Geometric Structures and Second-Order Differential Equations

Results and Open Problems

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Figure: Charles Eresmann

Jets

Let $\mathcal{F} = C^\infty(M)$ be the ring of smooth functions on a smooth manifold M:

$$\mu_a = \{f \in \mathcal{F} \mid f(a) = 0\}$$

$$\mu_a^2 = \left\{\sum f_1 f_2 \mid f_1, f_2 \in \mu_a\right\}, \qquad \mu_a^k = \left\{\sum f_1 \cdots f_k \mid f_i \in \mu_a\right\}.$$

$$\mathcal{F} \supset \mu_a \supset \mu_a^2 \supset \cdots \supset \mu_a^k \supset \cdots$$

The quotient ring

$$J_a^k(M) := \mathcal{F}/\mu_a^{k+1}$$

is called the *space of k-jets* at the point a. The equivalence class

$$[f]_a^k = \left\{g \in \mathcal{F} \mid f - g \in \mu_a^{k+1}\right\}$$

is called the *k-jet* of f at a.

$$[f]_a^k = \left\{ g \in \mathcal{F} \,|\, f - g \in \mu_a^{k+1} \right\}.$$

We can write:

$$[f]_a^k = f + \mu_a^{k+1}.$$

A vector space structure on J_a^k:

$$[f]_a^k + [g]_a^k := [f+g]_a^k, \qquad \lambda[f]_a^k := [\lambda f]_a^k.$$

The 1-jet $[f - f(a)]_a^1$ is called the *differential* of f at the point a.

The k-jet $[f]_a^k$ is called the *Taylor series segment* of f at the point a.
These definitions are coordinate-free.

Coordinate Notation

Let $x_1, \ldots, x_n$ be coordinates on M. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a multi-index, $|\sigma| = \sigma_1 + \cdots + \sigma_n$, and

$$\frac{\partial^{|\sigma|} f}{\partial x^\sigma} := \frac{\partial^{|\sigma|} f}{\partial x_1^{\sigma_1} \cdots \partial x_n^{\sigma_n}}.$$

Two functions f and g generate the same k-jet iff their values and all partial derivatives up to order k coincide at a:

$$f(a) = g(a), \qquad \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) = \frac{\partial^{|\sigma|} g}{\partial x^\sigma}(a), \quad 1 \leqslant |\sigma| \leqslant k.$$

Example: $n = 1$ and $k = 1$. Let $[f]_a^1 = [g]_a^1$. Then

$$g = f + \sum h_1 h_2,$$

where the function $h_1, h_2 \in \mu_a$.

$$g' = f' + \sum h_1' h_2 + h_1 h_2'$$

$$\begin{aligned} g'(a) =& f'(a) + \sum h_1'(a) h_2(a) + h_1(a) h_2'(a) = \\ & f'(a) + \sum h_1'(a) \cdot 0 + 0 \cdot h_2'(a) = f'(a). \end{aligned}$$

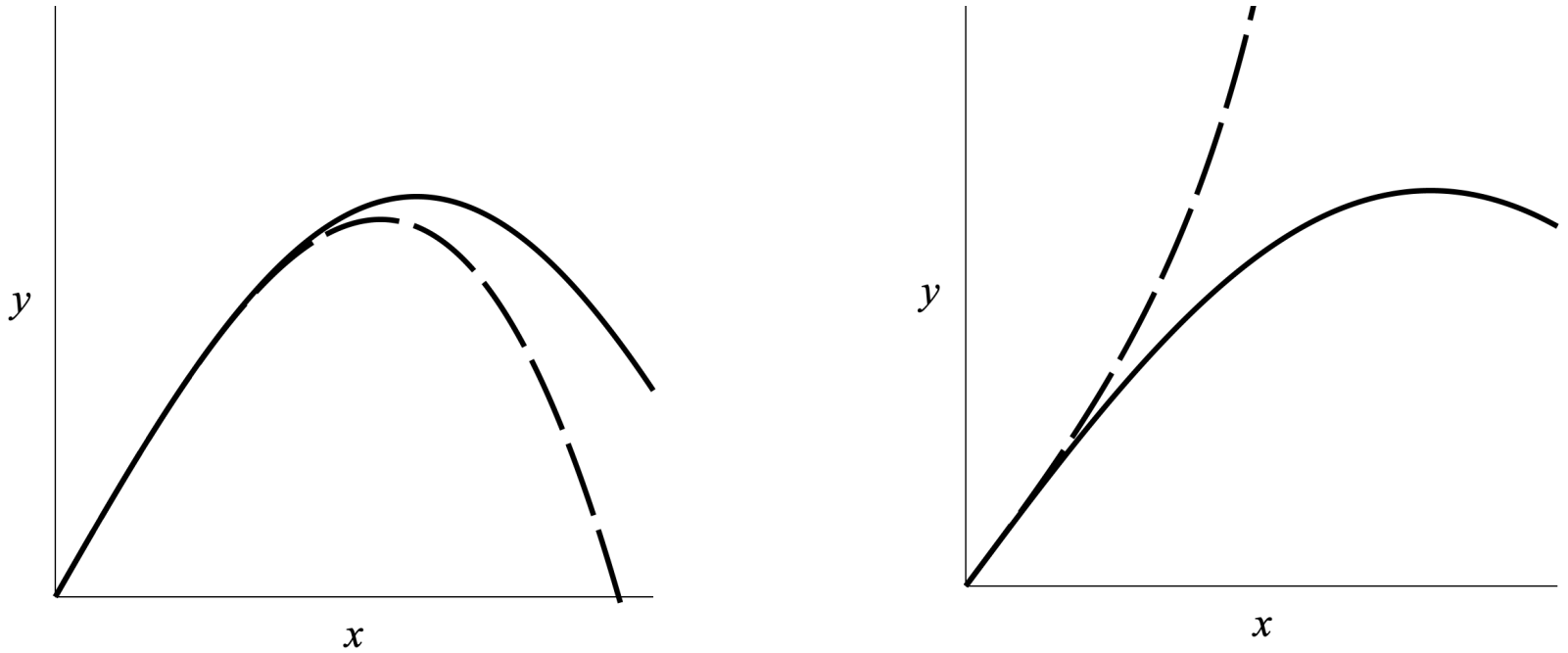


Figure: The functions $\sin x$ and $x - \frac{1}{6}x^3$ generate the same 4-jet at the point $x = 0$. The functions $\sin x$ and $\tan x$ generate the same 2-jet at the point $x = 0$.

Taylor's formula:

$$f(x) = f(a) + \sum_{|\sigma| \leqslant k} \frac{1}{\sigma!} \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a)(x-a)^\sigma + o\big((x-a)^\sigma\big).$$

Since $[(x-a)^\sigma]_a^{|\sigma|} = [0]_a^{|\sigma|}$ when $|\sigma| > k$,

$$\boxed{[f]_a^k = f(a)[1]_a^k + \sum_{|\sigma| \leqslant k} \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) \frac{[(x-a)^\sigma]_a^k}{\sigma!}}$$

Basis of $J_a^k(M)$:

$$[1]_a^k, \quad [x-a]_a^k, \quad \frac{1}{\sigma!}[(x-a)^\sigma]_a^k \quad (|\sigma| \leqslant k)$$

The numbers

$$f(a), \qquad \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) \quad (1 \leqslant |\sigma| \leqslant k)$$

are the coordinates of $[f]_a^k$ in the Taylor basis.

Invariant definition of the differential of a function

$$d_a f = \left.\frac{\partial f}{\partial x_1}\right|_a d_a x_1 + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_a d_a x_n.$$

On the other hand,

$$\boxed{[f - f(a)]_a^1 = \left.\frac{\partial f}{\partial x_1}\right|_a [x_1 - a_1]_a^1 + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_a [x_n - a_n]_a^1}$$

$$d_a x_i = [x_i - x_i(a)]_a^1$$

The formulas coincide!

Jet Bundle

Each point $a \in M$ is associated with $J^k_a(M)$. Define

$$J^k(M) := \bigcup_{a \in M} \{a\} \times J^k_a(M).$$

Coordinates on this set:

$$x_1, \ldots, x_n, \;\; u, \;\; p_\sigma \quad (1 \leqslant |\sigma| \leqslant k),$$

where for $\theta = [f]^k_a$

$$x_i(\theta) := a_i, \qquad u(\theta) := f(a), \qquad p_\sigma(\theta) := \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a).$$

These coordinates define a smooth manifold structure on $J^k(M)$:

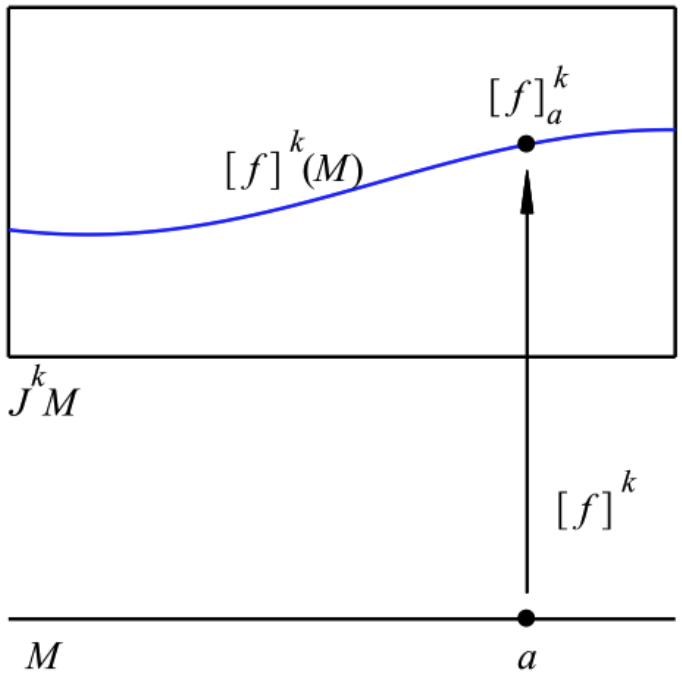
$$\dim J^k(M) = n + \binom{n+k}{k}.$$

The vector bundle $\pi : J^k(M) \to M$ is called the *k-jet bundle*.

$$J^k(M) \xrightarrow{\pi_{k,k-1}} J^{k-1}(M) \xrightarrow{\pi_{k-1,k-2}} \cdots \xrightarrow{\pi_{1,0}} J^0(M) \xrightarrow{\pi_{0,0}} M,$$

$$[f]^k_a \xmapsto{\pi_{k,k-1}} [f]^{k-1}_a \xmapsto{\pi_{k-1,k-2}} \cdots \xmapsto{\pi_{1,0}} [f]^0_a \xmapsto{\pi_{0,0}} a.$$

Any function f defines a section $[f]^k : M \to J^k(M), \quad a \mapsto (a, [f]^k_a)$.
Its image is the *k-graph* of f: $\Gamma^k_f := [f]^k(M)$



Cartan's Distribution

A distribution on a smooth manifold N:

$$P : N \ni \theta \mapsto P(\theta) \subset T_\theta N.$$

Methods of specification:

1. $P(\theta) = \operatorname{Span}(X_1|_\theta, \dots, X_k|_\theta)$,
2. $P(\theta) = \ker \omega_i|_\theta \cap \dots \cap \ker \omega_{m-k}|_\theta \quad (m = \dim N)$

$$\ker \omega_\theta := \{X_\theta \in T_\theta N \,|\, \omega_\theta(X_\theta) = 0\}\}$$

A submanifold $L \subset M$ is *integral* if $T_\theta L \subset P(\theta)$ for all $\theta \in L$.

The Cartan distribution $\mathcal{C}$ is a distribution on $J^k(\mathbb{R}^n)$ such that

1. the k-graph Γ_f^k of any function is an integral manifold;
2. $\dim \mathcal{C} = n + \binom{n+k-1}{k}$.

This definition is again coordinate-free.

In coordinates, $\mathcal{C}$ is defined by the 1-forms

$$\omega_\sigma = dp_\sigma - \sum_{i=1}^{n} p_{\sigma+1_i}\, dx_i, \qquad |\sigma| = 0, \dots, k-1.$$

Here $p_{0,\dots,0} = u$ and $\sigma + 1_i$ is obtained from σ by adding 1 to the i-th component.

Examples

1. $n = 1; k = 1$: $\omega_0 = du - pdx$
2. $n = 2; k = 1$: $\omega_{00} = du - p_{10}dx_1 - p_{01}dx_2$
3. $n = 2; k = 2$:

$$\begin{aligned}
\omega_{00} &= du - p_{10}dx_1 - p_{01}dx_2 \\
\omega_{10} &= dp_{10} - p_{20}dx_1 - p_{11}dx_2 \\
\omega_{01} &= dp_{01} - p_{11}dx_1 - p_{02}dx_2
\end{aligned}$$

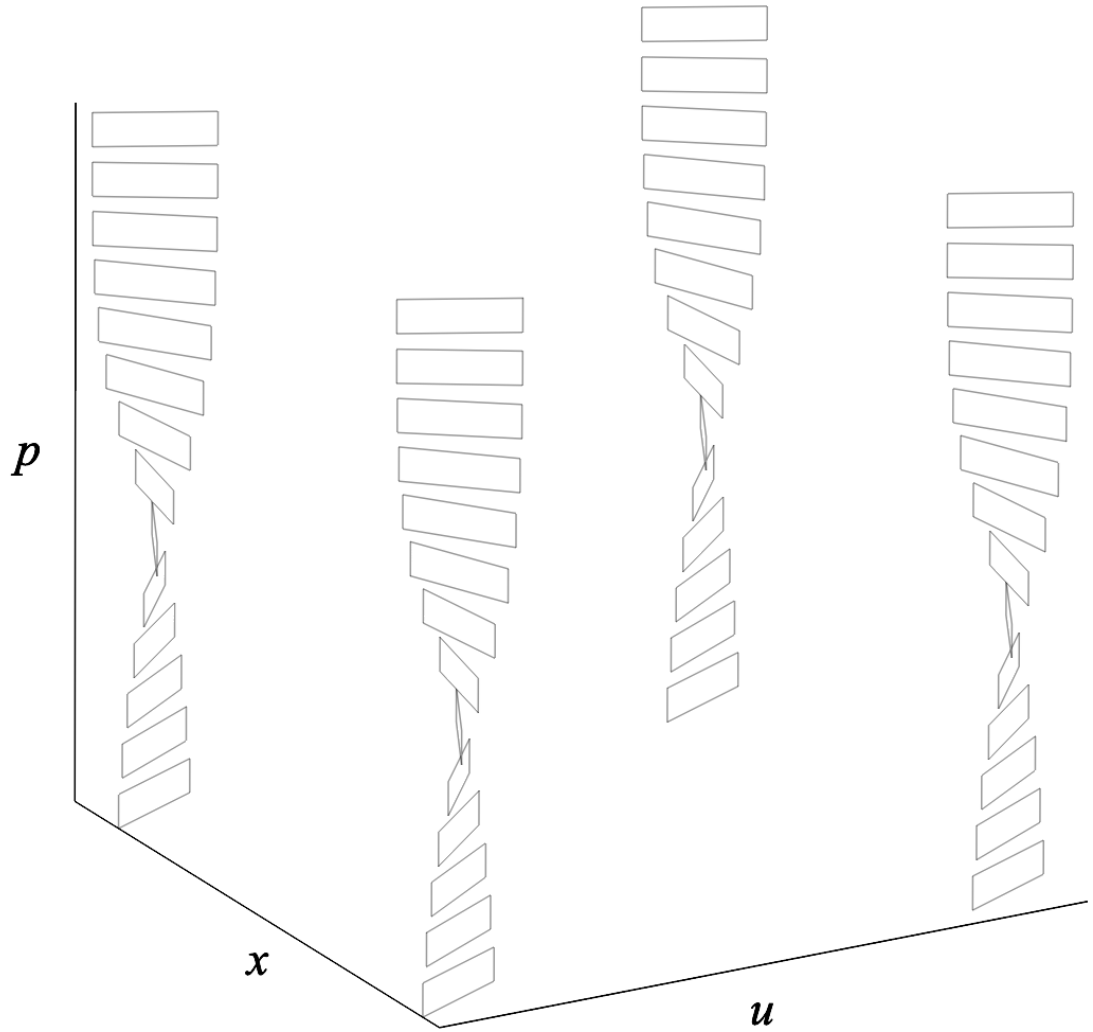


Figure: The Cartan distribution in $J^1(\mathbb{R})$

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Differential Equations: Invariant Definition



Figure: Alexander Vinogradov

A *differential equation* of k-th order with n independent variables is a submanifold $\mathcal{E} \subset J^k(\mathbb{R}^n)$.

$$\mathcal{E} = \begin{cases} F_1(x, u, p_\sigma) = 0, \\ \dots\dots\dots\dots\dots \\ F_m(x, u, p_\sigma) = 0, \end{cases} \qquad |\sigma| \leqslant k.$$

Restricting to Γ^k_v gives the equation in standard notation:

$$\begin{cases} F_1\left(x, v, \dfrac{\partial^{|\sigma|} v}{\partial x^\sigma}\right) = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ F_m\left(x, v, \dfrac{\partial^{|\sigma|} v}{\partial x^\sigma}\right) = 0 \end{cases}$$

If $m > 1$ the equation is *overdetermined*. We suppose $m = 1$.

A function $v = v(x)$ is a classical solution of the equation $F = 0$ iff its k-graph lies on the hypersurface $\mathcal{E} = \{F(x, u, p_\sigma) = 0\}$.

An n-dimensional submanifold L is a *multivalued solution* of $F(x, u, p_\sigma) = 0$ if

1. $L \subset \mathcal{E}$;
2. L is an integral manifold of the Cartan distribution.

Examples of Multivalued Solutions

Example 1. The parametric curve

$$L = \left\{ x = t^2,\ u = \frac{2}{3}t^3,\ p = t \right\}$$

is a multivalued solution of the Chibrario's equation $(y')^2 = x$.

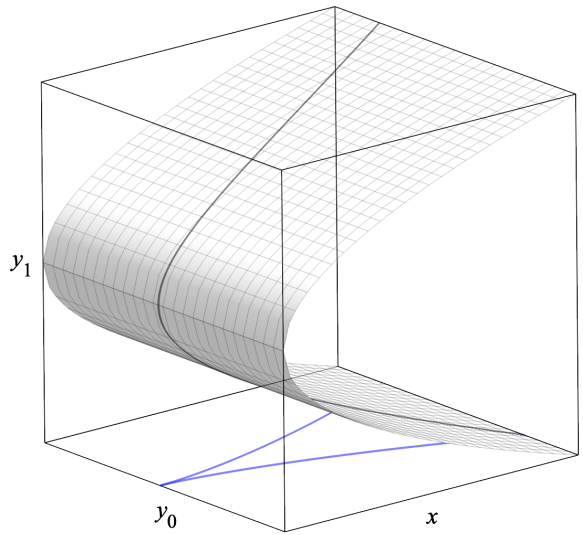


Figure: Chibrario's equation and its multi-valued solution.

Example 2.Euler–Hopf equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0.$$

The projection of a multivalued solution into J^0 is given implicitly by

$$V(x - ut) - u = 0.$$

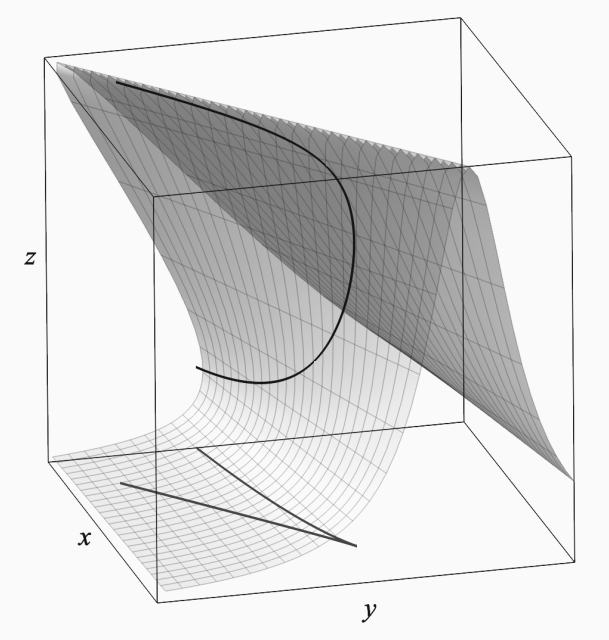


Figure: Projection to J^0 of the multi-valued solution to the Euler–Hopf equation.

Khokhlov–Zabolotskaya Equation

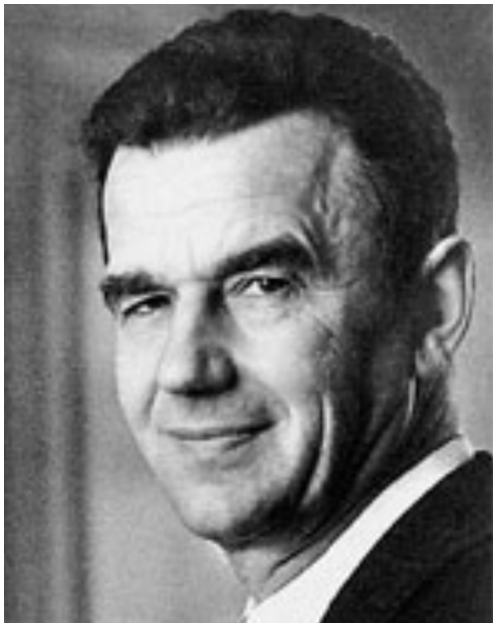


Figure: Rem Khokhlov

The Khokhlov–Zabolotskaya (Kh-Z) equation describes the propagation of a sound beam in 3D nonlinear media:

$$\frac{\partial^2 u}{\partial t \partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

The Symbol of the Kh-Z Equation

Let $\theta_2 \in J^2$ and $\theta_1 = \pi_{2,1}(\theta_2)$. The fiber

$$F_{\theta_1} = (\pi_{2,1})^{-1}(\theta_1)$$

consists of all 2-jets with a fixed 1-jet.

In coordinates, u_t, u_x, u_y, u_z are fixed, while $u_{tt}, u_{tx}, \ldots, u_{zz}$ are free, subject to

$$u_{tx} = u_{xt}, \quad u_{ty} = u_{yt}, \quad \ldots$$

Only the symmetric combinations

$$u_{(ij)} = \frac{1}{2}(u_{ij} + u_{ji})$$

are independent; the antisymmetric part $u_{[ij]}$ vanishes on any 2-graph.

Let $p = \pi_2(\theta_2) \in \mathbb{R}^4$. Then

$$F_{\theta_1} \cong T^*_p(\mathbb{R}^4) \odot T^*_p(\mathbb{R}^4), \quad \iota : u_{ij} \mapsto \partial_i \odot \partial_j.$$

Coordinate notation:

$$t = x_1,\ x = x_2,\ y = x_3,\ z = x_4, \quad u_t = p_{1000},\ u_{tx} = p_{1100},\ \ldots$$

Define on $J^2(\mathbb{R}^4)$ the function

$$F := u_{tx} - uu_{xx} - u_{yy} - u_{xx} - u_x^2.$$

$$\iota(F) = \partial_t \odot \partial_x - u\,\partial_x \odot \partial_x - \partial_y \odot \partial_y - \partial_z \odot \partial_z.$$

This tensor is non-degenerate and induces an isomorphism $T^*_p(\mathbb{R}^4) \to T_p(\mathbb{R}^4)$. The inverse isomorphism is defined the Lorentz metric:

$$g(\theta_2) = 4udt^2 - 4dtdx - dy^2 - dz^2 \in T^*_p(\mathbb{R}^4) \odot T^*_p(\mathbb{R}^4).$$

The Lorentz Metric on Solutions

We obtain the Lorentz metric:

$$g : J^2(\mathbb{R}^4) \to T^*_p(\mathbb{R}^4) \odot T^*_p(\mathbb{R}^4), \qquad \theta_2 \mapsto g(\theta_2).$$

Restriction to a solution

If $u = U(t, x, y, z)$ solves the Kh-Z equation and Γ^2_U is its 2-graph, then

$$g|_{\Gamma^2_U} = 4U(t, x, y, z)\, dt^2 + 4\, dt\, dx - du^2 - dz^2$$

is a Lorentz metric on Γ^2_U associated with the solution.

Locally Flat Solutions

A metric is locally flat iff its curvature tensor vanishes.

Locally flat solutions of the Kh-Z equation are those for which $g|_{\Gamma^2_U}$ has vanishing Riemann curvature tensor with respect to the Levi–Civita connection.

The conditions

$$u_{xx} = u_{xy} = u_{xz} = u_{yy} = u_{yz} = u_{zz} = 0,$$

combined with the Kh-Z equation, yield

$$u = h_1(t)yz - \frac{x}{t} + C + h_2(t)y + h_3(t)z + h_4(t).$$

Ricci-Flat Solutions

For Ricci-flat solutions the Ricci tensor of $g|_{\Gamma^2_U}$ is zero:

$$u_{xx} = 0, \quad u_{xy} = 0, \quad u_{xz} = 0, \quad u_{yy} + u_{zz} = 0.$$

The corresponding Ricci-flat solutions are

$$u = -\frac{x}{t + C} + h(t, x, z),$$

where h solves the Laplace equation

$$h_{yy} + h_{zz} = 0.$$

The described method for constructing a metric from a symbol was proposed by Valentin Lychagin and Valeriy Yumaguzhin.

Open Problems

1. The constructed Lorentz metric can be used to define a Gromov–Hausdorff distance between solutions of the Kh-Z equation. For this goal we need a Lorentz analog of the Gromov–Hausdorff distance.
2. We have seen that the Lorentz metric allows one to obtain exact solutions to a nonlinear differential equation. It will be interesting to repeat this algorithm for an elliptical model. Such equations generate a Riemannian metric. We need that this metric depends on solutions to the equation.

Thank you for your attention!