# Exercises 27 Oct. 2026

## 1 Hausdorff convergence of graphs

**Exercise 1.** *Let $(U, d_U)$ and $(V, d_V)$ be compact metric spaces. Consider a series of continuous functions $\{f_n\}_{n\in\mathbb{N}_+} \subset C(U;V)$ and the corresponding graphs*

$$\Gamma_n := \mathrm{graph}(f_n) \subset U \times V,\ n \in \mathbb{N}_+.$$

*Assume that $\Gamma_n$ converges to some closed set $\Gamma \subset U \times V$ as $n \to +\infty$ in the Hausdorff topology.*
*1) Prove that:*
*1.1) Under the canonical projection $\pi_U : U \times V \to U$, $\pi_U(\Gamma) = U$;*
*1.2) If the following condition:*

*(*) There exists a number $r_0 > 0$ and a monotonic function $m = m(t) : [0, +\infty) \to [0, +\infty)$ with $m(0^+) = 0$ such that for every $x \in U$,*

$$\mathrm{osc}_{B_x(r)}(f_n) \le m(r),\ \forall n \in \mathbb{N}_+,\ r \le r_0,$$

*holds, then $\Gamma$ is the graph of certain $f \in C(U;V)$.*
*2) Please give an example to show if the Condition (*) fails, $\Gamma$ may not be a graph of any function.*

(**Hint:** This is a variant of Arzéla-Ascoli's Lemma, where Condition (*) is a variant of equicontinuity.)

## 2 Equivariant Gromov–Hausdorff convergence

The following exercises are on Lim–Mémoli's equivariant Gromov–Hausdorff theory. Assume in the following that $G$ is a finite group. By "$G$-acts on $(X, d)$" we mean $G$ acts on $X$ as an isomorphism, i.e. morphism $\mu : G \to \mathrm{Iso}(X, d)$ is given. The $G$-equivariant Gromov–Hausdorff distance of two metric spaces with $G$-action is defined as

$$d_{GH}^G := \{\frac{1}{2}\mathrm{dis}(R) | R \text{ is a } G\text{-invariant correspondence in } X \times Y\}.$$

Suppose that $(X, d_X)$ is a metric space with $G$-action, the metric on quotient space $X/G$ is defined as following:

$$d_{X/G}([x_1], [x_2]) := \inf_{g\in G} d_X(x_1, g \cdot x_2).$$

**Exercise 2** (Lim–Mémoli)**.** *Suppose that $X$ and $Y$ are two metric spaces with $G$-action. Prove that*

$$d_{GH}^G(X, Y) \ge d_{GH}(X/G, Y/G),$$

*and give an example to show the inequality can be strict.*

(**Hint:** See: [arXiv:2506.15414, Prop. 3.3]. If $X$, $Y$ are $G$-homogeneous spaces then the quotient spaces are singleton.)

**Exercise 3.** *Suppose that one has two different morphisms $\mu_i : G \to \mathrm{Iso}(X, d)$, $i = 1, 2$, and denote by $X_i$ the two spaces $(X, d)$ equipped with different $G$-actions, resp. Give an example to show that $d_{GH}^G(X_1, X_2)$ can be non-zero.*

(**Hint:** An example not too trivial: Consider two $\mathbb{Z}^2$-actions on $\mathbb{R}^2$. $\mu_1((x, y), (m, n)) = (x + m, y + n)$ and $\mu_2((x, y), (m, n)) = (x + m + n, y + n)$.)

**Exercise 4.** *Consider the standard 3-dimensional unit sphere $S^3(1)$ centred at the origin in $\mathbb{R}^4 \cong \mathbb{C}^2$. Consider the following $\mathbb{Z}_p$-action ($p \geq 2$):*

$$e^{\frac{2\pi\sqrt{-1}}{p}}(z^1, z^2) = (e^{\frac{2\pi\sqrt{-1}}{p}} z^1, e^{\frac{2\pi\sqrt{-1}}{p}q} z^2), \text{ where } p \text{ and } q \text{ are relatively prime.}$$

*Suppose that $x$ is a sufficiently generic point in $S^3(1)$ and its $\mathbb{Z}_p$-orbit $O_x$. Calculate (at least an upper and a lower bounds of) $\mathrm{d}_{GH}^{\mathbb{Z}_p}(S^3(1), O_x)$.*

(**Hint:** This equals to the diameter of $S^3/\mathbb{Z}_p$ - the lens space $L(p, q)$, whose diameter is $\pi/2$. Similar to this exercise one may also get $d_{GH}^{\mathbb{Z}^n}(\mathbb{R}^n, \mathbb{Z}^n) = \frac{1}{2}\sqrt{n}$.)

**Exercise 5.** *Consider the same (as above) problem in the following case: $z_1$, $z_2 \in S^1(1) \subset \mathbb{C}^1$, and $\mathbb{Z}_p$ acts on the flat torus $S^1(1) \times S^1(1)$.*

(**Hint:** Also consider the diameter of quotient space.)