Gromov–Hausdorff Convergence with Group Actions

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Backgrounds from Algebraic Geometry

Let X be a compact Fano manifold (i.e. $c_1(X) > 0$) and consider (normalized) Kähler-Ricci flow on it

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) + \omega(t),\ \omega(0) \in c_1(X),$$

where $\omega(t)$ is the Kähler form, $\mathrm{Ric}(\omega(t))$ its Ricci (curvature) form. There is a famous result:

Theorem 1 (Hamilton–Tian Conjecture (confirmed))**.** *For any sequence $\{t_n\}_{n\in\mathbb{N}_+}$ with $t_n \to +\infty$, the sequence $\{(X, \omega(t_n))\}_{n\in\mathbb{N}_+}$ converges to certain $(X_\infty, \omega_\infty)$ in the Gromov–Hausdorff topology, where X_∞ is a normal $\mathbb{Q}$-Fano variety with klt singularities, ω_∞ a (possibly singular) Kähler-Ricci soliton on X_∞.*

Problem 1. *Is it possible to derive $(X_\infty, \omega_\infty)$ for concrete (explicit) initial data $(X, \omega(0))$? Especially if $X\infty \ncong X$?*

One possible way is do on $(X, \omega(0))$ with large symmetry, i.e. with a large group action, and expect X_∞ has at least a symmetry "with the same large".

Algebraic Approach

In algebraic geometry, the problem is solved via a different specific approach. It is proved by a series of works by Chen–Sun–Wang, Han–Li and Blum–Liu–Xu–Zhuang that X_∞ can be realized via algebraic deformation family called test configurations:

Definition 1. *Let (X, L) be a polarized variety. A test configuration of X consists of the following data:*

- *A scheme $\mathcal{X}$ with a $\mathbb{C}^\times$-action;*
- *A $\mathbb{C}^\times$-equivariant flat and proper morphism $\mathrm{pr} : \mathcal{X} \to \mathbb{P}^1$ so that $\mathcal{X} \setminus \mathrm{pr}^{-1}(0)$ is $\mathbb{C}^\times$-equivariantly isomorphic to $X \times \mathbb{P}^1 \setminus \{0\}$.*

A (semi-)test configuration of (X, L) consists of a test configuration $\mathcal{X}$ and a $\mathbb{C}^\times$-linearized $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ so that:

- *$\mathcal{L}$ is relative (semi-)ample on $\mathcal{X}$ with respect to the morphism pr;*
- *With respect to the $\mathbb{C}^\times$-action on $\mathcal{X}$, $(\mathcal{X}, \mathcal{L})|_{\mathbb{P}^1\setminus\{0\}}$ is $\mathbb{C}^\times$-equivariantly isomorphic to $(X, L^{r_0}) \times (\mathbb{P}^1 \setminus \{0\})$ for some fix $r_0 \in \mathbb{N}_+$ (called the index of $(\mathcal{X}, \mathcal{L})$).*

A test configuration (more precisely, its fibres) can be realized as

$$\mathcal{X}_t = \rho(t)\iota(X) \subset \mathbb{P}^N,$$

where $\iota : X \to \mathbb{P}^N$ is a projective (Kodaira) embedding by sections of $K_X^{-r_0}$, $\rho(t) \subset \mathrm{PSL}_{N+1}$ is an (algebraic) one-parameter group. In this case X_t converges even in the Hausdorff topology to $\mathcal{X}_0$.

Algebraic Approach (Continued)

- For the case of Hamilton–Tian Conjecture, X_∞ is given by a test configuration $\mathcal{X}$ that minimizes certain *strictly convex* functional (hence unique). Thus $X_\infty \cong \mathcal{X}_0$ is G-equivariant if X admits a G-action.

How to measure symmetry of a group action?

Definition 2 (Vinberg, 1985)**.** *Let X be a normal variety with a regular action of a connected, complex, reductive group G. Fix any Borel subgroup B of G, the complexity*

$$c_G(X) := \mathrm{tr.deg}(\mathbb{C}(X)^B) = \textit{codim of a generic } B\textit{-orbit.}$$

- Example-1: When G is a torus, then $G = B$. This has been well-studied in the theory of T-varieties.
- For any G-invariant subvariety Y of X, $c_G(Y) \leq c_G(X)$;
- Example-2: A variety of one-parameter type:

$$X \dashrightarrow (G/H \times \tilde{C})/A \xrightarrow{\pi} \tilde{C}/A =: C,$$

where $c_G(G/H) = 0$, $\tilde{C}$ a smooth projective curve, A a discrete group whose action commutes with G. If it is considered as a deformation family of some $X_{t_0} := \overline{\pi^{-1}(t_0)}$, then as t varies on C, the fibre always has complexity 0 ($= c_G(X_{t_0})$).

Problem 2. *Suppose that there is a sequence of metric spaces $\{(X_n, d_n)\}_{n\in\mathbb{N}_+}$, each has the action of a Lie group G-action, and $\{(X_n, d_n)\}_{n\in\mathbb{N}_+}$ converges to (X_∞, d_∞), is it possible to prove (X_∞, d_∞) can admits a G-action? (Which should be "compatible" with K_n-actions on sequence in certain sense.)*

Problem 3. *If so, can we compare codimension of generic G- (or maximal torus T)-orbits on the sequence and on the limit? (If the terminology* codim *matters).*

- If $G = T$ is a compact torus, then we gets a real analogous of $c_G(X)$;
- When G is compact, the "G-cohomogeneity" ($=:\dim X - 2\dim(\text{generic } G\text{-orbit})$) appears in several geometrico-analytical problems, such as classification of manifolds with certain canonical metrics.

Gromov–Hausdorff Spaces with Finite Group Actions

Results from arXiv:2506.15414 by Lim–Mémoli:

- Consider two compact metric spaces (X, d), (Y, ρ) with a finite group K-action (as isometry). In this article a K-equivariant Gromov–Hausdorff metric is defined:

$$d_{GH}^K((X,d),(Y,\rho)) := \frac{1}{2}\inf\{\mathrm{dis}(\mathfrak{R})|\mathfrak{R}\text{–}K\text{-invariant correspondence between } X \text{ и } Y\};$$

- It is also proved that

$$d_{GH}^K((X,d),(Y,\rho)) \geq d_{GH}((X/K, d/K),(Y/K,\rho/K)).$$

  Here by d/K and ρ/K we mean metrics on quotient spaces;

- The space

$$\mathfrak{M}_K := \{(X,d)|\text{compact metric spaces with } K\text{-action}\}$$

  with metric d_{GH}^K is complete, and the precompactness theorem holds. In particular, any limit space of sequence from $\mathfrak{M}_K$ has a K-action;

- It is crucial to use K-invariant ϵ-nets in the proof, where the finiteness of K plays a role.

Problems

1) Can one cancel the assumption that K-is finite? One possible case this instead to assume K is compact (confirmed by Lang Lang recently);

2) Consider an *approximating sequence* $K_1 \subset ... \subset K_n \subset K_{n+1} \subset ... \subset K$ so that: 1) each K_n is finite and $\cup_n K_n$ is dense in K. When K is compact, such a sequence exists only if and only if $K = (S^1)^r$ is a torus.

In the following we hope to show:

Theorem 2. *Let $K_1 \subset ... \subset K_n \subset K_{n+1} \subset ... \subset K$ be an approximating sequence of $K = (S^1)^r$, equipped with canonical metric on K. Assume that $\{(X_i, d_i)\}_{i \in \mathbb{N}_+}$ is a sequence of compact metric spaces that converges to some (X, d) in the Gromov-Hausdorff topology, and each (X_i, d_i) admits a K_i-action. Assume further that the sequence satisfies the following Condition:*

(*) There is a monotonic decreasing function $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\omega(0) = 0$ such that

$$d_n(x, \mu_n(g_n, x)) \leq \omega(\varrho_0(g, \mathrm{Id})), \ \forall x \in X_n, \ g \in K_n, \ n \in \mathbb{N}_+,$$

where μ_n denotes the K_n-action on (X_n, d_n). Then (X, d) admits at least one K-action (not necessarily unique), as limit of the K_n-actions on X_n.

Our approach

- In general, let X be a topological and K a group, X admits a K-action if there is a continuous map

$$\begin{aligned} \mu : K \times X &\to X \\ (k, x) &\to k \cdot x, \end{aligned}$$

  satisfies the group law. Also, μ is uniquely characterized by its graph;

- By Gromov's theorem, up to replace $\{(X_n, d_n)\}_{n\in\mathbb{N}_+}$ by its subsequence, there is some metric space (Z, d_Z) and isometry $\phi_n : X_n \to Z, n = 1, 2, ...$ so that $\{X_n\}_{n\in\mathbb{N}_+} \subset \mathcal{H}(Z)$ converges in Hausdorff topology in Z (identify $\phi_k(X_k)$ with X_k below);

- Consider sequence of graphs:

$$\Gamma_k := \{(g, x, \mu_k(g, x))|g \in K_k, x \in X_k\} \subset K \times Z \times Z.$$

  As each Γ_k is compact, by Blaschke's theorem, there is a subsequence (stilled denoted by $\{\Gamma_k\}$), converges in Hausdorff topology to some $\Gamma \subset K \times Z \times Z$;

- It remains to prove Γ is the graph of some isometric action $\mu : K \times X \to X$.

Our approach (Continuous)

Consider Hausdorff limits of graphs in a compact metric space:

Lemma 1. *Let (U, d_U) and (V, d_V) be compact metric spaces, $\{K_n \Subset U\}_{n\in\mathbb{N}_+}$ a sequence of compact subsets and $\{f_n \in C(K_n; V)\}_{n\in\mathbb{N}_+}$ a sequence of continuous functions. Suppose that the graphs $\Gamma_n := \mathrm{graph}(f_n)$ converges to some closed set $\Gamma \subset U \times V$ and the following condition holds:*

(*) There is a function $m = m(t) \searrow 0^+$ as $t \searrow 0^+$ and an $r_0 > 0$ such that

$$\mathrm{osc}_{B_x(r)} f_n \leq m(r),\ \forall x \in K_n,\ r < r_0,\ n \in \mathbb{N}_+,$$

$$\Gamma_k := \{(g, x, \mu_k(g, x)) | g \in K_k, x \in X_k\} \subset K \times Z \times Z.$$

Then $\Gamma = \mathrm{graph}(\varphi)$ for some $f \in C(K; V)$, where K_n converges to some compact $K \subset U$.

- Apply the above Lemma to $\{\Gamma_n\}_{n\in\mathbb{N}_+}$, it is also not hard to check the resulting f defines a K-action μ.

Uniqueness Problem

Suppose that in the above progress we have two subsequences $\{\Gamma_n^{(\alpha)}\} \subset \{\Gamma_n\}$ that gives different K-actions $\mu^{(\alpha)}$ for $\alpha = 1, 2$, can one find a $\varphi \in \mathrm{Iso}(X, d)$ so that

$$\varphi \circ \mu^{(1)} \circ \varphi^{-1} = \mu^{(2)} \tag{1}$$

holds?

- φ (if exists) is determined up to elements in $Z_{\mathrm{Iso}(X,d)}(\mu^{(1)})$;
- φ (if exists) is characterized by a graph in $X \times X$.

Theorem 3. *Consider*

$$C_{n,m} := d_{\mathrm{GH}}^{K_{\min\{n,m\}}}(X_n^{(1)}, X_m^{(2)}),\ n, m \in \mathbb{N}_+.$$

Then (1) *holds iff*

$$\lim_{n,m\to+\infty} C_{n,m} = 0.$$

- In the above we use the notation of Lim–Mémoli. Our assumption can be roughly phrased as "K_n-equivariantly converges for all $n \in \mathbb{N}_+$".

Our approach

Lemma 2. *Let (Z, d_Z) be a compact metric space and $\{X_n\}$, $\{Y_n\} \subset Z$ to sequences of compact subspaces. Assume $X_n \to X$ and $Y_n \to Y$ in Hausdorff topology. Suppose that $\{R_n \in \mathfrak{R}(X_n, Y_n)\} \subset Z$ is a sequence of correspondences with $R_n \to R \subset Z \times Z$ in Hausdorff topology. Then $R \in \mathfrak{R}(X, Y)$ is a correspondence and*

$$\lim_{n \to +\infty} \mathrm{dis}(R_n) = \mathrm{dis}(R).$$

- Apply the above lemma to $\{X_n^{(1)}\}$, $\{X_n^{(2)}\} \subset Z$, and consider

  $$R_n \in \mathfrak{R}^{K_n}(X_n^{(1)}, X_n^{(2)}).$$

  With $C_{n,m} \to 0$ one gets a limit correspondence $R \in \mathfrak{R}(X, X)$ that is the graph of some $\varphi \in \mathrm{Iso}(X, d)$;

- Abstractly, R admits a K-action. However one should prove it is induced from the K-action $\mu^{(1)} \times \mu^{(2)}$ on $X \times X$!

Our approach (Continuous)

Lemma 3. *Given compact metric spaces Z, W, A, B and sequences of compact subspaces $\{X_n\} \subset Z, \{Y_n\} \subset W, \{U_n\} \subset A, \{V_n\} \subset B$ with*

$$X_n \to X,\ Y_n \to Y,\ U_n \to U,\ V_n \to V,\ \textit{(all are compacts)}$$

in Hausdorff topology. Consider $f_n \in C(X_n; Y_n)$ and $g_n \in C(U_n; V_n)$ with convergence of graphs (in Hausdorff topology)

$$\text{graph}(f_n) \to \text{graph}(f),\ \text{graph}(g_n) \to \text{graph}(g)$$

for $f \in C(X; Y)$, $g \in C(U; V)$. Then

$$\text{graph}(f_n \times g_n) \to \text{graph}(f \times g)$$

in $(X \times U) \times (Y \times V)$.

- The restriction of graph of functions also converges to the graph of restriction of the limiting function.

Our approach (Continuous)

- There is a subsequence of

$$\{(X_n \times X_m) \times K_n \times (X_n \times X_m)\}_{n,m\in\mathbb{N}_+} \subset (Z \times Z) \times K \times (Z \times Z),$$

  so that the graphs

$$\hat{\Gamma}_n := \mathrm{graph}(\hat{\rho}_n), \text{ where } \hat{\rho}_n(x, x', k) = (\mu_{i_n}(x, k), \mu_{j_n}(x', k)),$$

  converges to the graph of $\mu^{(1)} \times \mu^{(2)}$ (μ_{i_n}, μ_{j_n} converges to $\mu^{(1)}$ and $\mu^{(2)}$, resp.);

- Consider (taking subsequence)

$$((X_{i_n} \times X_{j_n}) \times K_n \times (X_{i_n} \times X_{j_n}) \supset) R_n \times K_n \times R_n \to R \times K \times R,$$

  the resulting K-action on this R is induced from the K-action $\mu^{(1)} \times \mu^{(2)}$ on $X \times X$.

Examples

Example 1. *The Condition (*) can not be dropped: Let* $X_1 = X_2 = ... = X_n = ... = S^1 \cong \{z \in \mathbb{C}||z| = 1\}$, *and* $d_1 = d_2 = ... = d_n = ... = \varrho_0$. *Set*

$$\mu_n(t_j, x) = t^{2^{n-1}} \cdot x,\ t_j = e^{\frac{\pi\sqrt{-1}j}{2^{n-1}}} \in K_n.$$

Then it holds $\mu_n(t_1, x) = -x$ *and*

$$(1, x, x), (1, x, -x) \in \Gamma,\ \forall x \in X,$$

which shows that Γ *can not be a graph of any mapping.*

Example 2. *The* K*-action on* X *derived from different subsequences can be different: Let* $X_1 = X_2 = ... = X_n = ... = S^1 \cong \{z \in \mathbb{C}||z| = 1\}$ *and* $d_1 = d_2 = ... = d_n = ... = \varrho_0$. *Set*

$$\mu_{2m}(t, x) = t \cdot x,\ \mu_{2m-1}(t, x) = \bar{t} \cdot x.$$

Then for $n = 2m$ *and* $n = 2m - 1$ *we get on* X *different* μ*-actions. Note that in this case the two actions differ from each other by a reflection in* $\mathrm{Iso}(X)$.

Example 3. *One more example: Let* $X_1 = X_2 = ... = X_n = ... = S^1 \cong \{z \in \mathbb{C}||z| = 1\}$ *and* $d_1 = d_2 = ... = d_n = ... = \varrho_0$. *Set*

$$\mu_{2m}(t, x) = t \cdot x,\ \mu_{2m-1}(t, x) = x.$$

Then for $n = 2m$ *and* $n = 2m - 1$ *we get on* X *different* S^1*-actions. However in this case the condition "*K_n*-equivariantly converges for all* $n \in \mathbb{N}_+$*" fails, the two actions are obviously non-conjugate.*

Thank You for Your Attention!