Cheeger–Colding Theory

Lectures 2–3: Splitting maps and cones

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Parallel Vector field and rigidity

Splitting Theorem: rigidity

Throughout: M^n is a complete connected Riemannian manifold. **Assume**

$$u \in C^\infty(M), \qquad \operatorname{Hess} u = 0, \qquad |\nabla u| = 1.$$

Set $N := u^{-1}(0)$, a smooth hypersurface, with $g_N := g|_{TN}$. **Then there is a Riemannian product isometry:**

$$\Longrightarrow \quad \exists\, \Phi : (\mathbb{R} \times N, dt^2 + g_N) \xrightarrow{\text{isometry}} (M, g),$$
$$u(\Phi(t, z)) = t \quad \Longrightarrow \quad \Phi(\{t\} \times N) = u^{-1}(t).$$

Product distance: the Pythagorean identity

$$d_M(\Phi(s, y), \Phi(t, z))^2 = |s - t|^2 + d_N(y, z)^2.$$

Note: The conclusion is elegant, but its applications are rather limited, as they require quite restrictive assumptions.

Question: Any weak version? or almost rigidity version?

Splitting function

Let us recall the definition of $(1,\delta)$-splitting function introducing by Cheeger-Colding:

Definition 2.2: $(1,\delta)$-splitting function

Let $B_R := B_R(p)$, $u \in C^\infty(B_R)$ and $0 < \delta < 1$. u is $(1,\delta)$-**splitting** if

$$\Delta u = 0 \text{ (harmonic)}, \qquad \sup_{B_R} |\nabla u| \le 1 + \delta \text{ (controlled stretch)},$$

$$\fint_{B_R} \left||\nabla u|^2 - 1\right| \le \delta \qquad \text{(almost unit gradient)},$$

$$R^2 \fint_{B_R} |\operatorname{Hess} u|^2 \le \delta^2 \qquad \text{(almost parallel gradient)}.$$

$\fint_A f := \frac{1}{\operatorname{Vol}(A)} \int_A f$. Adding a constant preserves the definition.

Theorem 2.3: $(1,\delta)$-splitting implies ϵ-GH splitting

For every $\varepsilon > 0$, there exists $\delta = \delta(n,\varepsilon) \in (0,1)$.

Assume

$$\mathrm{Ric} \geq -(n-1)\delta g, \qquad u : B_{100}(p) \to \mathbb{R} \text{ is } (1,\delta)\text{-splitting}, \qquad u(p) = 0.$$

Set $Y := u^{-1}(0) \cap \overline{B}_4(p)$, $d_Y := d_M|_{Y\times Y}$: the transverse metric space. **Then:**

$$\Longrightarrow \quad d_{\mathrm{GH}}\big(\overline{B}_1(p), \overline{B}_1((0,p)) \subset \mathbb{R}\times Y\big) < \varepsilon,$$

All ball distances are ambient.

Set $Y := u^{-1}(0) \cap \overline{B}_4(p)$, $\quad \pi(x) \in \underset{z\in Y}{\operatorname{argmin}}\, d(x,z)$, $\qquad F(x) := (u(x), \pi(x))$, $\qquad x \in \overline{B}_2(p)$.

Need to show the following:

- **Almost isometric (Pythagoras):** $d(x,y)^2 \approx |u(x)-u(y)|^2 + d_Y(\pi(x),\pi(y))^2$.
- **Almost onto:** $(t,z) \in \overline{B}_1((0,p)) \quad \Longrightarrow \quad \exists x:\ u(x) \approx t, \quad \pi(x) \approx z.$

Almost Pythagorean: ϵ-isometry

Lemma 2.4(Key Lemma): Almost Pythagorean

Assume

$$u \in C^2(B_{100}), \quad \sup_{B_{100}} |\nabla u| \le 1+\eta, \quad x, w, z \in B_{10}, \quad 0 < \eta < 1,$$

$$a := d(w,z) > 0, \qquad |u(x) - u(z)| \le \eta, \qquad |u(z) - u(w) - a| \le \eta.$$

Choose unit-speed minimizing geodesics and define

$$\sigma : [0,a] \to M,\ w \to z; \qquad \gamma_s : [0, \ell(s)] \to M,\ x \to \sigma(s),$$

$$\mathcal{H}(x,\sigma) := \int_0^a \int_0^{\ell(s)} |\operatorname{Hess} u|(\gamma_s(t))\, dt\, ds.$$

Then the distance error satisfies

$$\implies \quad \left| d(x,w)^2 - d(x,z)^2 - a^2 \right| \le C\left(\sqrt{\eta} + \mathcal{H}(x,\sigma)\right),$$

in particular, $\mathcal{H}(x,\sigma) \le \eta \quad \implies$ Pythagorean error $\le C\sqrt{\eta}$.

Proof of Lemma 2.4: the vertical direction

Input: $u(z) - u(w) \geq a - \eta$, $|\nabla u| \leq 1 + \eta$, and $a \leq 20$.

- **The minimizing side is almost tangent to ∇u.**

$$\int_0^a |\sigma'(s) - \nabla u|^2 \, ds = a + \int_0^a |\nabla u|^2 \, ds - 2\big(u(z) - u(w)\big) \leq (2\eta + \eta^2)a + 2\eta \leq C\eta.$$

- **Cauchy–Schwarz:**

$$\int_0^a |\sigma' - \nabla u| \, ds \leq a^{1/2} \left(\int_0^a |\sigma' - \nabla u|^2 \, ds \right)^{1/2} \leq C\sqrt{\eta}.$$

- **Fix s, consider $\gamma_s(t)$.**

$$\ell(s) := d(x, \sigma(s)), \qquad f_s(t) := u \circ \gamma_s(t), \qquad \frac{d^2}{dt^2} f_s(t) = \operatorname{Hess} u(\dot\gamma_s, \dot\gamma_s),$$

$$\int_0^\ell t \, \operatorname{Hess} u(\dot\gamma_s, \dot\gamma_s) \, dt = \int_0^\ell \Big(\frac{d}{dt}(t \frac{d}{dt} f_s(t)) - \frac{d}{dt} f_s(t) \Big) dt = \ell \langle \nabla u(\sigma(s)), \dot\gamma_s(\ell) \rangle - u(\sigma(s)) + u(x)$$

Proof of Lemma 2.4: first variation

Recall $\int_0^\ell t\ \mathrm{Hess}\, u(\dot\gamma_s,\dot\gamma_s)\,dt = \ell\langle\nabla u(\sigma(s)),\dot\gamma_s(\ell)\rangle - u(\sigma(s)) + u(x)$.

- **By the first variation of length:** $\ell'(s) = \langle\sigma(s)', \dot\gamma_s(\ell)\rangle$, therefore

$$\big|\ell\ell' - \big(u(\sigma(s)) - u(x)\big)\big| \le \ell|\sigma' - \nabla u| + \ell\int_{\gamma_s} |\,\mathrm{Hess}\, u|.$$

Noting $|\sigma'| = 1$, we have

$$|u(\sigma(s)) - u(x) - (s-a)| = |\int_s^a \Big(\langle\nabla u, \sigma'\rangle - |\sigma'|^2\Big)dt \le \int_s^a |\nabla u - \sigma'|\,dt \le C\sqrt{\eta}.$$

Hence

$$|\ell(s)\ell'(s) - (s-a)| \le C\sqrt{\eta} + \ell|\sigma' - \nabla u| + \ell\int_{\gamma_s} |\,\mathrm{Hess}\, u|$$

This gives

$$|(\ell(a)^2 - \ell(0)^2)/2 - \int_0^a (s-a)\,ds| \le C\sqrt{\eta} + C\mathcal{H}(x,\sigma)$$

Noting $\ell(0) = d(x,w)$, $\ell(a) = d(x,z)$, $\int_0^a (s-a)\,ds = -a^2/2$, hence

$$\big|d(x,w)^2 - d(x,z)^2 - a^2\big| \le C\big(\sqrt{\eta} + \mathcal{H}(x,\sigma)\big). \quad \square$$

Proof of Theorem 2.3: Find condition of Lemma 2.4 (I)

Under Theorem 2.3:

$$\mathrm{Ric} \geq -(n-1)\delta g, \qquad 100^2 \fint_{B_{100}} |\operatorname{Hess} u|^2 \leq \delta^2, \qquad V := \mathrm{Vol}(B_{10}),$$
$$\text{Let: } K_x(v) := \int_{\gamma_{xv}} |\operatorname{Hess} u|, \qquad \mathcal{H}(x, \gamma_{wz}) := \int_{\gamma_{wz}} K_x.$$

- **Fix $x \in B_{10}$; average the moving endpoints (w, z).**
$$\int_{B_{10}\times B_{10}} \mathcal{H}(x, \gamma_{wz})\, dw\, dz \leq C(n) V \int_{B_{20}} K_x(v)\, dv.$$

- **Apply the segment inequality again to (x, v).**
$$\int_{B_{10}\times B_{20}} K_x(v)\, dx\, dv \leq C(n) V \int_{B_{100}} |\operatorname{Hess} u|.$$

- **Volume comparison, Cauchy–Schwarz; divide by V^3.**
$$\int_{B_{10}^3} \mathcal{H}(x, \gamma_{wz})\, dx\, dw\, dz \leq C(n) V^2 \int_{B_{100}} |\operatorname{Hess} u| \leq C(n) V^3 \left(\fint_{B_{100}} |\operatorname{Hess} u|^2 \right)^{1/2}$$
$$\Longrightarrow \quad \fint_{B_{10}^3} \mathcal{H}(x, \gamma_{wz}) \leq C(n)\delta.$$

Proof of Theorem 2.3: Find condition of Lemma 2.4 (II)

A GH map needs distance control at every point; move the endpoints slightly.

$$\mathrm{Ric} \geq -(n-1)\delta g, \qquad x, w, z \in B_8, \qquad 0 < \rho < 1/10, \qquad V = \mathrm{Vol}(B_{10}).$$

- **Relative volume comparison:** $\mathrm{Vol}(B_\rho(x))\,\mathrm{Vol}(B_\rho(w))\,\mathrm{Vol}(B_\rho(z)) \geq c(n)\rho^{3n}V^3$.
- **Divide the global triple bound by this product volume:**

$$⨍_{B_\rho(x)\times B_\rho(w)\times B_\rho(z)} \mathcal{H}(x', \gamma_{w'z'}) \leq \frac{\int_{B_{10}^3} \mathcal{H}(x', \gamma_{w'z'})\, dx'\, dw'\, dz'}{\mathrm{Vol}(B_\rho(x))\,\mathrm{Vol}(B_\rho(w))\,\mathrm{Vol}(B_\rho(z))} \leq C(n)\delta\rho^{-3n}.$$

- **Hence some triple within ρ of (x, w, z) satisfies $\mathcal{H}(x', \gamma_{w'z'}) \leq C(n)\delta\rho^{-3n}$.**

Special ϵ-isometry

Special ϵ-isometry Lemma 2.5

Under Theorem 2.3, take $x, w, z \in B_8$ with

$$|u(x) - u(z)| \le \eta, \qquad \big||u(z) - u(w)| - d(w,z)\big| \le \eta.$$

Then, uniformly,

$$\Longrightarrow \quad \big|d(x,w)^2 - d(x,z)^2 - d(z,w)^2\big| \le \Psi(\eta + \delta \mid n), \qquad \Psi(s \mid n) \longrightarrow 0 \ (s \longrightarrow 0).$$

Proof.

- Replace u by $-u$ if needed; **transfer to a good nearby triple:**

  coordinate and distance errors $\le C(\eta + \delta + \rho)$, $\qquad \mathcal{H} \le C(n)\delta\rho^{-3n}$.

- **Apply Lemma 2.4 and move the endpoints back.**

$$\big|d(x,w)^2 - d(x,z)^2 - d(z,w)^2\big| \le C(n)\left(\sqrt{\eta + \delta + \rho} + \delta\rho^{-3n} + \rho\right).$$

- **Choose ρ small first, then η and δ small.** This gives the stated uniform bound.

Two Claims without proofs

The proof of the following depends on geodesic flow:

Claim 1:

For any $x, y \in B_2(p)$ with $u(y) = a$, we have

$$\left| d(x, u^{-1}(a)) - |u(x) - a| \right| \leq \Psi(\delta | n)$$

Application: If $z_1, z_2 \in u^{-1}(a)$ such that $d(x, z_1) = d(x, z_2) = d(x, u^{-1}(a))$, then by Lemma 2.5, we have $d(z_1, z_2) \leq \Psi(\delta | n)$.

Claim 2:

For any $x \in B_2(p)$ with $|s| \leq 2$, there exists $y \in B_5(p)$ such that

$$u(y) = u(x) + s \text{ and } \left| d(x, y) - |u(x) - u(y)| \right| \leq \Psi(\delta | n)$$

Proof of Theorem 2.3 (ϵ-isometry)

Set $Y := u^{-1}(0) \cap \overline{B}_4(p)$, $\pi(x) \in \operatorname{argmin}_{z \in Y} d(x,z)$, $F(x) := (u(x), \pi(x))$, $x \in \overline{B}_2(p)$.

For $x, y \in B_2(p)$, the distance in the product space:

$$d(F(x), F(y))^2 = |u(x) - u(y)|^2 + d(\pi(x), \pi(y)|^2.$$

We need to estimate $|d(x,y) - d(F(x), F(y))|$. Choose $z \in u^{-1}(u(x))$ such that $d(y,z) = d(y, u^{-1}(u(x))$. Then noting Claim 1 and using Lemma 2.5 we have

$$\begin{aligned} d(x, \pi(y))^2 &= d(\pi(x), \pi(y))^2 + u(x)^2 + \Psi, \\ d(x, \pi(y))^2 &= d(x,z)^2 + u(x)^2 + \Psi, \\ d(x,y)^2 &= d(x,z)^2 + |u(x) - u(y)|^2 + \Psi. \end{aligned}$$

Thus

$$d(x,y)^2 = |u(x) - u(y)|^2 + d(\pi(x), \pi(y))^2 + \Psi. \ \square$$

Proof of Theorem 2.3 (ϵ-onto)

We need to show

For any $z \in Y \cap B_1(p)$ and $|t| \leq 1$, there exists $x \in B_2(p)$ such that $d(F(x), (t, z)) \leq \Psi$.

Proof. By Claim 2, there exists $x \in B_5(p)$ such that

$$u(x) = t \text{ and } \Big| d(x, z) - |u(x) - u(z)| \Big| \leq \Psi.$$

By Claim 1, we have $d(z, \pi(x)) \leq \Psi$. Thus

$$d\big(F(x), (t, z)\big) = d\big((t, \pi(x), (t, z)\big) \leq \Psi.$$

Using ϵ-isometry esimate one can show that $x \in B_2(p)$. Hence we complete the whole proof. □

Long geodsic and almost splitting

Cheeger-Gromoll splitting Theorem

Assume $\mathrm{Ric} \geq 0$ and $\gamma : \mathbb{R} \to M$ is a line:

$$d(\gamma(s), \gamma(t)) = |s - t| \qquad \forall s, t \in \mathbb{R}.$$

Then there exist a complete Riemannian manifold (N, g_N) with $\mathrm{Ric}_N \geq 0$, a point $z_0 \in N$, and an isometry

$$(M, g) \cong (\mathbb{R} \times N, dt^2 + g_N), \qquad \gamma(t) = (t, z_0).$$

Theorem 2.6: Almost splitting with a long geodsic

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(n, \varepsilon) > 0.$$

For $p, q^+, q^- \in M$, define the excess $e(x) := d(x, q^+) + d(x, q^-) - d(q^+, q^-)$.

Assume $\mathrm{Ric} \geq -(n-1)\delta g$, $d(p, q^\pm) \geq \delta^{-1}$, and $e(p) \leq \delta$.

Then there exists a compact metric space (Y, y) such that

$$d_{\mathrm{GH}}\big(\overline{B}_1(p), \overline{B}_1((0, y)) \subset \mathbb{R} \times Y\big) < \varepsilon.$$

Proof of 2.6: harmonic replacement

Assume $\mathrm{Ric} \geq -(n-1)\delta g$, $d(p, q^{\pm}) \geq \delta^{-1}$, and $e(p) \leq \delta$.

- **1. Distance coordinates:**

$$b^{+}(x) := d(x, q^{+}) - d(p, q^{+}), \qquad b^{-}(x) := d(p, q^{-}) - d(x, q^{-}).$$

  Thus $b^{+} - b^{-} = e - e(p)$ and $|\nabla b^{\pm}| = 1$ almost everywhere.

- **Excess estimates:** for small δ, $\eta = \Psi(\delta \mid n) \to 0$,

$$\sup_{B_{400}} |b^{+} - b^{-}| \leq \eta,$$

  and by Laplacian comparison:

$$\Delta b^{+} \leq \eta, \qquad \Delta b^{-} \geq -\eta \quad \text{weakly on } B_{400}.$$

- **2. Harmonic replacement:** Let u solve

$$\Delta u = 0 \text{ in } B_{400} \text{ with } \ u = b^{+} \text{ on } \partial B_{400}.$$

  The interior regularity gives $u \in C^{\infty}(B_{400})$.

Proposition 2.7: Harmonic replacement

Assume $p, q^+, q^- \in M$, $\mathrm{Ric} \geq -(n-1)\delta g$, and $0 < \delta \leq \delta_0(n)$, where $\delta_0(n)$ is sufficiently small, with

$$d(p, q^\pm) \geq \delta^{-1}, \qquad d(p, q^+) + d(p, q^-) - d(q^+, q^-) \leq \delta.$$

Set $b^+(x) := d(x, q^+) - d(p, q^+)$, and let u solve

$$\Delta u = 0 \text{ in } B_{400}, \qquad u - b^+ \in W^{1,2}_0(B_{400}).$$

Then, for $\eta = \Psi(\delta \mid n) \to 0$ as $\delta \to 0$,

$$\sup_{B_{400}} |u - b^+| \leq C(n)\eta, \qquad \fint_{B_{300}} \big||\nabla u|^2 - 1\big| \leq C(n)\eta,$$

$$\Longrightarrow$$

$$\fint_{B_{200}} |\operatorname{Hess} u|^2 \leq C(n)(\eta + \delta), \qquad \sup_{B_{100}} |\nabla u|^2 \leq 1 + C(n)(\eta + \delta).$$

Hence u is $(1, \eta)$-splitting function on B_{100}.

Theorem 2.3 $\Longrightarrow$ $d_{\mathrm{GH}}\big(\overline{B}_1(p), \overline{B}_1((0, y)) \subset \mathbb{R} \times Y\big) < \varepsilon$. Complete the proof of Theorem 2.6.

Proof of 2.7: the energy estimate

Input: the hypotheses of Proposition 2.7, $\mathrm{Ric} \geq -(n-1)\delta g$, and $\eta = \Psi(\delta \mid n)$.

- **1. Barrier estimate(prove this later):**
$$w := u - b^+, \qquad \|w\|_{L^\infty(B_{400})} \leq C\eta, \qquad -\Delta w \leq \eta.$$

- **2. Test with** $(w + C\eta)\phi^2 \geq 0$**.** Choose $\phi = 1$ on B_{300}, $\operatorname{supp}\phi \subset B_{350}$, and $|\nabla\phi| \leq C$. Integration by parts and Young give
$$\int \phi^2 |\nabla w|^2 \leq \frac{1}{2}\int \phi^2 |\nabla w|^2 + C\eta^2 \operatorname{Vol}(B_{350}).$$

Absorb and use volume doubling: $\Longrightarrow$ $\fint_{B_{300}} |\nabla u - \nabla b^+|^2 \leq C\eta^2$.

By elliptic estimate we have $\sup_{B_{300}} |\nabla u| \leq C$. Thus
$$\fint_{B_{300}} ||\nabla u|^2 - 1| = \fint_{B_{300}} ||\nabla u|^2 - |\nabla b^+|^2| \leq \sup_{B_{300}} (|\nabla u| + 1) \fint_{B_{300}} |\nabla u - \nabla b^+| \leq C\eta.$$

Proof of 2.7: Hessian energy

Input: $\mathrm{Ric} \geq -(n-1)\delta g$, $\Delta u = 0$, and the preceding energy estimate.

- **3. Set:** $G := |\nabla u|^2 - 1$, we have
$$\fint_{B_{300}} |G| \leq C\eta.$$
- **4. Bochner with a Laplacian cutoff.** Choose $\chi|_{B_{200}} = 1$, $\operatorname{supp}\chi \subset B_{300}$, $0 \leq \chi \leq 1$, and $|\Delta\chi| \leq C$. Then
$$2\int \chi |\operatorname{Hess} u|^2 \leq \int G\Delta\chi + 2(n-1)\delta \int \chi |\nabla u|^2.$$

Volume comparison $\Longrightarrow$ $\fint_{B_{200}} |\operatorname{Hess} u|^2 \leq C(n)(\eta + \delta)$.

Proof of 2.7: the gradient upper bound

Input: $\mathrm{Ric} \geq -(n-1)\delta g$, $G := |\nabla u|^2 - 1$, and $\fint_{B_{300}} |G| \leq C\eta$.

- **5. Positive part:** $G_+ := \max\{G, 0\}$. Bochner and $|\nabla u| \leq C$ give
$$\Delta G \geq -C\delta \quad \Longrightarrow \quad \Delta G_+ \geq -C\delta \quad \text{weakly.}$$

- **Mean-value inequality (Lecture 1):**
$$\sup_{B_{100}} G_+ \leq C \fint_{B_{200}} G_+ + C\delta \leq C(\eta + \delta).$$

$$\sup_{B_{100}} |\nabla u|^2 \leq 1 + C(n)(\eta + \delta).$$

Hence we will finish the proof of Proposition 2.7 after proving the barrier estimate.

Proof of 2.7: the barrier estimate I

Set $\Omega = B_R(p)$, $R = 400$. Recall that

$$\Delta u = 0, \quad u - b^+ \in W_0^{1,2}(\Omega), \quad |b^+ - b^-| \le \eta, \quad \Delta b^+ \le \eta, \quad -\Delta b^- \le \eta.$$

We want to prove the barrier estimate: $|u - b^+| \le C\eta$. Recall the general maximum principle:

Weak Dirichlet maximum estimate

If $-\Delta v \le f$ weakly on Ω and $(v - a)_+ \in W_0^{1,2}(\Omega)$, then

$$\sup_\Omega v \le a + C(n)R^2 \|f_+\|_{L^\infty(\Omega)}.$$

- **Upper bound for $u - b^+$.** Put $w = u - b^+$. Then

$$-\Delta w = \Delta b^+ \le \eta, \qquad w \in W_0^{1,2}(\Omega).$$

The maximum estimate with $a = 0$ gives $\sup_\Omega (u - b^+) \le C(n)\eta$.

Proof of 2.7: the barrier estimate II

The lower bound uses the other distance coordinate b^-.

- **Control $b^- - u$.** Put $v = b^- - u$. We have
$$-\Delta v = -\Delta b^- \le \eta, \qquad v = b^- - b^+ \le \eta \quad \text{on } \partial\Omega.$$
More precisely, $(v-\eta)_+ \in W_0^{1,2}(\Omega)$.
- **Apply the same maximum estimate, now with $a = \eta$.**
$$\sup_\Omega (b^- - u) \le \eta + C(n)\eta.$$
Since $b^+ - b^- \le \eta$, this gives
$$b^+ - u = (b^+ - b^-) + (b^- - u) \le 2\eta + C(n)\eta.$$

Therefore, combining the upper and lower bounds, we have:

The barrier estimate:

$$\boxed{\|u - b^+\|_{L^\infty(B_{400})} \le C(n)\eta}.$$

Splitting at Ricci limit space

Corollary: Splitting at limit space

Let (X, d, p) be a GH limit space of a sequence of complete manifolds (M_i^n, g_i, p_i) with $\mathrm{Ric}_{g_i} \geq -(n-1)\delta_i g_i \to 0$. If there exists a line in X, i.e., $\gamma : \mathbb{R} \to X$ such that $\gamma(0) = p$ and

$$d(\gamma(s), \gamma(t)) = |s - t|,$$

then X isometrically splits an $\mathbb{R}$ factor, $X = Y \times \mathbb{R}$.

Proof. Exercise. Using Theorem 2.6.

Definition: (k, ϵ)-splitting maps

Several almost Euclidean coordinates must also be almost orthonormal.

Definition 2.8: (k, ϵ)-splitting map

Let $u = (u^1, \dots, u^k) \in C^\infty(B_R, \mathbb{R}^k)$, $1 \le k \le n$, and $\varepsilon > 0$. We call u (k, ε)-**splitting** if for all $a, b \in \{1, \dots, k\}$,

$$\Delta u^a = 0 \text{ (harmonic)}, \qquad \sup_{B_R} |du| \le 1 + \varepsilon \,,$$

$$\fint_{B_R} |\langle \nabla u^a, \nabla u^b \rangle - \delta_{ab}| \le \varepsilon \qquad \text{(almost orthonormal)},$$

$$R^2 \fint_{B_R} \sum_{a=1}^{k} |\operatorname{Hess} u^a|^2 \le \varepsilon^2 \qquad \text{(almost parallel)}.$$

δ_{ab} are the identity-matrix entries.

Theorem 2.9: splitting maps and GH products

Theorem 2.9: Anysis VS Geometry

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(n, \varepsilon) \in (0, 1).$$

Assume throughout: $\mathrm{Ric} \geq -(n-1)\delta g$ and $1 \leq k \leq n$.

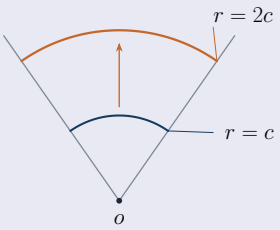
- **If** $u : B_{100}(p) \to \mathbb{R}^k$ is (k, δ)-splitting, **then** there exists a proper length space (Y, y) with
$$d_{\mathrm{GH}}\left(\overline{B}_1(p), \overline{B}_1((0, y)) \subset \mathbb{R}^k \times Y\right) < \varepsilon.$$
- **Conversely**, for a proper length space (Y, y):
$$\text{If} \quad d_{\mathrm{GH}}\left(\overline{B}_{\delta^{-1}}(p), \overline{B}_{\delta^{-1}}((0, y)) \subset \mathbb{R}^k \times Y\right) < \delta,$$
$$\text{then} \quad \Longrightarrow \quad \exists u : B_1(p) \to \mathbb{R}^k \quad \text{a } (k, \varepsilon)\text{-splitting map.}$$

Metric cones

Two Hessian equations produce two different kinds of metric flow.

- **Isometric flow:**
$$\operatorname{Hess} u = 0 \quad \Longrightarrow \quad \mathcal{L}_{\nabla u} g = 0.$$

- **Radial dilations:**
$$\operatorname{Hess} h = g \quad \Longrightarrow \quad \mathcal{L}_{\nabla h} g = 2g.$$



The Lie derivative is the infinitesimal change of the metric under a flow.

Assume the smooth cone metric and define the radial quadratic function:
$$g = dr^2 + r^2 g_Z, \qquad h := \tfrac{1}{2} r^2 \quad (r > 0).$$
$$\operatorname{Hess} h = dr \otimes dr + r \operatorname{Hess} r = g \quad \Longrightarrow \quad \Delta h = n, \quad |\nabla h|^2 = 2h.$$

Metric cones

Definition: metric cone

Given a compact metric space (Z, d_Z) with $\operatorname{diam} Z \le \pi$, identify all points $(0, z)$ to a single vertex:

$$C(Z) := ([0,\infty) \times Z)/(\{0\} \times Z), \qquad o := [(0, z)].$$

Define the cone distance by

$$d_C((r, z), (s, w))^2 := r^2 + s^2 - 2rs \cos d_Z(z, w).$$

If (Z, g_Z) is Riemannian, then away from the vertex $g = dr^2 + r^2 g_Z$ $(r > 0)$.

Theorem 2.11: metric cone rigidity

Assume $\Omega \subset M$ is open, $h \in C^\infty(\Omega)$, and

$$h > 0, \qquad \operatorname{Hess} h = g, \qquad |\nabla h|^2 = 2h.$$

Then $r := \sqrt{2h}$ is a local radial coordinate, with $g = dr^2 + r^2 g_Z$ $(r > 0)$ in local flow coordinates, where Z is a smooth level hypersurface.

Volume ratios and almost metric cone theorem

For $\kappa \geq 0$, let $V^n_{-\kappa}(t)$ denote the model ball volume in constant sectional curvature $-\kappa$. **Define**

$$\Phi_\kappa(t) := \frac{\mathrm{Vol}(B_t(p))}{V^n_{-\kappa}(t)}.$$

Bishop–Gromov consequence:

$$\mathrm{Ric} \geq -(n-1)\kappa g \quad \Longrightarrow \quad \Phi_\kappa(t) \text{ is nonincreasing.}$$

Theorem 2.12(Cheeger-Colding): Almost volume cone implies almost metric cone

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(n, \varepsilon) > 0 \quad \forall R > 0.$$

Assume

$$\mathrm{Ric} \geq -(n-1)\kappa g, \quad 0 \leq \kappa R^2 \leq \delta, \quad 0 \leq \log \Phi_\kappa(R) - \log \Phi_\kappa(2R) \leq \delta.$$

Then there is a compact metric space (Z, d_Z) with $\mathrm{diam}\, Z \leq \pi$ such that

$$d_{\mathrm{GH}}\big(\overline{B}_R(p), \overline{B}_R(o) \subset C(Z)\big) < \varepsilon R.$$

Definition 2.13: conical functions

An analytic approximation to a radial quadratic function.

$$\rho := d(p, \cdot), \qquad f := \tfrac{1}{2}\rho^2, \qquad h \in C^\infty(B_R), \quad \varepsilon > 0.$$

We call h ε-**conical** if and only if $\Delta h = n$ and all four bounds hold:

$$\text{Values:} \qquad R^{-2}\|h - f\|_{L^\infty(B_R)} \le \varepsilon,$$

$$\text{Gradient:} \qquad R^{-2} \fint_{B_R} |\nabla h - \nabla f|^2 \le \varepsilon,$$

$$\text{Hessian:} \qquad \fint_{B_R} |\operatorname{Hess} h - g|^2 \le \varepsilon,$$

$$\text{Radial identity:} \qquad R^{-2} \fint_{B_R} \left||\nabla h|^2 - 2h\right| \le \varepsilon.$$

Derivatives of f are interpreted almost everywhere.

Existence of ϵ-conical function.

The volume pinching also supplies a smooth analytic cone coordinate.

Lemma 2.14 (existence of ϵ-conical function)

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(n, \varepsilon) > 0 \quad \forall R > 0.$$

Assume the same two bounds as in Theorem 2.12:

$$\mathrm{Ric} \geq -(n-1)\kappa g, \quad 0 \leq \kappa R^2 \leq \delta, \quad 0 \leq \log \Phi_\kappa(R) - \log \Phi_\kappa(2R) \leq \delta.$$

Then $\exists h \in C^\infty(B_R)$, an ε-conical function.

Thus $\Delta h = n$ and all four errors in Definition 2.13 are controlled.

Remark: Similar argument as the almost splitting theorem implies Theorem 2.12.