Cheeger–Colding Theory

Lectures 4–5: Ricci limits and Einstein geometry

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Noncollapse and the geometry of a GH limit

Main Gaol: The Structure of the limit space X

$$(M_i^n, g_i, p_i) \xrightarrow{\text{pGH}} (X, d, p), \qquad \mathrm{Ric}_{g_i} \geq -(n-1)g_i, \qquad \mathrm{Vol}_{g_i} B_1(p_i) \geq v > 0.$$

A lower Ricci bound gives compactness. A volume lower bound prevents loss of dimension.

Example

$$M_\varepsilon = \mathbb{R}^{n-1} \times S^1(\varepsilon), \quad \mathrm{Ric} = 0, \quad \varepsilon \downarrow 0 \quad \Longrightarrow \quad M_\varepsilon \xrightarrow{\text{pGH}} \mathbb{R}^{n-1}, \quad \mathrm{Vol}\, B_1 \to 0.$$

- **Questions:** How do we study a signular space X? Local analysis? What are the tangent cones?
- **Main route:** Local regularity $\Rightarrow$ conical tangents $\Rightarrow$ structure of regular points and singular strata.
- **Einstein case:** almost Euclidean volume $\Rightarrow$ curvature bounds $\Rightarrow$ smooth local geometry.

Hausdorff measure and dimension

A metric measures the cost of covers, even when no coordinates are available.

Definition

For $E \subset (X, d)$ and $s > 0$, define

$$\mathcal{H}^s_\rho(E) = c_s \inf \left\{ \sum_j (\operatorname{diam} U_j)^s : E \subset \bigcup_j U_j, \quad \operatorname{diam} U_j < \rho \right\}, \qquad \mathcal{H}^s(E) = \lim_{\rho \downarrow 0} \mathcal{H}^s_\rho(E),$$

where covers are countable and $c_k = \omega_k / 2^k$ for integers $k \geq 1$. Set $\mathcal{H}^0(E) = \#E$ and

$$d_H := \dim_{\mathrm{H}} E = \inf\{s > 0 : \mathcal{H}^s(E) = 0\}.$$

$$s < d_H \Rightarrow \mathcal{H}^s(E) = \infty, \qquad s > d_H \Rightarrow \mathcal{H}^s(E) = 0.$$

$$\mathcal{H}^k|_{\mathbb{R}^k} = \mathcal{L}^k, \qquad \mathcal{H}^n|_{(M^n, g)} = \mathrm{Vol}_g, \qquad \mathcal{H}^s_{ad}(E) = a^s \mathcal{H}^s_d(E) \quad (a > 0).$$

At the critical exponent, the measure may be zero, finite positive, or infinite. Thus $\dim_{\mathrm{H}} E \leq k$ alone does not imply $\mathcal{H}^k(E) < \infty$.

Euclidean volume convergence

Theorem 4.1: Euclidean volume convergence

For every $\varepsilon > 0$, there exists $\delta = \delta(n, \varepsilon) > 0$. **Assume**

$$\mathrm{Ric} \geq -(n-1)\delta g, \qquad d_{\mathrm{GH}}\big(\overline{B}_2(p), \overline{B}_2(0^n)\big) < \delta$$
$$\Longrightarrow \qquad \big|\mathrm{Vol}\, B_1(p) - \omega_n\big| < \varepsilon.$$

Proof. $\Longrightarrow$ (n, ϵ)-splitting u and by a degree argument give

$$u = (u^1, \dots, u^n), \qquad J_u = \sqrt{\det\big(\langle \nabla u^a, \nabla u^b \rangle\big)} \leq 1 + \Psi, \qquad B_{1-\Psi}(0^n) \subset u(B_1(p)).$$

By Area formula:

$$\omega_n (1-\Psi)^n \leq \mathcal{L}^n\big(u(B_1(p))\big) \leq \int_{B_1(p)} J_u \, dV \leq (1+\Psi)\,\mathrm{Vol}\, B_1(p).$$

Bishop–Gromov: $\mathrm{Vol}\, B_1(p) \leq V^n_{-\delta}(1) = \omega_n + O_n(\delta)$.

$$\Longrightarrow \qquad \omega_n - \Psi(\delta \mid n) \leq \mathrm{Vol}\, B_1(p) \leq \omega_n + C(n)\delta.$$

Almost maximal volume

Theorem 4.2: almost maximal volume

For every $\varepsilon > 0$, there exists $\delta = \delta(n, \varepsilon) > 0$, uniformly for $r > 0$:

$$\mathrm{Ric} \geq -(n-1)\delta r^{-2} g, \qquad \mathrm{Vol}\, B_{2r}(p) \geq (1-\delta)\omega_n (2r)^n$$
$$\Longrightarrow \qquad d_{\mathrm{GH}}\big(\overline{B}_r(p), \overline{B}_r(0^n)\big) < \varepsilon r.$$

- Almost constant volume ratio gives a cone; almost maximal volume gives $\mathbb{R}^n$.
- Rescaling r to 1 makes the error relative to the observation scale.

Proof of Theorem 4.2: compactness

Rescale to $r = 1$ and argue by contradiction. Use the almost volume-cone theorem from Lectures 2–3

- If the assertion fails, choose (M_i^n, g_i, p_i) and $\delta_i \downarrow 0$ with
$$\mathrm{Ric}_{g_i} \geq -(n-1)\delta_i g_i, \quad \mathrm{Vol}\, B_2(p_i) \geq (1-\delta_i)\omega_n 2^n,$$
while $d_{GH}(\overline{B}_1(p_i), \overline{B}_1(0^n)) \geq \varepsilon_0 > 0$.
- Bishop–Gromov gives uniform noncollapse on $B_1(p_i)$; take a pointed GH limit (X, d, p). For every fixed $0 < s < 2$,
$$\mathrm{Vol}\, B_s(p_i) \geq \frac{V^n_{-\delta_i}(s)}{V^n_{-\delta_i}(2)} \mathrm{Vol}\, B_2(p_i) \longrightarrow \omega_n s^n.$$
- The preceding estimates and Bishop–Gromov imply, for each fixed $0 < s < 2$,
$$\frac{\mathrm{Vol}\, B_s(p_i)}{\mathrm{Vol}\, B_2(p_i)} \longrightarrow \left(\frac{s}{2}\right)^n.$$
Thus the volume ratios are asymptotically constant on every fixed annulus. The almost volume-cone-implies-almost metric-cone theorem, applied on the smooth manifolds before passing to the limit, shows that the GH limit is a metric cone $C(Z)$ with vertex o.

Proof of Theorem 4.2: every link point is a cone vertex

Let $B_2(p_i) \to B_2(o) \subset C(Z)$ be the limit obtained above.

- Fix $z \in Z$. Choose q_i with $d(p_i, q_i) = s_i \downarrow 0$ sufficiently slowly, so that
$$(B_{20s_i}(q_i), s_i^{-2} g_i) \longrightarrow B_{20}((1, z)) \subset C(Z).$$

- Since $B_1(p_i) \subset B_{1+s_i}(q_i)$ and $\operatorname{Vol} B_1(p_i) \to \omega_n$, the ball $B_{1+s_i}(q_i)$ has asymptotically maximal volume. Bishop–Gromov therefore gives
$$\frac{\operatorname{Vol} B_{20s_i}(q_i)}{V^n_{-\delta_i}(20s_i)} \longrightarrow 1.$$

- Applying Theorem 4.1 at the rescaled points shows that $B_2((1, z))$ is a metric cone with vertex $(1, z)$.
- Since $z \in Z$ was arbitrary, every point $(1, z)$ on the link sphere is also a cone vertex.

Proof of Theorem 4.2: splitting and dimension

- Every ray from the original cone vertex o passes through a point $(1, z)$, which is itself a cone vertex. Hence each such ray extends through o to a line.
- By the splitting theorem, the cone splits off a Euclidean factor. Repeating the argument gives
$$C(Z) \cong \mathbb{R}^k$$
for some $k \geq 1$.
- The noncollapsing and volume-doubling estimates imply $\dim_{\mathrm{H}} C(Z) = n$. Therefore $k = n$, so $C(Z) \cong \mathbb{R}^n$.
- In particular, $\overline{B}_1(p)$ is isometric to the Euclidean unit ball, contradicting the assumed GH separation.

$$\Longrightarrow \quad d_{GH}\left(\overline{B}_1(p_i), \overline{B}_1(0^n)\right) \to 0.$$

Noncollapsed volume convergence

The limit metric determines the limiting volume measure.

Theorem 4.3: noncollapsed volume convergence

Assume (NC):

$$(M_i^n, g_i, p_i) \xrightarrow{\mathrm{pGH}} (X, d, p), \qquad \mathrm{Ric}_{g_i} \geq -(n-1)g_i, \qquad \mathrm{Vol}_{g_i} B_1(p_i) \geq v > 0.$$

Then

$$dV_{g_i} \rightharpoonup \mu = \mathcal{H}^n_X, \qquad \dim_{\mathrm{H}} X = n,$$

$$x_i \to x, \quad r > 0 \quad \Longrightarrow \quad \mathrm{Vol}_{g_i} B_r(x_i) \longrightarrow \mu(B_r(x)).$$

The measure μ satisfies Bishop–Gromov comparison.

The measure and ball-volume convergence are quoted as the noncollapsed volume convergence theorem; below we prove only the dimension assertion. The notation **(NC)** is retained below.

Proof of the dimension assertion in Theorem 4.3

We prove only $\dim_{\mathrm{H}} X = n$. The measure-convergence statement in Theorem 4.3 is quoted and is not proved here.

- By Bishop–Gromov and $\operatorname{Vol} B_1(p_i) \geq v$, we have for any x_i and r,
$$c\, r^n \leq \operatorname{Vol}_{g_i} B_r(x_i) \leq C\, r^n,$$
uniformly for centers x_i in that region.
- Pass to a weak limit ν of the volume measures under GH approximations. The same bounds pass to ν:
$$c\, r^n \leq \nu(B_r(x)) \leq C\, r^n$$
- Choose any covering $\{B_{r_j}(y_j), r_j \leq \delta\}$ of $B_1(x)$, we have
$$c \leq \nu(B_1(x)) \leq \sum_j \nu(B_{r_j}(y_j)) \leq C \sum_j r_j^n. \tag{1}$$
This implies $\mathcal{H}^n_\delta(B_1(x)) \geq c > 0$ for any δ. In particular, $\mathcal{H}^n(B_1(x)) \geq c > 0$ and $\dim_H X \geq n$.

Proof of the dimension assertion in Theorem 4.3

- for any $\delta > 0$ and $0 < r < \delta$ we can choose a maximal disjoint balls $\{B_{r/2}(y_j) \subset B_2(x)\}$. In particular
$$B_1(x) \subset \cup_j B_r(y_j).$$

- We can compute that
$$C \geq \nu(B_2(x)) \geq \sum_j \nu(B_{r/2}(y_j)) \geq c \sum_j r^n.$$

- This implies for any $s > n$ that
$$\mathcal{H}^s_\delta(B_1(x)) \leq \sum_j r^s \leq r^{s-n} \sum_j r^n \leq C r^{s-n}.$$
Letting $r \to 0$, we get $\mathcal{H}^s_\delta(B_1(x)) = 0$. Thus $\mathcal{H}^s(B_1(x)) = 0$ and $\dim_H X \leq n$. Hence
$$\dim_H X = n.$$

Volume density at a point

Lemma

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. For $x \in B_L(p)$, set

$$\Phi_x(r) = \frac{\mu(B_r(x))}{V_{-1}^n(r)}.$$

Then Φ_x is nonincreasing and

$$\frac{v}{V_{-1}^n(L+2)} \leq \Phi_x(r) \leq 1 \quad (0 < r \leq 1), \qquad \theta(x) := \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_n r^n} \in (0, 1].$$

Proof.

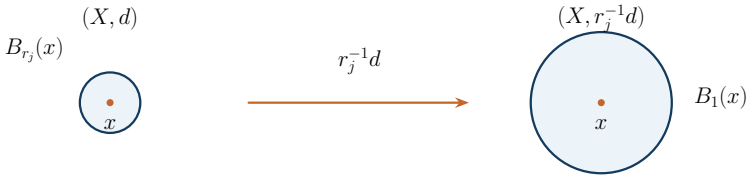
- $B_1(p) \subset B_{L+2}(x)$ and Bishop–Gromov give the lower bound. Smooth volume comparison and Theorem 4.3 give $\Phi_x \leq 1$.
- $V_{-1}^n(r)/(\omega_n r^n) \to 1$ and monotonicity give $\theta(x) = \lim_{r \downarrow 0} \Phi_x(r)$.

Tangent cones

Definition 4.4: tangent cones

A **tangent cone** at $x \in X$ is any pointed limit

$$r_j \downarrow 0, \qquad (X, r_j^{-1} d, x) \xrightarrow{\text{pGH}} (Y, d_Y, o).$$



Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1) g_i$ gives covering bounds and tangent subsequences. The definition itself asserts neither conical structure nor uniqueness.

Tangent cones are metric cones

At two fixed radii in a blow-up, the normalized volumes approach the same density.

Theorem 4.5: tangent cones are metric cones

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. Let

$$r_j \downarrow 0, \qquad (X, r_j^{-1} d, x) \xrightarrow{\mathrm{pGH}} (Y, o).$$

Then

$$(Y, o) \cong (C(Z), o), \qquad Z \text{ compact}, \qquad \operatorname{diam} Z \leq \pi,$$

Here $\cong$ denotes a pointed isometry.

Proof route. Density $\Rightarrow$ zero volume drop $\Rightarrow$ almost cone models on expanding balls $\Rightarrow$ one global cone.

Proof of Theorem 4.5: volume in the blow-up

- **Fixed radii:** for $0 < a < b < \infty$,
$$\frac{\mu(B_{ar_j}(x))}{\omega_n (ar_j)^n}, \quad \frac{\mu(B_{br_j}(x))}{\omega_n (br_j)^n} \to \theta(x) > 0 \quad \Rightarrow \quad \log \Phi_x(ar_j) - \log \Phi_x(br_j) \to 0.$$

- **Smooth diagonal:** choose $i(j)$ after fixing the scale and accuracy. With $h_j = r_j^{-2} g_{i(j)}$,
$$(M_{i(j)}, h_j, x_{i(j)}) \xrightarrow{\text{pGH}} (Y, o), \qquad \mathrm{Ric}_{h_j} \geq -(n-1) r_j^2 h_j,$$
By volume convergence,
$$\log \Phi_{x_{i(j)}}(a) - \log \Phi_{x_{i(j)}}(b) \to 0.$$

- **By Almost volume cone $\Rightarrow$ almost metric cone:** For any $m > 0$ and $j >> 1$ we have
$$d_{\mathrm{GH}}\big(\overline{B}_m(x_{i(j)}), \overline{B}_m^{C(Z_m)}(o_m)\big) < m^{-1}.$$
for some compact Z_m, $\operatorname{diam} Z_m \leq \pi$. This implies
$$d_{\mathrm{GH}}\big(\overline{B}_m^{Y}(o), \overline{B}_m^{C(Z_m)}(o_m)\big) < 2m^{-1}.$$

Proof of Theorem 4.5: compatible cone models

- **Compactness of the links:** on the unit cone sphere,
$$\frac{2}{\pi} d_{Z_m}(z,w) \le d_{C(Z_m)}((1,z),(1,w)) = 2\sin\frac{d_{Z_m}(z,w)}{2} \le d_{Z_m}(z,w).$$
Covering bounds give $Z_m \xrightarrow{\mathrm{GH}} Z$ along a subsequence.
- **Cone distance formula:** $Z_m \to Z \Rightarrow C(Z_m) \xrightarrow{\mathrm{pGH}} C(Z)$. So $(Y,o) \cong (C(Z),o)$.

Remark. Different sequences $r_j \downarrow 0$ may give nonisometric cones. The density fixes their ball volumes, not their links.

Nonuniqueness examples: Colding–Naber, GAFA (2013), arXiv:1108.3244.

Almost Reifenberg theorem and unique tangent

A Euclidean approximation at one scale gives Euclidean control at all smaller interior scales.

Theorem 4.6 (Cheeger–Colding)

For every $\varepsilon > 0$, there exists $\delta = \delta(n, \varepsilon) > 0$ such that, for $R > 0$,

$$\mathrm{Ric} \geq -(n-1)\delta R^{-2} g, \qquad d_{\mathrm{GH}}\big(\overline{B}_{2R}(p), \overline{B}_{2R}(0^n)\big) \leq \delta R$$
$$\Longrightarrow \qquad d_{\mathrm{GH}}\big(\overline{B}_s(q), \overline{B}_s(0^n)\big) < \varepsilon s \qquad \big(q \in B_R(p),\ 0 < s \leq R/2\big).$$

As a consequence we have

Proposition 4.7: one Euclidean tangent

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. For $x \in X$,

$$\text{one tangent at } x \text{ is } \mathbb{R}^n \quad \Longrightarrow \quad \text{every tangent at } x \text{ is } \mathbb{R}^n.$$

Proof of Theorem 4.6: volume at a fixed interior scale

Rescale to $R = 1$. Fix the threshold $\eta = \eta(n, \varepsilon/2) \in (0, 1)$ from Theorem 4.2 *before* choosing δ. Set $a = 1/4$.

- **Restrict the large-ball GH approximation.** For $q \in B_1(p)$, we have:
$$d_{\mathrm{GH}}\big(\overline{B}_t(q), \overline{B}_t(0^n)\big) \le C\delta \qquad (1/8 \le t \le 1/2).$$
This is an absolute error at fixed radii.
- **Use Theorem 4.1 at radius $a = 1/4$.** Hence, uniformly in $q \in B_1(p)$,
$$\mathrm{Vol}_g B_a(q) \ge \big(1 - \Psi(\delta \mid n)\big)\omega_n a^n, \qquad V^n_{-\delta}(a) = \big(1 + O_n(\delta)\big)\omega_n a^n.$$
In particular,
$$\frac{\mathrm{Vol}\, B_a(q)}{V^n_{-\delta}(a)} \ge 1 - \Psi(\delta \mid n) \ge 1 - \eta \qquad (q \in B_1(p)).$$
- **For $0 < s \le 1/8$:** $2s \le a = 1/4$, so Bishop–Gromov gives
$$\frac{\mathrm{Vol}\, B_{2s}(q)}{V^n_{-\delta}(2s)} \ge \frac{\mathrm{Vol}\, B_a(q)}{V^n_{-\delta}(a)} \ge 1 - \eta \quad \Longrightarrow \quad \mathrm{Vol}\, B_{2s}(q) \ge (1 - \eta)\omega_n (2s)^n.$$

$$\text{Theorem 4.2} \quad \Longrightarrow \quad d_{\mathrm{GH}}\big(\overline{B}_s(q), \overline{B}_s(0^n)\big) < (\varepsilon/2)s.$$

Regular points and nearby scales

Definition: Singular-Regular decomposition

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. Define

$$\mathcal{R} = \{x : \text{every tangent at } x \text{ is } \mathbb{R}^n\}, \qquad \mathcal{S} = X \setminus \mathcal{R}.$$

Consequence

For every $x \in \mathcal{R}$ and $\eta > 0$, some $r_0 = r_0(x, \eta) > 0$ satisfies

$$d_{\mathrm{GH}}\big(\overline{B}_s(y), \overline{B}_s(0^n)\big) < \eta s \qquad \big(y \in B_{2r_0}(x),\ 0 < s \leq r_0\big).$$

Question: What is the geometry and regularity of $\mathcal{R}$ and $\mathcal{S}$?

Regularity of regular set: Local bi-Hölder

Uniformly Euclidean small balls can be assembled into coordinates.

Theorem 4.8: local metric Reifenberg

For every $\alpha \in (0,1)$, there exists $\varepsilon = \varepsilon(n,\alpha) > 0$. Let Y be a proper length space. **Assume**

$$d_{\mathrm{GH}}\big(\overline{B}_s(y), \overline{B}_s(0^n)\big) < \varepsilon s \qquad \big(y \in B_{2r_0}(q),\ 0 < s \le r_0\big).$$

Then there are open sets $q \in U \subset Y$, $V \subset \mathbb{R}^n$ and a homeomorphism $F : U \to V$ such that, for $x, y \in U$,

$$C^{-1}\left(\frac{d(x,y)}{r_0}\right)^{1/\alpha} \le |F(x) - F(y)| \le C\left(\frac{d(x,y)}{r_0}\right)^{\alpha}, \qquad C = C(n,\alpha).$$

Thus, under (NC) and $\mathrm{Ric}_{g_i} \ge -(n-1)g_i$, each $x \in \mathcal{R}$ has a bi-Hölder neighborhood.

Cheeger–Colding (1997), Theorem 5.14 and Appendix A. Statement only.

Regularity of regular set: Local bi-Lipschitz

Bi-Lipschitz coordinates control distances linearly on an open neighborhood.

Theorem 4.9: local bi-Lipschitz

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. For every $x \in \mathcal{R}$, there exist open sets $x \in U \subset X$, $V \subset \mathbb{R}^n$, a homeomorphism $F : U \to V$, and a finite constant $L_x \geq 1$ such that

$$L_x^{-1} d(y,z) \leq |F(y) - F(z)| \leq L_x d(y,z) \qquad (y, z \in U).$$

Here d is the restricted ambient distance, and the neighborhood depends on x.

The chart domain may contain singular points; this does not assert that $\mathcal{R}$ is open.

OpenAI, *Bi-Lipschitz Coordinates at Regular Points of Noncollapsed RCD Spaces*, 2026 preprint. Ricci-limit consequence; statement only.

Singular set: Symmetry and the singular strata

Definition

A pointed cone is k-**symmetric** if

$$(Y, o_Y) \cong \big(\mathbb{R}^k \times C(W), (0, o)\big).$$

The remaining factor may be a point. Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$, the k-strata is defined as

$$\mathcal{S}^k = \{x \in X : \text{no tangent at } x \text{ is } (k+1)\text{-symmetric}\}.$$

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n-1} = \mathcal{S}.$$

Definition: Quantitative strata

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. For $\varepsilon > 0$, define

$$\mathcal{S}^k_\varepsilon = \left\{ x \in X : \begin{array}{l} d_{\mathrm{GH}}\big(\overline{B}_s(x), \overline{B}^C_s(o)\big) \geq \varepsilon s \\ \text{for every } 0 < s \leq 1 \text{ and every } (k+1)\text{-symmetric cone } (C, o) \end{array} \right\}.$$

Then $\mathcal{S}^k = \bigcup_{m=1}^{\infty} \mathcal{S}^k_{1/m}$.

Consequently, a dimension bound for each fixed $\mathcal{S}^k_\varepsilon$ gives the same bound for $\mathcal{S}^k$.

Proof of the quantitative-strata identity

The inclusion $\bigcup_m \mathcal{S}^k_{1/m} \subset \mathcal{S}^k$ follows directly from the definition.

- If $x \notin \bigcup_m \mathcal{S}^k_{1/m}$, choose $(k+1)$-symmetric cones C_j and $s_j \in (0,1]$ with
$$d_{\mathrm{GH}}\big(s_j^{-1}\overline{B}_{s_j}(x), \overline{B}_1^{C_j}(o_j)\big) < j^{-1}.$$
Cone-model compactness gives $s_j \to s_* \in [0,1]$, $C_j \to C$, with C still $(k+1)$-symmetric.
- **If** $s_* = 0$: take a tangent Y along these scales. Then
$$\overline{B}_1^{Y}(o) \cong \overline{B}_1^{C}(o_C), \quad Y \text{ is a cone} \quad \Longrightarrow \quad Y \cong C.$$
- **If** $s_* > 0$: $\overline{B}_{s_*}(x) \cong \overline{B}_{s_*}^{C}(o_C)$, hence $(X, t^{-1}d, x) \xrightarrow[t\downarrow 0]{\mathrm{pGH}} C$.

In either case, a tangent at x is $(k+1)$-symmetric, so $x \notin \mathcal{S}^k$. □

We will show $\dim \mathcal{S}^k \le k$ below.

An additional Euclidean direction

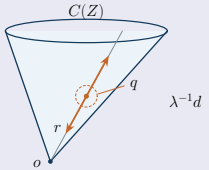
Away from a cone vertex, the inward and outward radial directions become a line.

Lemma 4.10: an additional Euclidean direction

Let $Y = \mathbb{R}^\ell \times C(Z) = \mathrm{pGH}\text{-}\lim_i (N_i, h_i, q_i)$, with $\mathrm{Ric}_{h_i} \geq -(n-1)h_i$. Then

$$q \notin \mathbb{R}^\ell \times \{o\} \quad \Longrightarrow$$

every tangent at q splits off $\mathbb{R}^{\ell+1}$.



Proof. Fix $q = (a, (r, z))$, $r > 0$, and any tangent scales $\lambda_j \downarrow 0$.

- On bounded rescaled balls, $|b| + |c| = O(1)$ and $d_Z(u, v) = O(\lambda_j)$:

$$\begin{aligned} \lambda_j^{-2} d_C^2\big((r + \lambda_j b, u), (r + \lambda_j c, v)\big) &= (b - c)^2 + 2(r + \lambda_j b)(r + \lambda_j c)\lambda_j^{-2}(1 - \cos d_Z(u, v)) \\ &= (b - c)^2 + r^2 \lambda_j^{-2} d_Z(u, v)^2 + o(1). \end{aligned}$$

- Angular covering bounds give $(Z, r\lambda_j^{-1} d_Z, z) \to W$ along a subsequence. Since $-r/\lambda_j \to -\infty$, the limit is $\mathbb{R}^{\ell+1} \times W$.

Dimension of the strata

Theorem 4.11: dimension of the strata

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. Then

$$\dim_{\mathrm{H}} \mathcal{S}^k \leq k \qquad (0 \leq k \leq n-1).$$

Proof.

- **Contradiction:** fix $\varepsilon > 0$, $s > k$, and assume $\mathcal{H}^s(\mathcal{S}^k_\varepsilon \cap B_R(p)) > 0$. A quoted content-selection lemma produces successive tangent data
$$Y_j = \mathbb{R}^{\ell_j} \times C(Z_j), \qquad o_j \in E_j \subset \mathcal{S}^k_{\varepsilon_j}(Y_j), \qquad \mathcal{H}^s_\infty(E_j \cap B_1(o_j)) > 0,$$
where ℓ_j is maximal and $\mathcal{H}^s_\infty$ denotes Hausdorff content.
- **Select off the axis:** $\ell_j \leq k < s$ implies $\mathcal{H}^s_\infty((\mathbb{R}^{\ell_j} \times \{o\}) \cap B_1) = 0$. Choose a marked point off this axis and blow up again.
- **Gain a line:** Lemma 4.10 gives $\ell_{j+1} \geq \ell_j + 1$, but the obstruction gives $\ell_{j+1} \leq k$.

$$0 \leq \ell_0 < \ell_1 < \cdots < \ell_{k+1} \leq k \quad \text{is impossible;} \qquad \dim_{\mathrm{H}} \mathcal{S}^k_\varepsilon \leq k \Rightarrow \dim_{\mathrm{H}} \mathcal{S}^k \leq k.$$

Codimension-one rigidity and codimension two

Dimension reduction alone permits a half-space model. Limits of manifolds without boundary satisfy an additional rigidity theorem.

Theorem 4.12: codimension-one rigidity

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. For every tangent cone Y at $x \in X$,

$$Y \cong \mathbb{R}^{n-1} \times W \quad \Longrightarrow \quad Y \cong \mathbb{R}^n.$$

Corollary 4.13: codimension two

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. Then

$$\mathcal{S} = \mathcal{S}^{n-2}, \qquad \dim_{\mathrm{H}} \mathcal{S} \leq n-2, \qquad \mathcal{H}^n(\mathcal{S}) = 0.$$

Proof. By Theorem 4.12 and Proposition 4.6,

$$x \notin \mathcal{S}^{n-2} \Longleftrightarrow \text{some tangent at } x \text{ splits off } \mathbb{R}^{n-1}$$
$$\Longleftrightarrow \text{some tangent at } x \text{ is } \mathbb{R}^n \Longleftrightarrow x \in \mathcal{R}.$$

Theorem 4.11 gives $\dim_{\mathrm{H}} \mathcal{S} = \dim_{\mathrm{H}} \mathcal{S}^{n-2} \leq n-2 < n$, hence $\mathcal{H}^n(\mathcal{S}) = 0$.

Rectifiable strata

Rectifiability describes the geometry of a stratum, beyond its dimension.

Definition

A set E is k-**rectifiable** if there are Lipschitz maps $f_j : A_j \subset \mathbb{R}^k \to X$ with

$$\mathcal{H}^k\left(E \setminus \bigcup_{j=1}^{\infty} f_j(A_j)\right) = 0.$$

Theorem 4.14(Cheeger–Jiang–Naber 2021): rectifiable strata

Under (NC), $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$. For $0 \leq k \leq n-2$,

$$\mathcal{S}^k \text{ is } k\text{-rectifiable}, \qquad \mathcal{H}^k(\mathcal{S}^k_\varepsilon \cap B_1(p)) \leq C(n, v, \varepsilon) \quad (\varepsilon > 0).$$

For $\mathcal{H}^k$-a.e. $x \in \mathcal{S}^k$, **every** tangent at x is k-symmetric.

Inputs: quantitative splitting, neck decompositions, and summable errors.

Structure theorem for noncollapsed Ricci limits

This collects the main conclusions used in the preceding part.

Structure theorem: Noncollapsing Ricci lower bound limit space

Assume

$$(M_i^n, g_i, p_i) \xrightarrow{\text{pGH}} (X, d, p), \qquad \mathrm{Ric}_{g_i} \geq -(n-1)g_i, \qquad \mathrm{Vol}\, B_1(p_i) \geq v > 0.$$

Then X has Hausdorff dimension n and the Riemannian measures converge to $\mathcal{H}^n$ (with the standard normalization). Moreover:

- Every tangent cone is a metric cone; $X = \mathcal{R} \cup \mathcal{S}$, where $x \in \mathcal{R}$ iff a tangent cone at x is $\mathbb{R}^n$.
- $\dim_{\mathrm{H}} \mathcal{S} \leq n-2$ and $\mathcal{H}^n(\mathcal{S}) = 0$; the regular set is locally bi-lipschitz regular, but need not be open.
- For the quantitative strata $\mathcal{S}^k_\varepsilon$,

  $$\mathcal{S}^k_\varepsilon \text{ is } k\text{-rectifiable}, \quad \mathcal{H}^k(\mathcal{S}^k_\varepsilon \cap B_1(p)) \leq C(n, v, \varepsilon), \quad \mathrm{Vol}\left(T_r(\mathcal{S}^k_\varepsilon) \cap B_1(p)\right) \leq C(n, v, \varepsilon) r^{n-k}.$$

- The full singular set is countably $(n-2)$-rectifiable up to an $\mathcal{H}^{n-2}$-null subset; the quantitative measure bounds above are uniform only for fixed ε.

Cheeger–Colding (1997); Cheeger–Jiang–Naber (2021); quantitative stratification estimates of Cheeger–Naber.

Geometry of Smooth Einstein compactness

The Einstein equation turns curvature bounds into bounds for all derivatives.

Theorem(Cheeger-Gromov): smooth Einstein compactness

Assume

$$\mathrm{Ric}_{g_i} = \lambda_i g_i, \quad \lambda_i \to \lambda, \quad \mathrm{Vol}_{g_i} B_1(p_i) \geq v > 0, \quad \sup_{B_R(p_i)} |\mathrm{Rm}_{g_i}| \leq C_R \quad (0 < R < \infty).$$

Then, after a subsequence,

$$(M_i, g_i, p_i) \xrightarrow{C^\infty_{\mathrm{loc}}} (N, g_\infty, p_\infty), \qquad \mathrm{Ric}_{g_\infty} = \lambda g_\infty,$$

where (N, g_∞) is a complete smooth manifold. On each relatively compact $U \subset N$, there are diffeomorphisms onto their images

$$F_i : U \longrightarrow M_i, \qquad F_i^* g_i \longrightarrow g_\infty \quad \text{in } C^\infty(U).$$

Inputs: curvature + noncollapse $\Rightarrow$ injectivity radius; Einstein equation $\Rightarrow$ elliptic derivative estimates.

Einstein ε-regularity

Nearly Euclidean volume excludes curvature concentration in the Ricci-flat case.

Proposition 4.16: Einstein ε-regularity

There exist $\varepsilon_E(n), C_E(n) > 0$ such that, for every $r > 0$,

$$\mathrm{Ric} = 0, \qquad \mathrm{Vol}\, B_{2r}(p) \geq (1 - \varepsilon_E)\omega_n (2r)^n \quad \Longrightarrow \quad \sup_{B_{r/2}(p)} |\mathrm{Rm}| \leq C_E r^{-2}.$$

- **Contradiction:** rescale at a point of large curvature, so $|\mathrm{Rm}|(q) = 1$ in the smooth limit.
- **Volume comparison:** near-maximal volume survives at every smaller scale; the limit must be Euclidean.
- The above also holds for general Einstein $\mathrm{Ric} = \lambda g$.

Proof of Proposition 4.16: point selection

- **Normalize** $r = 1$. If the proposition fails, choose
$$\mathrm{Ric}_{g_i} = 0, \quad \mathrm{Vol}\, B_2(p_i) \geq (1 - i^{-1})\omega_n 2^n, \quad \sup_{B_{1/2}(p_i)} |\operatorname{Rm}_{g_i}| \longrightarrow \infty.$$

- Maximize the weighted curvature on $\overline{B}_{3/4}(p_i)$:
$$q_i \in \arg\max\big(\tfrac{3}{4} - d(p_i, \cdot)\big)^2 |\operatorname{Rm}_{g_i}|, \qquad s_i := \tfrac{3}{4} - d(p_i, q_i), \quad Q_i := |\operatorname{Rm}_{g_i}|(q_i).$$
Then $s_i > 0$ and
$$s_i^2 Q_i \geq \tfrac{1}{16} \sup_{B_{1/2}(p_i)} |\operatorname{Rm}_{g_i}| \to \infty.$$

- If $y \in B_{s_i/2}(q_i)$, then $\frac{3}{4} - d(p_i, y) \geq s_i/2$; hence
$$\tfrac{1}{4} s_i^2 |\operatorname{Rm}_{g_i}|(y) \leq s_i^2 Q_i \quad \Longrightarrow \quad \sup_{B_{s_i/2}(q_i)} |\operatorname{Rm}_{g_i}| \leq 4Q_i.$$

Proof of Proposition 4.16: volume at the new center

- **Theorem 4.2:** $\operatorname{Vol} B_2(p_i) \to \omega_n 2^n$ and $\operatorname{Ric}_{g_i} = 0$ imply
$$\overline{B}_1(p_i) \xrightarrow{\mathrm{GH}} \overline{B}_1(0^n).$$

- **Interior volume convergence (Theorem 4.3).** For $a := 1/32$,
$$\inf_{q \in B_{3/4}(p_i)} \frac{\operatorname{Vol} B_a(q)}{\omega_n a^n} \longrightarrow 1.$$
Otherwise a sequence of bad centers converges to an interior Euclidean center, contradicting volume convergence.

- **Bishop–Gromov at q_i:**
$$1 - o(1) \le \frac{\operatorname{Vol} B_t(q_i)}{\omega_n t^n} \le 1 \qquad (0 < t \le a).$$
The same error controls the shrinking curvature scale $Q_i^{-1/2}$.

Proof of Proposition 4.16: the smooth blow-up

- **Rescale:** $\widetilde{g}_i := Q_i g_i$, $R_i := s_i\sqrt{Q_i}/2 \to \infty$. Then
$$\mathrm{Ric}_{\widetilde{g}_i} = 0, \qquad |\mathrm{Rm}_{\widetilde{g}_i}|(q_i) = 1, \qquad \sup_{B^{\widetilde{g}_i}_{R_i}(q_i)} |\mathrm{Rm}_{\widetilde{g}_i}| \le 4.$$
For fixed $R > 0$, $Q_i \to \infty$ gives
$$\frac{\mathrm{Vol}_{\widetilde{g}_i} B_R(q_i)}{\omega_n R^n} = \frac{\mathrm{Vol}_{g_i} B_{R/\sqrt{Q_i}}(q_i)}{\omega_n (R/\sqrt{Q_i})^n} \longrightarrow 1.$$

- **Smooth compactness:** after local diffeomorphisms,
$$(M_i, \widetilde{g}_i, q_i) \xrightarrow{C^\infty_{\mathrm{loc}}} (N, g_\infty, q), \quad \mathrm{Ric}_{g_\infty} = 0, \quad |\mathrm{Rm}_{g_\infty}|(q) = 1.$$

- **Equality in Bishop–Gromov:**
$$\mathrm{Vol}_{g_\infty} B_R(q) = \omega_n R^n \ (R > 0) \quad \Longrightarrow \quad (N, g_\infty) \cong \mathbb{R}^n \quad \Longrightarrow \quad |\mathrm{Rm}_{g_\infty}|(q) = 0,$$
contradicting the normalization $|\mathrm{Rm}_{g_\infty}|(q) = 1$.

The Einstein regular set

At a regular point, density one gives a volume margin at a fixed small scale.

Corollary 4.17: the Einstein regular set

Assume

$$(M_i^n, g_i, p_i) \xrightarrow{\text{pGH}} (X, d, p), \qquad \operatorname{Vol}_{g_i} B_1(p_i) \geq v > 0,$$
$$\operatorname{Ric}_{g_i} = \lambda_i g_i, \qquad |\operatorname{Ric}_{g_i}| \leq n-1, \qquad \lambda_i \to \lambda.$$

Then

$$\Longrightarrow \quad \mathcal{R} \subset X \text{ is open}, \qquad (\mathcal{R}, g_\infty) \text{ is a smooth Einstein manifold}, \qquad \operatorname{Ric}_{g_\infty} = \lambda g_\infty.$$

After a subsequence, convergence is smooth on compact subsets of $\mathcal{R}$:

$$F_i : U \subset \mathcal{R} \xrightarrow{\text{diffeomorphism}} U_i \subset M_i, \qquad F_i^* g_i \longrightarrow g_\infty \quad \text{in } C^\infty(U).$$

Proof. It is a consequence of Proposition 4.16 and the definition of regular set $\mathcal{R}$.

Convexity of the regular set and codimension four

Smoothness is local; convexity controls minimizing paths through the limit.

Theorem 4.18(Colding-Naber 2013): convexity of the regular set

Assume

$$(M_i^n, g_i, p_i) \xrightarrow{\text{pGH}} (X, d, p), \qquad \operatorname{Vol} B_1(p_i) \geq v > 0,$$
$$\operatorname{Ric}_{g_i} = \lambda_i g_i, \qquad |\operatorname{Ric}_{g_i}| \leq n - 1.$$

Then $\mathcal{R}$ is geodesically convex:

$$x, y \in \mathcal{R} \quad \Longrightarrow \quad \exists\, \gamma : [0, d(x,y)] \to \mathcal{R} \text{ minimizing in } X,$$
$$\gamma(0) = x, \qquad \gamma(d(x,y)) = y.$$

Theorem 4.19(Tian, Cheeger, Cheeger-Naber 2015): codimension four

Assume $n \geq 4$ and

$$(M_i^n, g_i, p_i) \xrightarrow{\text{pGH}} (X, d, p), \qquad |\operatorname{Ric}_{g_i}| \leq n - 1, \qquad \operatorname{Vol} B_1(p_i) \geq v > 0.$$

Then

$$\Longrightarrow \qquad \dim_{\mathrm{H}} \mathcal{S} \leq n - 4.$$

Curvature energy and critical measure

Jiang–Naber obtain the endpoint estimates, using neck decompositions.

Theorem 4.20(Jiang-Naber 2021): curvature energy and critical measure

Assume $n \geq 4$.

- **On a manifold:**

$$|\operatorname{Ric}_g| \leq n-1, \quad \operatorname{Vol} B_1(p) \geq v > 0 \quad \Longrightarrow \quad \fint_{B_1(p)} |\operatorname{Rm}_g|^2 \, \mathrm{d}V_g \leq C(n,v).$$

- **On a pointed limit** $(M_i^n, g_i, p_i) \to (X, d, p)$ with $|\operatorname{Ric}_{g_i}| \leq n-1$ and $\operatorname{Vol} B_1(p_i) \geq v$:

$$\Longrightarrow \quad \mathcal{S} \text{ is countably } (n-4)\text{-rectifiable}, \qquad \mathcal{H}^{n-4}(\mathcal{S} \cap B_1(p)) \leq C(n,v).$$

Moreover, for $\mathcal{H}^{n-4}$-a.e. $x \in \mathcal{S}$, the tangent is unique:

$$\operatorname{Tan}_x X = \{\mathbb{R}^{n-4} \times C(S^3/\Gamma_x)\}, \qquad \Gamma_x \leq O(4) \text{ finite, free on } S^3.$$

Scaling: $\displaystyle\int_{B_1^{r^{-2}g}} |\operatorname{Rm}_{r^{-2}g}|^2 \, \mathrm{d}V_{r^{-2}g} = r^{4-n} \int_{B_r^g} |\operatorname{Rm}_g|^2 \, \mathrm{d}V_g$. For $n = 4$, $\mathcal{H}^0$ counts singular points.

Structure theorem for noncollapsed Einstein limits

Structure theorem: Noncollapsing Einstein Limit Space

Assume $n \geq 4$ and

$$(M_i^n, g_i, p_i) \xrightarrow{\text{pGH}} (X, d, p), \quad \text{Ric}_{g_i} = \lambda_i g_i, \quad |\lambda_i| \leq n-1, \quad \text{Vol}\, B_1(p_i) \geq v > 0.$$

Writing $X = \mathcal{R} \cup \mathcal{S}$, the following hold:

- $\mathcal{R}$ is open and geodesically convex; $(\mathcal{R}, g_\infty)$ is a smooth Einstein manifold, and the convergence is smooth locally on $\mathcal{R}$.
- The singular set has codimension at least four:

$$\dim_{\text{H}} \mathcal{S} \leq n-4, \qquad \mathcal{S} \text{ is countably } (n-4)\text{-rectifiable}, \qquad \mathcal{H}^{n-4}(\mathcal{S} \cap B_1(p)) \leq C(n, v).$$

- The critical curvature-energy estimate holds:

$$\int_{B_1(p_i)} |\,\text{Rm}_{g_i}\,|^2 \, dV_{g_i} \leq C(n, v).$$

- For $\mathcal{H}^{n-4}$-a.e. $x \in \mathcal{S}$, the tangent cone is unique and has the form $\mathbb{R}^{n-4} \times (\mathbb{R}^4/\Gamma_x)$, where $\Gamma_x \subset O(4)$ is finite and acts freely on S^3.

Curvature estimates and collapsed fibrations

$\mathrm{Ric} = 0$ leaves sectional curvature uncontrolled. Fibration theory needs more.

Theorem (Cheeger–Fukaya–Gromov)

Uniform sectional-curvature bound $\Rightarrow$ local nilpotent structures. Over a smooth base of bounded geometry: fibrations with infranil fibers.

Theorem (Cheeger–Tian, dimension four)

Assume $\mathrm{Ric} = \lambda g$, $|\mathrm{Ric}| \le 3$ and $0 < r \le 1$. There are universal $\varepsilon_0, C > 0$ such that

$$\int_{B_r(p)} |\mathrm{Rm}|^2 \, \mathrm{d}V \le \varepsilon_0 \quad \Longrightarrow \quad \sup_{B_{r/2}(p)} |\mathrm{Rm}| \le C r^{-2}.$$

Unnormalized integral; no volume lower bound is required.

Question

Can neck decompositions, splitting maps, and local-cover estimates describe higher-dimensional Ricci-flat collapse?

Optional discussion. Cheeger–Fukaya–Gromov (1992); Cheeger–Tian (2006).

Collapsing Ricci limits: what is known?

Assume $(M_i^n, g_i, p_i) \to (X, d, p)$ in pointed GH sense, $\mathrm{Ric}_{g_i} \geq -(n-1)$, and $\mathrm{Vol}\, B_1(p_i) \to 0$. The limit carries a renormalized measure ν.

Known structure (Cheeger–Colding; Colding–Naber). There is an integer $k \leq n$ such that X is k-rectifiable and ν is concentrated on the k-regular set

$$\mathcal{R}^k = \{x : \text{every tangent cone at } x \text{ is } \mathbb{R}^k\}.$$

At ν-a.e. point, the tangent cone is unique and Euclidean. The regular set is weakly convex and almost every pair of points can be joined by a minimizing geodesic whose interior lies in the regular set.

Important distinction. In the collapsed case, k need not equal $\dim_{\mathrm{H}} X$; the Hausdorff dimension can be non-integer.

Cheeger–Colding (2000); Colding–Naber (2012); Pan–Wei (2022).

Open question: regular points in collapsed limits

The noncollapsed theory gives a strong relation between Euclidean tangents, volume, and local geometry. Collapse removes the volume lower bound that drives many of these arguments.

Question 1. Under a uniform lower Ricci bound, what additional hypotheses ensure that the regular set is dense, or that a dense subset consists of manifold points?

- A point may have a unique Euclidean tangent without having a neighborhood bi-Lipschitz or smoothly equivalent to Euclidean space.
- Which quantitative hypotheses on the convergence, the renormalized measure, or the geometry of tangent cones yield stronger local regularity?

Motivated by Cheeger, "Ricci curvature: some recent progress and open questions" (2016).

Open question: singular sets and dimension in the collapsed case

Question 2. Can the singular set of a collapsed Ricci limit be described by a useful stratification with quantitative size estimates and geometric models for its different pieces?

- Noncollapsed limits satisfy strong codimension and quantitative-stratification estimates; what remains valid when the renormalized measure replaces volume?
- The Hausdorff dimension may exceed the rectifiable dimension, and may be non-integer. Thus a general bound comparing the two dimensions is false.
- Can one characterize which dimension pairs and singular-set geometries are compatible with a lower Ricci bound?

Pan–Wei (2022), "Examples of Ricci limit spaces with non-integer Hausdorff dimension".

Open question: geometric models for collapse

With two-sided sectional-curvature bounds, Cheeger–Fukaya–Gromov theory gives local nilpotent structures and infranil fibers. A lower Ricci bound alone allows much more singular behavior.

Question 3. Under a lower Ricci bound alone, when can a collapsed sequence be described by local fibrations or approximate fibrations over the regular part of its limit? What controls the fibers and the singular locus of the base?

- Seek hypotheses weaker than sectional-curvature bounds that still control topology and geometry of collapsing fibers.
- Relate splitting maps and quantitative stratification to local models for the collapse, including regions where the dimension of tangent cones changes.

Compare Cheeger–Fukaya–Gromov (1992) with Cheeger–Colding and Colding–Naber.

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