Geometric Structures and Second-Order Differential Equations: Results and Open Problems

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1. Intro

In this lecture, I will discuss geometric structures that arise from differential equations. In particular, I will discuss metric structures. I believe this approach can be used to extend Gromov-Hausdorff theory to differential equations. But geometric structures have to live somewhere. As a rule, these are smooth manifolds. Differential equations are no exception. They also live on manifolds. These are so called jet spaces. Their elements are jets of smooth functions. I will show that differential equations of order k can be considered as submanifolds in the space of k-jets.

In other words, jets form the basis for the geometry of differential equations. They make it possible to formalize the concept of derivatives independently of any specific coordinate system.

By the way, in ancient Egyptian mythology, Jet is the supreme primordial serpent, the father and creator of all that exists. Here you can see the hieroglyph representing this serpent:



2. Jets

2.1. Invariant definitions

The concept of a jet was introduced by Charles Ehresmann in the middle of the last century [1]. He was one of the members of the Nicolas Bourbaki group.



Figure 2.1: Charles Eresmann

Let $\mathcal{F} = C^\infty(M)$ be the ring of smooth functions on an n-dimensional smooth manifold M. Let a be a fixed point of M and let $\mu_a \subset \mathcal{F}$ be the subset such that all its function are vanishing at the point $a \in M$.

If we take one function f from μ_a and other one g from $\mathcal{F}$ then their product fg lies in μ_a. Indeed, $f(a)g(a) = 0 \cdot g(a) = 0$. This means that μ_a is an ideal of the ring $\mathcal{F}$.

We now form the second degree of the ideal as all possible finite sums of products of two functions from μ_a:

$$\mu_a^2 = \left\{ \sum f_1 f_2 \,|\, f_1, f_2 \in \mu_a \right\}.$$

Similarly we can define degrees of μ_a:

$$\mu_a^{k+1} = \left\{ \sum f_1 \dots f_{k+1} \,|\, f_1 \dots f_{k+1} \in \mu_a \right\}.$$

We obtain the following sequence of ideals:

$$\mathcal{F} \supset \mu_a \supset \mu_a^2 \supset \dots \supset \mu_a^k \supset \dots .$$

Definition The quotient ring

$$J_a^k(M) := \mathcal{F}/\mu_a^{k+1}$$

is called the *space of k-jets at the point a*.

Let $f \in \mathcal{F}$. The corresponding element of the quotient ring we denote by $[f]^k_a$. This class is called the k-jet of the function f at the point a.

As a set, it consists of functions which differ from f by a function of μ_a^{k+1}:

$$[f]^k_a = \left\{ g \in \mathcal{F} \,|\, f - g \in \mu_a^{k+1} \right\}.$$

So, we can write:

$$[f]^k_a = f + \mu_a^{k+1}.$$

Let us introduce a structure of a vector space on J^k_a. We define the sum of two jets and the product of a jet by a real number:

$$[f]^k_a + [g]^k_a := [f+g]^k_a, \quad \lambda[f]^k_a := [\lambda f]^k_a, \tag{2.1}$$

Now we can define differentials of functions without any coordinates.

Definition The 1-jet $[f - f(a)]^1_a$ is called the *differential* of f at the point a.

Similarly, one can propose the following definition of Taylor's series segments that is independent of the choice of coordinates.

Definition The k-jet $[f]^k_a$ is called the *Taylor series segment* of f at the point a.

These definitions likely seem strange. But if we introduce coordinates, everything becomes obvious.

2.2. Coordinate notations

Let $x_1, \ldots, x_n$ be coordinates on M. Two functions f and g generate the same k-jet if their values and values of all partial derivatives up to order k coincide at the point a:

$$f(a) = g(a), \quad \frac{\partial^{|\sigma|} f}{\partial x^\sigma}(a) = \frac{\partial^{|\sigma|} g}{\partial x^\sigma}(a), \qquad 1 \le |\sigma| \le k$$

Here $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multy-index, $|\sigma| := \sigma_1 + \cdots + \sigma_n$ is its weight,

$$\frac{\partial^{|\sigma|} f}{\partial x^\sigma} := \frac{\partial^{|\sigma|} f}{\partial x_1^{\sigma_1} \ldots \partial x_n^{\sigma_n}}.$$

Indeed, let us consider, for example, the case when $n = k = 1$. Suppose that the 1-jets of the functions f and g coincide:

$$[f]^1_a = [g]^1_a.$$

Then

$$g = f + \sum h_1 h_2,$$

where $h_1, h_2 \in \mu_a$. Let's calculate the derivative of g:

$$g' = f' + \sum (h_1' h_2 + h_1 h_2').$$

At the point a we have:

$$g'(a) = f'(a) + \sum (h_1'(a) h_2(a) + h_1(a) h_2'(a)) = f'(a) + \sum (h_1'(a) \cdot 0 + 0 \cdot h_2'(a)) = f'(a).$$

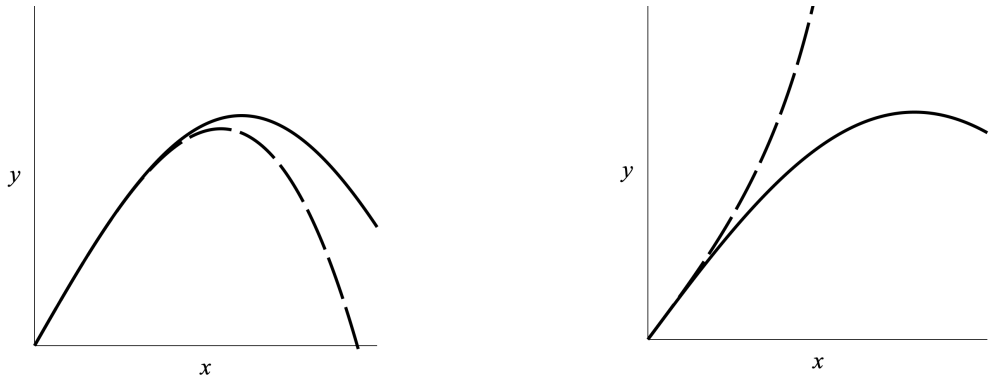


Figure 2.2: The functions $\sin x$ and $x - \frac{1}{6}x^3$ generate the same 4-jet at the point $x = 0$. The functions $\sin x$ and $\tan x$ generate the same 2-jet at the point $x = 0$.

For a smooth function f, we can write the Taylor serie:

$$f(x) = f(a) + \sum_{|\sigma| \le k} \frac{1}{\sigma!} \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a)(x-a)^\sigma + o((x-a)^\sigma).$$

Here $o((x-a)^\sigma)$ is an infinitesimal of order higher than k: $|\sigma| > k$. Of course, we do not require a converge of the serie.

Note that the function $(x-a)^\sigma$ is equivalent to the zero function if $|\sigma| > k$. Therefore,

$$[o((x-a)^\sigma]^k = [0]_a^k.$$

Using (2.1) we can write the corresponding k-jet:

$$[f]_a^k = f(a)[1]_a^k + \sum_{|\sigma| \le k} \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a) \frac{[(x-a)^\sigma]_a^k}{\sigma!}.$$

So, the k-jet of the function f can be viewed as a segment of its Taylor series.

From the last formula it follos that the jets

$$[1]_a^k, \quad [x-a]_a^k, \ldots, \frac{1}{\sigma!}[(x-a)^\sigma]_a^k \quad (|\sigma| \le k)$$

form a basis of the vector space $J_a^k(M)$. The numbers

$$f(a), \quad \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a) \quad (1 \le |\sigma| \le k)$$

are the *coordinates* of the jet $[f]_a^k$ in this basis.

What about differential? We know that

$$d_a f = \left.\frac{\partial f}{\partial x_1}\right|_a d_a x_1 + \cdots \left.\frac{\partial f}{\partial x_n}\right|_a d_a x_n.$$

On the other hand

$$[f - f(a)]_a^1 = \left.\frac{\partial f}{\partial x_1}\right|_a [x_1 - a_1]_a^1 + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_a [x_n - a_n]_a^1.$$

Since $d_a x_i = [x_i - x_i(a)]_a^1$, we see that the formulas coincide.

3. Jet Bundle

We see that each point $a \in M$ is associated with the space J^k_a. Let's introduce the set

$$J^k(M) := \bigcup_{a \in M} \{a\} \times J^k_a(M).$$

On this set, we define coordinates $x_1, \ldots, x_n, u, p_\sigma \quad (1 \leq |\sigma| \leq k)$. For $\theta = [f]^k_a$ we suppose:

$$x_i(\theta) := a_i, \quad u(\theta) := f(a), \quad p_\sigma(\theta) := \frac{\partial f^{|\sigma|}}{\partial x^\sigma}(a).$$

These coordinates define a structure of a smooth manifold on $J^k(M)$.

$$\dim J^k(M) = n + \binom{n+k}{k}$$

Here the second term is the binomial coefficient, that is the number of independent partial derivatives of order from 0 to k.

Since the derivatives of functions transform linearly when changing coordinates on a manifold, we obtain the vector bundle

$$\pi : J^k(M) \to M.$$

It is called *k-jet bundle*. We obtain the following sequence of projections:

$$J^k(M) \overset{\pi_{k,k-1}}{\longrightarrow} J^{k-1}(M) \overset{\pi_{k-1,k-2}}{\longrightarrow} \cdots \overset{\pi_{1,0}}{\longrightarrow} J^0(M) \overset{\pi_{0,0}}{\longrightarrow} M,$$

where

$$[f]^k_a \overset{\pi_{k,k-1}}{\longmapsto} [f]^{k-1}_a \overset{\pi_{k-1,k-2}}{\longmapsto} \cdots \overset{\pi_{1,0}}{\longmapsto} [f]^0_a \overset{\pi_{0,0}}{\longmapsto} a.$$

We see that $\pi : J^k(M) \to M$ is the composition of these projections:

$$\pi = \pi_{0,0} \circ \pi_{1,0} \circ \cdots \circ \pi_{k,k-1}.$$

Definition Any function f defines a section of this bundle by the following formula:

$$[f]^k \colon M \to J^k(M), \quad a \mapsto (a, [f]^k_a).$$

This section is called the *k-jet of the function f*. The image of M is called the *k-graph of f*:

$$\Gamma^k_f := [f]^k(M).$$

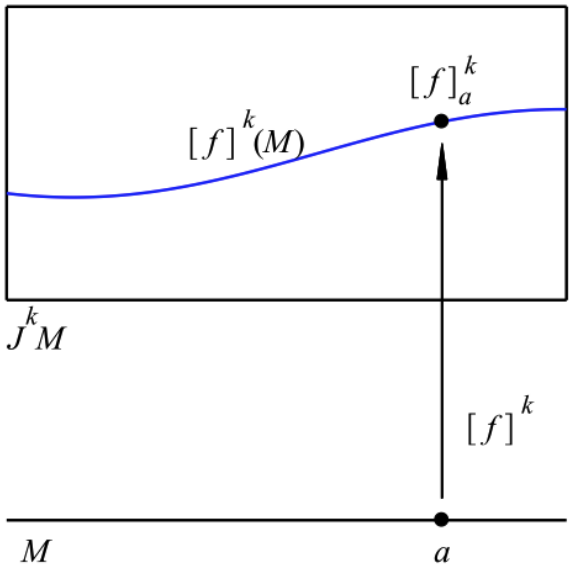


Figure 3.1: The k-graph of f

4. The Cartan Distribution

Recall that a distribution P on a smooth manifold N is a rule that assigns a k-dimensional subspace $P(\theta)$ of the tangent space to each point of the space:

$$P : N \ni \theta \mapsto P(\theta) \subset T_\theta N.$$

The number k is called the dimension of the distribution.

Distributions can be specified either by a set of k vector fields $X_1, \ldots, X_k$:

$$P(\theta) = \operatorname{Span}(X_1|_\theta, \ldots, X_k|_\theta),$$

or by a set of $m - k$ differential 1-forms $\omega_1, \ldots, \omega_{m-k}$, $m = \dim N$:

$$P(\theta) = \ker \omega_i|_\theta \cap \cdots \cap \ker \omega_{m-k}|_\theta.$$

Here

$$\ker \omega|_\theta := \{X_\theta \in T_\theta N \mid \omega|_\theta(X_\theta) = 0\}$$

is the kernel of the covector $\omega|_\theta$. A submanifold $L \subset N$ is said to be *integral* if its tangent space $T_\theta L$ is a subspace of $P(\theta)$ at any point θ: $T_\theta L \subset P(\theta)$

Definition The *Cartan*[1] *distribution* $\mathcal{C}$ is a distribution on the k-jet space $J^k(M)$ if the following conditions hold:

1. the k-graph Γ_f^k of any function is an integral manifold;
2. $\dim \mathcal{C} = n + \binom{n+k-1}{k}$.

We didn't use coordinates again in this definition! In coordinates, this distribution is defined by a set of differential 1-forms

$$\omega_\sigma = dp_\sigma - \sum_{i=1}^{n} p_{\sigma+1_i} dx_i, \qquad |\sigma| = 0, \ldots, k-1.$$

Here $p_{(0,\ldots,0)} = u$. The multi-index $\sigma + 1_i$ is obtained by adding one to the component with number i of the multi-index σ:

$$\sigma + 1_i = (\sigma_i, \ldots, \sigma_{i-1}, \sigma_i + 1, \sigma_{i+1}, \ldots, \sigma_n).$$

Example

[1] It should be noted that neither Elie Cartan nor Henri Cartan is the author of this definition. That happens quite often.

1. $n = 1; k = 1$:

$$\omega_0 = du - pdx$$

2. $n = 2; k = 1$:

$$\omega_{00} = du - p_{10}dx_1 - p_{01}dx_2$$

3. $n = 2; k = 2$:

$$\begin{aligned}\omega_{00} =& du - p_{10}dx_1 - p_{01}dx_2\\ \omega_{10} =& dp_{10} - p_{20}dx_1 - p_{11}dx_2\\ \omega_{01} =& dp_{01} - p_{11}dx_1 - p_{02}dx_2\end{aligned}$$

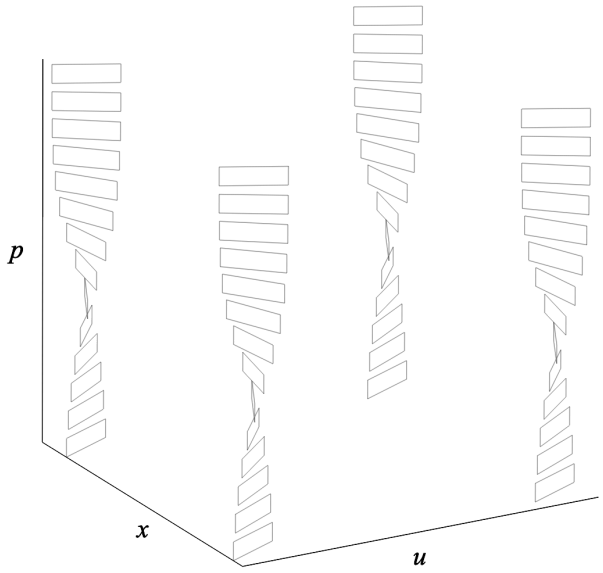


Figure 4.1: The Cartan distribution in $J^1(\mathbb{R})$

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5. Differential equations and multivalued solutions

Let's move on to differential equations. Jet theory is the natural language for describing differential equations. This was noted by Alexander Vinogradov in the second half of the last century [5]. He was a professor Moscow State University.



Figure 5.1: Alexander Vinogradov

Definition A differential equations of k-order with n independent variables is a submanifolds $\mathcal{E}$ of the space $J^k(\mathbb{R}^n)$.

Indeed, any submanifold of codimension m can be defined by common zero level of m smooth functions:

$$\mathcal{E} = \begin{cases} F_1(x, u, p_\sigma) = 0 \\ \dots\dots\dots\dots\dots \\ F_m(x, u, p_\sigma) = 0, \end{cases}$$

where $|\sigma| \leq k$. If we restrict all these functions to the k-graph Γ^k_v, we obtain a differential equation in the standard notation:

$$\begin{cases} F_1\left(x, v, \dfrac{\partial^{|\sigma|} v}{\partial x^\sigma}\right) = 0 \\ \dots\dots\dots\dots\dots \\ F_m\left(x, v, \dfrac{\partial^{|\sigma|} v}{\partial x^\sigma}\right) = 0 \end{cases}$$

If $m > 1$ then the equation is called *over defined*. We consider only the case $m = 1$.
Definition A function $v = v(x)$ is a (classical) solution of the equation

$$F\left(x, v, \frac{\partial^{|\sigma|} v}{\partial x^\sigma}\right) = 0$$

iff its k-graph lies on the hypersurface $\mathcal{E} = \{F(x, u, p_\sigma) = 0\}$. The following definition is a generalization of a notion of classical solution.

Definition An n-dimensional submanifold L is called a *multivalued solution* of equation $F(x, u, p_\sigma) = 0$ if

1. $L \subset \mathcal{E}$;
2. L is an integral manifold of the Cartan distribution.

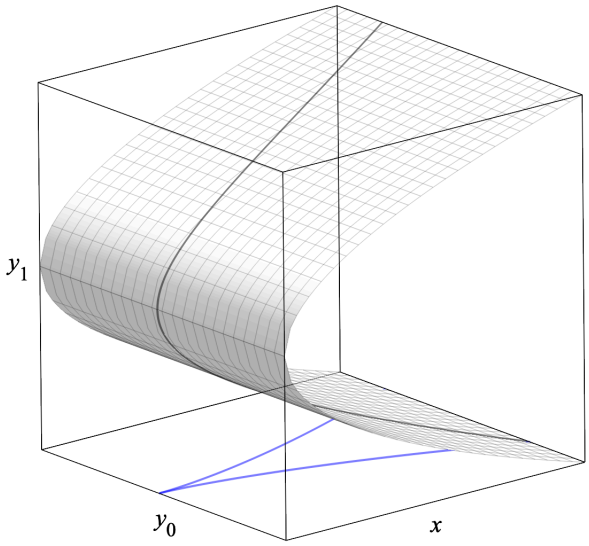


Figure 5.2: Chibrario's equation and its multi-valued solution.

Example Chibrario's equation. The parametric curve

$$L = \left\{x = t^2,\ u = \frac{2}{3}t^3,\ p = t\right\}$$

is a multi-valued solution of the Chibrario equation

$$(y')^2 = x.$$

Example Euler–Hopf equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0.$$

The projection of a multi-valued solution into the space of 0-jets in implicit form:

$$V(x - ut) - u = 0.$$

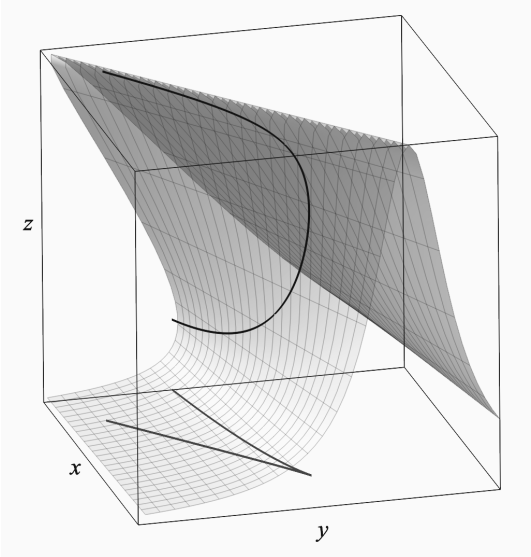


Figure 5.3: Projection to J^0 of the multi-valued solution to the Euler–Hopf equation.

The class of multi-valued solutions is broader than the class of classical solutions: every classical (i.e. one-valued) solution to the equation generates a multi-valued solution. But not every multi-valued solution is the graph of a function.

Remark Let $L \subset J^k(\mathbb{R}^n)$ be a multi-valued solution. If we remove singular points of the projection $L \to \mathbb{R}^n$, then L decomposes into the graphs of classical solutions.

6. The Khokhlov–Zabolotskaya equation and exact solutions

6.1. Kh-Z equation

The Khokhlov–Zabolotskaya equation describes the propagation of a sound beam in 3D nonlinear media:

$$\frac{\partial^2 u}{\partial t \partial x} - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

The beam propagates in x-direction. Here $u = u(t, x, y, z)$ is an acoustic pressure.

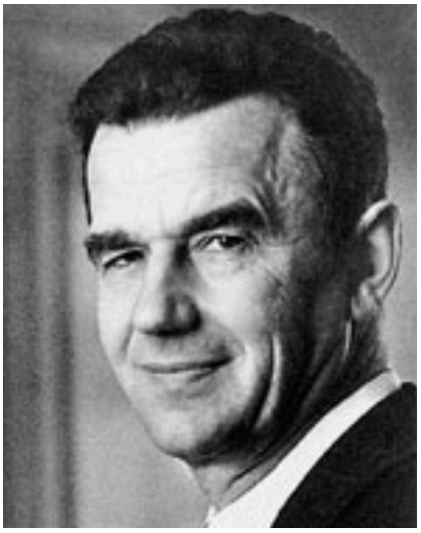


Figure 6.1: Rem Khokhlov

Rem Khokhlov was the rector of Moscow State University. He died in 1977 following a climb in the Pamir Mountains.

6.2. The symbol of Kh-Z equation

Let $\theta_2 \in J^2(\mathbb{R}^4)$ be an arbitrary point and $\theta_1 = \pi_{2,1}(\theta_2)$. Let

$$F_{\theta_1} = (\pi_{2,1})^{-1}(\theta_1)$$

be the fiber of the projection $\pi_{2,1}$ over the point θ_1 (see [4]).

Here, we use the following coordinate notations on the 2-jet space $J^2(\mathbb{R}^4)$:

$$t = x_1, x = x_2, y = x_3, z = x_4, u_t = p_{1000}, u_{tx} = p_{1100}, \dots$$

The fiber consists of all 2-jets with a fixed 1-jet. In coordinates, this means that u_t, u_x, u_y, u_z are fixed, while $u_{tt}, u_{tx}, \dots, u_{zz}$ are free. However, not all second derivatives are independent as coordinates on the fiber:

$$u_{tx} = u_{xt}, \quad u_{ty} = u_{yt}, \dots$$

Thus, among the coordinates u_{ij}, only the symmetric combinations

$$u_{(ij)} := \frac{1}{2}(u_{ij} + u_{ji})$$

are independent, The antisymmetric part

$$u_{[ij]} := \frac{1}{2}(u_{ij} - u_{ji})$$

is identically zero on any 2-graph.

Let $p = \pi_2(\theta_2)$ be a projection of θ_2 to $\mathbb{R}^4$. The fiber F_{θ_1} is isomorphic to the symmetric product $T^*_p(\mathbb{R}^4) \odot T^*_p(\mathbb{R}^4)$:

$$\iota : u_{ij} \mapsto \partial_i \odot \partial_j.$$

Let's construct the function F on $J^2(\mathbb{R}^4)$:

$$F := u_{tx} - uu_{xx} - u_{yy} - u_{xx} - u_x^2$$

Then

$$\iota(F) = \partial_t \odot \partial_x - u\partial_x \odot \partial_x - \partial_y \odot \partial_y - \partial_z \odot \partial_z.$$

This tensor is non-degenerate, and therefore it induces an isomorphism $T^*_p(\mathbb{R}^4) \to T_p(\mathbb{R}^4)$. The inverse isomorphism is defined a metric of signature $(+,-,-,-)$:

$$g(\theta_2) = 4udt^2 - 4dtdx - dy^2 - dz^2 \in T^*_p(\mathbb{R}^4) \odot T^*_p(\mathbb{R}^4).$$

6.3. The Lorentz metric on solutions

So, we get the Lorentz metrics on the 2-jet space $J^2(\mathbb{R}^4)$:

$$g : J^2(\mathbb{R}^4) \to T^*_p(\mathbb{R}^4) \odot T^*_p(\mathbb{R}^4), \qquad g : \theta_2 \mapsto g(\theta_2).$$

Let $u = U(t,x,y,x)$ be a solution of Kh-Z equation and Γ^2_U be its 2-graph. Then the restriction

$$g|_{\Gamma^2_U} = 4U(t,x,y,x)dt^2 + 4dtdx - du^2 - dz^2$$

gives a Lorentz metric on Γ^2_U, associated with the solution.

6.4. Exact solutions

Here we use constructed metric to find exact solutions to the Kh-Z equation.

6.4.1. Locally-flat solutions

A metric is locally flat if and only if its curvature tensor is zero.

Locally flat solutions of the Kh-Z equation are solutions for which the metric $g|_{\Gamma_U^2}$ has a vanishing Riemann curvature tensor with respect to the Levi–Civita connection (which is automatically torsion-free).

The condition $u_{xx} = u_{xy} = u_{xz} = u_{yy} = u_{yz} = u_{zz} = 0$, combined with the Kh-Z equation, yields:

$$u = h_1(t)yz - \frac{x}{t} + C + h_2(t)y + h_3(t)z + h_4(t).$$

6.4.2. Ricci-flat solutions

For Ricci–flat solutions the Ricci tensor of $g|_{\Gamma_U^2}$ is zero. This means that

$$u_{xx} = 0, u_{xy} = 0, u_{xz} = 0, u_{yy} + u_{zz} = 0.$$

Corresponding Ricci-flat solutions are

$$u = -\frac{x}{t + C} + h(t, x, z),$$

where h is a solution of the Laplace equation

$$h_{yy} + h_{zz} = 0.$$

7. Open problems

1. The constructed Lorenz metric can be used to define Gromov-Hausdorff distance between solutions of Kh-Z equation. For this goal we need construct Lorenz analog of Gromov-Hausdorff distance.
2. We have seen that the Lorentz metric allows one to obtain exact solutions to a nonlinear differential equation. It will be interesting to repeat this algorithm for an elliptical model. Such equations generate a Riemannian metric. We need that this metric depends on the solution spaces of differential equations.

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