any edge of the tree connecting different components lies among the removed ones, and $\sigma_k$ is the length of the smallest of the removed ones.
3. For an arbitrary $D = \{X_1, \dots, X_{k+1}\} \in \mathcal{D}_{k+1}(X)$ we contract each part in the graph $T$ into one vertex. The resulting graph is connected and has at least $k$ edges; among the $k$ smallest of them there is an edge of weight $\leq \sigma_k$ (otherwise $T$ has more than $k$ edges of weight $> \sigma_k$). Therefore $\alpha(D) \leq \sigma_k$.
4. So $\sigma_k = \max_{D \in \mathcal{D}_{k+1}(X)} \alpha(D)$.

# Main theorem: $\mathrm{MST}$-spectrum through GH-distance

**Theorem**

Let $X$ be a finite metric space, $\#X = n$, and $\lambda \geq 2 \operatorname{diam} X$. Then for each $k = 1, \ldots, n-1$

$$\sigma_k(X) = \lambda - 2d_{GH}(\lambda\Delta_{k+1}, X).$$

The combinatorial characteristic (the lengths of the edges of $\mathrm{MST}$) turned out to be geometric: it is equal to a linear function of the distance from $X$ to the simplex $\lambda\Delta_{k+1}$.

# Idea for proving the theorem

1. **Partitions.** By the cut-characterization lemma $\sigma_k = \max_{D \in \mathcal{D}_{k+1}(X)} \alpha(D)$.
2. **Formula for the distance to the simplex.** For $m = k+1 \le n$

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D \in \mathcal{D}_m(X)} \max\bigl\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\bigr\}.$$

3. **Simplification for** $\lambda \ge 2\operatorname{diam} X$. Since $\operatorname{diam} D \le \operatorname{diam} X \le \lambda - \operatorname{diam} X \le \lambda - \alpha(D)$ and $\operatorname{diam} X - \lambda \le 0$, the maximum is simply $\lambda - \alpha(D)$. So,

$$2d_{GH}(\lambda\Delta_m, X) = \lambda - \sup_{D \in \mathcal{D}_m(X)} \alpha(D).$$

4. **Substitution** $m = k+1$. We get $2d_{GH}(\lambda\Delta_{k+1}, X) = \lambda - \sigma_k$.

# Note on the condition $\lambda \geq 2 \operatorname{diam} X$

**Note**

The condition $\lambda \geq 2 \operatorname{diam} X$ is essential: it guarantees that $\lambda - \alpha(D) \geq \operatorname{diam} D$ for all partitions of $D$, and therefore allows us to simplify the maximum. For $\lambda < 2 \operatorname{diam} X$ the formula becomes more complicated — at the maximum it is necessary to take into account $\operatorname{diam} D$.

# Example: three-point space

Let $X$ be a three-point space with distances $a \le b \le c$, where $c \le a + b$, and $\lambda \ge 2c$. Then $\operatorname{diam} X = c$, and the MST spectrum consists of two numbers: $\sigma_1 = b$ and $\sigma_2 = a$.

**Check via GH-distance.** For two three-point spaces $X_1$, $X_2$ with distances $a_i \le b_i \le c_i$ it is known that

$$2d_{GH}(X_1, X_2) = \max\bigl\{|a_1 - a_2|,\ |b_1 - b_2|,\ |c_1 - c_2|\bigr\}.$$

The simplex $\lambda\Delta_2$ is a three-point space with distances $0 \le \lambda \le \lambda$, and $\lambda\Delta_3$ is a space with distances $\lambda \le \lambda \le \lambda$. Therefore

$$2d_{GH}(\lambda\Delta_2, X) = \max\{a, \lambda - b, \lambda - c\} = \lambda - b,$$

$$2d_{GH}(\lambda\Delta_3, X) = \max\{\lambda - a, \lambda - b, \lambda - c\} = \lambda - a.$$

According to the theorem, $\sigma_1 = \lambda - 2d_{GH}(\lambda\Delta_2, X) = b$ and $\sigma_2 = \lambda - 2d_{GH}(\lambda\Delta_3, X) = a$, which is consistent with direct calculation.

# Example: four points on a line

Let $X = \{0, 1, 3, 6\} \subset \mathbb{R}$, $\operatorname{diam} X = 6$, $\lambda = 12 \geq 2 \operatorname{diam} X$. The minimum spanning tree is the chain 0–1–3–6, which means $\sigma = (3, 2, 1)$.

Let us check by cut-characterization:

- $\sigma_1 = \max_{D \in \mathcal{D}_2(X)} \alpha(D) = \max\{1, 2, 3\} = 3$ (partitioning $\{0, 1\} \cup \{3, 6\}$ gives 2, $\{0, 1, 3\} \cup \{6\}$ gives 3);
- $\sigma_2 = \max_{D \in \mathcal{D}_3(X)} \alpha(D) = 2$ (for example, $\{0, 1\} \cup \{3\} \cup \{6\}$ gives $\alpha = 2$).

And according to the theorem $\sigma_k = \lambda - 2d_{GH}(\lambda\Delta_{k+1}, X)$, that is

$$2d_{GH}(12\Delta_2, X) = 9, \qquad 2d_{GH}(12\Delta_3, X) = 10, \qquad 2d_{GH}(12\Delta_4, X) = 11.$$

# Borsuk's task: reminder

## Classical problem

Is it true that any bounded set in $\mathbb{R}^n$ can be divided into $n+1$ parts of strictly smaller diameter?

- Borsuk (1933): the hypothesis is correct for balls and "round enough" bodies.
- 1993: the hypothesis is refuted (Kan-Kalai) for $n \geq 1325$.
- Further reduction in the dimension of counterexamples:
  - Nilly (1994): $946$; Gray-Weisbach: $903$; Raigorodsky: $561$; Weisbach: $560$;
  - Hinrichs: $323$; Pikhurko: $321$; Hinrichs–Richter: $298$;
  - Bondarenko (2013): $416$ points in $\mathbb{R}^{65}$ are not divided into $84$ parts of smaller diameter;
  - Jenrich–Brauer: $352$ points in $\mathbb{R}^{64}$ are not split into $71$ parts;
  - Ji (2026): $321$ point in $\mathbb{R}^{63}$, not divided into $64$ parts, that is, $b(63) \geq 65$.
- **Borsuk number** $f(n)$ is the least $m$ such that every bounded set in $\mathbb{R}^n$ can be partitioned into $m$ parts of strictly smaller diameter.
- Asymptotics: $f(n) \geq \big(1.2255\ldots + o(1)\big)\sqrt{n}$ (Raigorodsky).
- Open range: $4 \leq n \leq 62$.

# Generalized Borsuk problem: formulation

## Definition (Borsuk partition)

Let $X$ be a bounded metric space and $m$ be a cardinal number, $m \leq \#X$. The partition $D = \{X_i\}_{i \in I} \in \mathcal{D}_m(X)$ is called *Borsuk partition* if all parts have a strictly smaller diameter: $\operatorname{diam} D < \operatorname{diam} X$, that is, $\operatorname{diam} X_i \leq \operatorname{diam} X - \varepsilon$ for some $\varepsilon > 0$ and all $i \in I$.

## Generalized Borsuk problem

Can a given bounded metric space $X$ be partitioned into a given (possibly infinite) number $m$ of subsets, each of which has a strictly smaller diameter than $X$?

# Criterion for the existence of a Borsuk partition

**Theorem**

Let $X$ be a bounded metric space, $m$ be a cardinal number, $m \le \#X$, and let $0 < \lambda < \operatorname{diam} X$. Then $X$ admits a partition into $m$ subsets of strictly smaller diameter if and only if

$$2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X.$$

**Idea of proof.** We use the formula $2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\}$ for $m \le \#X$.

- If $D$ is a Borsuk partition, then $\operatorname{diam} D < \operatorname{diam} X$, as well as $\lambda - \alpha(D) \le \lambda < \operatorname{diam} X$ and $\operatorname{diam} X - \lambda < \operatorname{diam} X$. This means that the maximum is $< \operatorname{diam} X$ and the infimum is $< \operatorname{diam} X$.
- Conversely, if for all $D$ there were $\operatorname{diam} D \ge \operatorname{diam} X$, then all maxima would be $\ge \operatorname{diam} X$, that is, the infimum $\ge \operatorname{diam} X$. This means that there is $D$ with $\operatorname{diam} D < \operatorname{diam} X$.

# Spaces with two distances

## Definition ($(a,b)$-space)

Let $0 < a < b$. A finite metric space $(X, d)$ is called a $(a,b)$-*space* if for any distinct $x, y \in X$ $d(x,y) \in \{a, b\}$ holds.

To each $(a,b)$-space $X$ we associate the graphs $G_X$ and $H_X$ on the set $X$ of its vertices:

- $x, y$ are adjacent in $G_X \iff d(x,y) = b$;
- $x, y$ are adjacent in $H_X \iff d(x,y) = a$.

A graph $H$ on the same vertex set as a graph $G$ is *complementary to* $G$ if its edges are exactly the non-edges of $G$, that is, $H = \overline{G}$.

## Note

The graph $H_X$ is complementary to $G_X$: $H_X = \overline{G_X}$ and any finite graph $G$ is realized as $G_X$ of some $(a,b)$-space with $0 < a < b \leq 2a$.

# Clique cover number

## Clique

*A clique* is a set $C$ of vertices of a (simple) graph $G = (V, E)$ any two distinct vertices of which are joined by an edge (equivalently, the subgraph induced on $C$ is complete); a one-point set is also a clique. By a clique we mean a set of vertices, not a subgraph.

## Clique cover number

*The clique cover number* of a graph $G = (V, E)$ is the smallest $k$ for which there are cliques $C_1, \ldots, C_k$ with $V = \bigcup_{i=1}^{k} C_i$; notation $\theta(G)$.

Let $0 < a < b \leq 2a$. Using $G$, we construct the $(a, b)$-space $Y_G = (V, d_G^a)$: put $d_G^a(u, v) = a$ for $\{u, v\} \in E$ and $d_G^a(u, v) = b$ for $\{u, v\} \notin E$, $u \neq v$. Edges of $G$ have *smaller* distance; if $G$ is complete, all distances equal $a$, otherwise $\operatorname{diam} Y_G = b$.

# Clique covering via GH-distance

**Theorem**

For any finite graph $G = (V, E)$ and any $m \geq 1$

$$\theta(G) \leq m \iff 2d_{GH}(a\Delta_m, Y_G) < b.$$

In particular, $\theta(G) = \min\{m \geq 1 : \ 2d_{GH}(a\Delta_m, Y_G) < b\}$.

**Idea of the proof** (here $n = \#V$) — reduction to the Borsuk criterion. If $G$ is non-complete, then $\operatorname{diam} Y_G = b$, and

- a subset $W \subseteq V$ is a clique if and only if $\operatorname{diam} W < b = \operatorname{diam} Y_G$; hence partitions of the vertex set into cliques are exactly Borsuk partitions of $Y_G$;
- for $m \leq n$ a covering of $V$ by at most $m$ cliques is equivalent to a partition into exactly $m$ cliques, and the Borsuk criterion with $X = Y_G$, $\lambda = a$ gives $2d_{GH}(a\Delta_m, Y_G) < b$;
- for $m \geq n$ and for complete $G$ a correspondence between $a\Delta_m$ and $Y_G$ gives $2d_{GH}(a\Delta_m, Y_G) \leq a < b$.

# Chromatic number

## Definition

The *chromatic number* of a graph $G = (V, E)$ is the smallest natural $k$ for which there exists a coloring $\gamma\colon V \to \{1, \dots, k\}$ with the condition $\gamma(u) \neq \gamma(v)$ for any edge $\{u, v\} \in E$.
Notation: $\chi(G)$.

## Note (duality)

A proper coloring of $G$ is the clique cover of the complement graph: $\chi(G) = \theta(\overline{G})$.

Let again $0 < a < b \leq 2a$. Using the graph $G$, we construct the $(a, b)$-space $X_G = (V, d^b_G)$:

$$d^b_G(u, v) = \begin{cases} b, & \{u, v\} \in E, \\ a, & \{u, v\} \notin E,\ u \neq v. \end{cases}$$

Here the edges $G$ have a *greater* distance, so $X_G = Y_{\overline{G}}$. If $G$ is empty, all nonzero distances are equal to $a$; otherwise $\operatorname{diam} X_G = b$.

# Chromatic number via GH-distance

**Theorem**

For any finite graph $G = (V, E)$ and any $m \geq 1$

$$\chi(G) \leq m \iff 2d_{GH}(a\Delta_m, X_G) < b,$$

and in particular

$$\chi(G) = \min\{m \geq 1 : \ 2d_{GH}(a\Delta_m, X_G) < b\}.$$

**Proof.** Reduction to the clique cover theorem: $\chi(G) = \theta(\overline{G})$ and $X_G = Y_{\overline{G}}$, it remains to apply the previous theorem to $\overline{G}$.

Direct check: if all distances in $X_G$ are equal to $a$, then $G$ is empty, $\chi(G) = 1$ and $2d_{GH}(a\Delta_m, a\Delta_n) < b$ for all $m \geq 1$; if $G$ is not empty, then $\operatorname{diam} X_G = b$, and the parts of the partition of diameter $< b$ are exactly independent sets.

# Example: cycle $C_5$

For a cycle of length $5$ it is known that $\chi(C_5) = 3$. Let us construct a $(a, b)$-space $X_{C_5}$: the distance $b$ between neighboring vertices, $a$ between non-neighboring ones; take $a = 1$, $b = 2$ (condition $b \le 2a$ is satisfied). By theorem

$$\chi(C_5) = \min\bigl\{m \ge 1 :\ 2d_{GH}(\Delta_m, X_{C_5}) < 2\bigr\}.$$

- $m = 1$: $2d_{GH}(a\Delta_1, X_{C_5}) = \operatorname{diam} X_{C_5} = 2$, condition $< 2$ is not satisfied.
- $m = 2$: $X_{C_5}$ does not allow partitioning into $2$ parts of smaller diameter, so $2d_{GH}(\Delta_2, X_{C_5}) = 2$; the condition is not satisfied.
- $m = 3$: for example, the partition $\{1, 3\}$, $\{2, 4\}$, $\{5\}$ has the diameter $1 < 2$, which means $2d_{GH}(\Delta_3, X_{C_5}) < 2$.

Therefore, $\chi(C_5) = 3$.

# Manhattan dimension

## Definition ($\ell_1$-dimension)

The *Manhattan dimension* (or $\ell_1$-*dimension*) of a finite metric space $(X, d)$ is the smallest $m \geq 1$ for which there is an isometric embedding $f\colon X \to \mathbb{R}_1^m$ into the space $\mathbb{R}^m$ with a distance given by the norm $\big|(x_1, \ldots, x_m)\big|_1 = \sum_{i=1}^m |x_i|$. Notation: $\dim_{\ell_1}(X)$.

## Definition (cut and cut pseudometric)

*A cut* of a set $X$ is an unordered pair $c = \{S, S'\}$, where $S$ is a proper non-empty subset, $S' = X \setminus S$. It corresponds to the *cut pseudometric*

$$\delta_c(x, y) = \begin{cases} 1, & \#\big(S \cap \{x, y\}\big) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

that is, $\delta_c(x, y) = 1$ exactly when $x$ and $y$ lie on opposite sides of the cut.

# Assouad criterion

## Definition (cone of cuts $\mathrm{CUT}_{\#X}$)

The *cone of cuts* $\mathrm{CUT}_{\#X}$ is the set of all sums $\sum_{c\in\mathcal{C}} \lambda_c \delta_c$ over all families $\mathcal{C}$ of cuts of $X$ and all $\lambda_c > 0$.

## Fact (Assouad criterion)

A finite metric space $(X, d)$ is isometrically embedded in some $\mathbb{R}_1^m$ if and only if $d \in \mathrm{CUT}_{\#X}$, that is, when there exists a family of cuts $\mathcal{C}$ and coefficients $\lambda_c > 0$ such that

$$d = \sum_{c\in\mathcal{C}} \lambda_c \delta_c.$$

So, $\ell_1$-embedding (isometric embedding in some $\mathbb{R}_1^m$) is the decomposability of a metric into a positive combination of cut pseudometrics. It remains to understand how to calculate the minimum embedding dimension from the family $\mathcal{C}$.

# Hypergraph (recall)

**Definition (hypergraph)**

A *hypergraph* is a pair $H = (V, E)$, where $V$ is a set of vertices and $E$ is a family of non-empty subsets of $V$, called *hyperedges*. Unlike the edges of a graph, a hyperedge may contain any number of vertices, at least $1$.

**Definition (proper coloring, chromatic number)**

A coloring of the vertices of $H$ is *proper* if no hyperedge is monochromatic, that is, every hyperedge contains at least two colors. The least number of colors is the *chromatic number* $\chi(H)$.

# Nested hypergraph

## Definition (incompatible cuts, asteroidal triples)

The cuts $\{A, A'\}$ and $\{B, B'\}$ are *incompatible* if all four intersections $A \cap B$, $A \cap B'$, $A' \cap B$, $A' \cap B'$ are non-empty. Three cuts form an *asteroidal triple* if one side from each cut can be selected so that the selected three sets do not intersect in pairs.

## Definition (nested hypergraph $\Gamma(\mathcal{C})$)

Vertices are the cuts from $\mathcal{C}$; hyperedges are all pairs of incompatible cuts and all asteroidal triples. This is a hypergraph with edges of size $2$ and $3$.

Hyperedges of size $3$ cannot be "forgotten": for an asteroidal triple

$$\Big\{\big\{\{a,b\},\{c\}\big\},\big\{\{b,c\},\{a\}\big\},\big\{\{c,a\},\{b\}\big\}\Big\},$$

none of the pairs $\big\{\{a,b\},\{c\}\big\}$, $\big\{\{b,c\},\{a\}\big\}$, $\big\{\{c,a\},\{b\}\big\}$ is an edge of $\Gamma(\mathcal{C})$.

# Bandelt–Chepoi–Laurent theorem

**Theorem**

Let $(X, d)$ be a finite metric space and

$$d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c, \qquad \lambda_c > 0.$$

Then $(X, d)$ is isometrically embedded in $\mathbb{R}_1^m$ if and only if the nested hypergraph $\Gamma(\mathcal{C})$ is $m$-colorable. In particular,

$$\dim_{\ell_1}(X) = \chi\big(\Gamma(\mathcal{C})\big).$$

To reduce the coloring of a hypergraph to the coloring of simple graphs (and apply the formula for the chromatic number), we replace each hyperedge of size $3$ with one ordinary edge.

# Family of simple graphs $\mathcal{G}(\mathcal{C})$

## Definition

Let $\{T_1, \ldots, T_k\}$ be all asteroidal triples in $\Gamma(\mathcal{C})$, and $E_2$ be the set of all hyperedges of size $2$. For each triple $T_i = \{a, b, c\}$, choose one of the pairs $\{a, b\}$, $\{b, c\}$, $\{a, c\}$ and put $E(G) = E_2 \cup \{e_1, \ldots, e_k\}$, where $e_i$ is the selected pair from $T_i$. The family of all resulting simple graphs is denoted by $\mathcal{G}(\mathcal{C})$; its power does not exceed $3^k$.

## Lemma

$\Gamma(\mathcal{C})$ $m$-colorable $\iff$ at least one graph from $\mathcal{G}(\mathcal{C})$ $m$-colorable. Therefore, $\chi\big(\Gamma(\mathcal{C})\big) = \min_{G \in \mathcal{G}(\mathcal{C})} \chi(G)$.

## Corollary

If $d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c$, $\lambda_c > 0$, then $(X, d)$ is embedded in $\mathbb{R}_1^m$ if and only if $\mathcal{G}(\mathcal{C})$ has a $m$-colorable graph, and thus $\dim_{\ell_1}(X) = \min_{G \in \mathcal{G}(\mathcal{C})} \chi(G)$.

# Manhattan dimension via GH-distance

Let us fix $0 < a < b \le 2a$. For each $G \in \mathcal{G}(\mathcal{C})$ we construct an $(a,b)$-space $C_G = (\mathcal{C}, d_G^b)$ on the set of cuts: the distance $b$ between adjacent vertices, $a$ between non-adjacent ones (this is the same space as $X_G$, but on the support $\mathcal{C}$).

**Theorem**

Let $(X,d)$ be a finite metric space, $d = \sum_{c\in\mathcal{C}} \lambda_c \delta_c$ with $\lambda_c > 0$, and $0 < a < b \le 2a$. Then

$$\dim_{\ell_1}(X) = \min_{G\in\mathcal{G}(\mathcal{C})} \chi(G) = \min_{G\in\mathcal{G}(\mathcal{C})} \min\bigl\{m \ge 1 :\ 2d_{GH}(a\Delta_m, C_G) < b\bigr\}.$$

# Manhattan dimension: equivalent formulation

**Theorem (continued)**

$\dim_{\ell_1}(X) = m$ if and only if $m$ is the smallest natural number for which

$$\min_{G \in \mathcal{G}(\mathcal{C})} 2d_{GH}(a\Delta_m, C_G) < b.$$

**Proof.** The equality $\dim_{\ell_1}(X) = \min_{G \in \mathcal{G}(\mathcal{C})} \chi(G)$ is a consequence of the Bandelt–Chepoi–Laurent theorem and the family lemma $\mathcal{G}(\mathcal{C})$, and the formula

$$\chi(G) = \min\bigl\{m \geq 1 : \ 2d_{GH}(a\Delta_m, C_G) < b\bigr\}$$

for every $G \in \mathcal{G}(\mathcal{C})$; this is the chromatic number theorem.

# Notes and summary of the chapter

**Note**

Constructing the family $\mathcal{G}(\mathcal{C})$ is difficult in the general case: the problem of $\ell_1$-embedding of a finite metric space is NP-complete. For special classes (trees, hypercubes, ultrametrics), the cut decomposition and nested hypergraph are described explicitly.

In this lecture we received:

1. expression of MST-spectrum through partitions and formula $\sigma_k = \lambda - 2d_{GH}(\lambda\Delta_{k+1}, X)$;
2. criterion for the existence of a Borsuk partition: $2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X$;
3. formulas for clique cover number and chromatic number: $\theta(G) = \min\{m : \ 2d_{GH}(a\Delta_m, Y_G) < b\}$, $\chi(G) = \min\{m : \ 2d_{GH}(a\Delta_m, X_G) < b\}$;
4. formula for Manhattan dimension: $\dim_{\ell_1}(X) = \min_{G\in\mathcal{G}(\mathcal{C})} \min\{m : \ 2d_{GH}(a\Delta_m, C_G) < b\}$.