Cheeger–Colding Theory

Lecture 1: Volume, segments and elliptic estimates

Wenshuai Jiang

Zhejiang University

The aim of the course: Compactness theory of Ricci lower bound

Goal: Use GH theory to study compactness and the geometry of limits for manifolds with a lower Ricci curvature bound.

- **Why GH is useful in this setting, Global geometry? Key point I: Volume comparison**

$$\text{volume comparison} \Longrightarrow \text{volume doubling} \Longrightarrow \text{covering bounds} \Longrightarrow \text{GH precompactness.}$$

- **How to estimate GH, local behavior? Key point II: The segment inequality**

  Our analytic information is often an integral estimate for a function. The segment inequality gives averaged control along joining geodesics, a key input for constructing GH approximations.

Bishop–Gromov volume comparison

Theorem 1.1 (Bishop-Gromov): Volume comparison

Assume: M^n is a complete Riemannian manifold and

$$n \geq 2, \qquad \mathrm{Ric} \geq -(n-1)Kg, \qquad K \geq 0.$$

Let $V^n_{-K}(s)$ be the s-ball volume in the simply connected model of sectional curvature $-K$.
Then: larger balls have smaller relative volume:

$$\Longrightarrow \quad \frac{\mathrm{Vol}\, B_R(x)}{V^n_{-K}(R)} \leq \frac{\mathrm{Vol}\, B_s(x)}{V^n_{-K}(s)} \qquad (x \in M,\ 0 < s < R).$$

Consequences when $KR^2 \leq A$:

$$\text{Doubling:} \quad \mathrm{Vol}\, B_{2R}(x) \leq C(n, A)\, \mathrm{Vol}\, B_R(x),$$

$$\text{Relative lower bound:} \quad \frac{\mathrm{Vol}\, B_s(x)}{\mathrm{Vol}\, B_R(x)} \geq c(n, A)(s/R)^n.$$

Note: No absolute volume lower bound is assumed. *Stated without proof.*

Volume doubling: Different centers and radii

Remark 1.2: different centers and radii

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \quad 0 \leq KR^2 \leq L, \quad d := d(x,y) \leq AR, \quad A \geq 0, \quad 0 < s \leq R.$$

Then

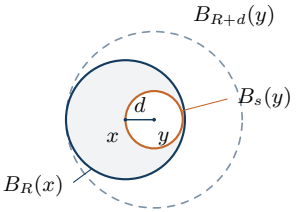
$$\Longrightarrow \quad c(n,A,L)(s/R)^n \leq \frac{\mathrm{Vol}\, B_s(y)}{\mathrm{Vol}\, B_R(x)} \leq C(n,A,L).$$

Thus comparable radii give comparable ball volumes.

Proof: inclusions and Bishop–Gromov.

- $B_R(x) \subset B_{R+d}(y)$; compare volumes at y.
- $B_s(y) \subset B_{R+d}(x)$; compare volumes at x.

$$\Longrightarrow \quad \frac{V^n_{-K}(s)}{V^n_{-K}(R+d)} \leq \frac{\mathrm{Vol}\, B_s(y)}{\mathrm{Vol}\, B_R(x)} \leq \frac{V^n_{-K}(R+d)}{V^n_{-K}(R)}.$$



Gromov precompactness

A uniform covering bound gives a convergent subsequence.

Gromov precompactness Theorem

Assume: (X_i, d_i, p_i) are proper pointed length spaces, and

$$\forall R, \rho > 0 \quad \exists N(R,\rho) < \infty \quad \forall i: \qquad \overline{B}_R(p_i) \subset \bigcup_{j=1}^{N(R,\rho)} B_\rho(x_{ij}).$$

Then: after a subsequence,

$$\Longrightarrow \quad (X_{i_j}, d_{i_j}, p_{i_j}) \xrightarrow{\text{pGH}} (X, d, p),$$

where (X, d, p) is again a proper pointed length space.

Proper means that every closed bounded ball is compact.

Note: Proof by TA this afternoon.

Doubling implies uniform covering numbers

Bishop–Gromov implies the hypothesis of Gromov's theorem.

Proposition 1.3: A uniform cover by small balls

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \qquad K \geq 0, \qquad 0 < \rho \leq R.$$

Then

$$\Longrightarrow \quad \overline{B}_R(p) \subset \bigcup_{j=1}^{N} B_\rho(x_j), \qquad N \leq C(n, KR^2)(R/\rho)^n.$$

Hence, for fixed n and K: pointed GH precompactness.

$$(M_i^n, g_i, p_i), \qquad \mathrm{Ric}_{g_i} \geq -(n-1)Kg_i$$

$$\Longrightarrow \quad \text{a pGH-convergent subsequence to a length space } (X, d, p).$$

Goal of compactness: Understand the geoemtry and regularity of (X, d, p).

Proof of Proposition 1.3

$$\mathrm{Ric} \geq -(n-1)Kg, \quad K \geq 0, \quad 0 < \rho \leq R, \qquad B_s := B_s(p).$$

- **Choose a maximal family in $\overline{B}_R$** with pairwise distances at least ρ. Then
$$\overline{B}_R \subset \bigcup_j B_\rho(x_j), \qquad B_{\rho/2}(x_j) \text{ are pairwise disjoint.}$$

- **Compare with the same reference volume.**
$$B_R(p) \subset B_{2R}(x_j) \quad \Longrightarrow \quad \mathrm{Vol}\, B_{\rho/2}(x_j) \geq \frac{V^n_{-K}(\rho/2)}{V^n_{-K}(2R)} \mathrm{Vol}\, B_R(p).$$

- **Sum the disjoint volumes.** The small balls lie in $B_{2R}(p)$, so
$$N \frac{V^n_{-K}(\rho/2)}{V^n_{-K}(2R)} \mathrm{Vol}\, B_R \leq \mathrm{Vol}\, B_{2R} \leq \frac{V^n_{-K}(2R)}{V^n_{-K}(R)} \mathrm{Vol}\, B_R.$$
$$\Longrightarrow \quad N \leq C(n, KR^2)(R/\rho)^n.$$

Motivation to Segment inequality: Recall GH approximations

For compact metric spaces X, Y, seek a map $\Phi : X \to Y$.

- **Control distance distortion: ϵ-isometry**

$$\sup_{x,y \in X} \left| d_X(x,y) - d_Y(\Phi(x), \Phi(y)) \right| \leq \varepsilon.$$

- **Check that the image is dense: ϵ-onto**

$$\sup_{z \in Y} d_Y(z, \Phi(X)) \leq \varepsilon.$$

Both conditions are required

If both hold, Φ is an ε-approximation, and

$$\implies \quad d_{\mathrm{GH}}(X,Y) \leq 2\varepsilon.$$

Question: How do we estimate the information between any pair of points?

Why the segment inequality?

- **Compare a smooth function along a joining geodesic:** Let $\gamma_{xy} : [0, \ell] \to M$ be unit-speed minimizing, where $\ell = d(x, y)$. Then

$$u(y) - u(x) = \int_0^\ell \langle \nabla u(\gamma_{xy}(s)), \dot{\gamma}_{xy}(s) \rangle \, ds.$$

- **Analysis gives a volume integral:** For a nonnegative error density F,

$$\underbrace{\int_{B_{2r}(p)} F \, dV}_{\text{volume information}} \quad \overset{?}{\Longrightarrow} \quad \underbrace{\int_{\gamma_{xy}} F \, ds}_{\text{geodesic information}} .$$

- **The segment inequality below gives geodesic information:** averaged control in x, y.

The segment inequality

Theorem 1.4 (Cheeger-Colding): Segment inequality

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \quad 0 < Kr^2 \leq L, \quad B_s := B_s(p),$$

$$A_1, A_2 \subset B_r \text{ measurable}, \qquad F : B_{2r} \to [0, \infty] \text{ Borel}.$$

For almost every (x, y), let γ_{xy} be unit-speed minimizing; set $I_F(x, y) := \int_{\gamma_{xy}} F$.

Then, with $C = C(n, L)$,

$$\Longrightarrow \quad \int_{A_1 \times A_2} I_F \leq Cr\big(\mathrm{Vol}\, A_1 + \mathrm{Vol}\, A_2\big) \int_{B_{2r}} F.$$

Average form:

$$\fint_{B_r \times B_r} I_F \leq Cr \fint_{B_{2r}} F, \qquad \fint_A F := \frac{1}{\mathrm{Vol}\, A} \int_A F.$$

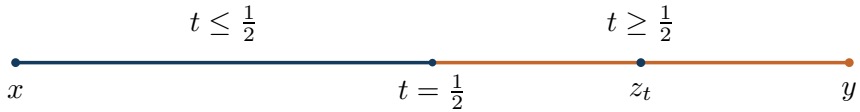
Note: The right hand side is a larger ball.

Euclidean proof of Theorem 1.4: split

$$A_1, A_2 \subset B_r(0) \subset \mathbb{R}^n, \qquad F \geq 0 \text{ on } B_r(0).$$

- **Parametrize the segment and split at $t = 1/2$.**

$$z_t = (1-t)x + ty, \qquad 0 \leq t \leq 1.$$



- **Use the length bound $|x - y| \leq 2r$.**

$$\int_{[x,y]} F = |x-y| \int_0^1 F(z_t)\, dt \leq 2r \int_0^1 F(z_t)\, dt.$$

- Convexity keeps the segment in B_r; extend F by zero outside B_r.

Euclidean proof: change of variables

For $A_1, A_2 \subset B_r(0)$ and $F \geq 0$, write $z_t = (1-t)x + ty$.

- **For $t \geq 1/2$, fix x and change y.**

$$z = (1-t)x + ty, \quad dz = t^n\, dy, \quad (1-t)x + tA_2 \subset B_r, \quad t^{-n} \leq 2^n.$$

$$\int_{A_2} F((1-t)x + ty)\, dy = t^{-n} \int_{(1-t)x+tA_2} F(z)\, dz \leq 2^n \int_{B_r} F(z)\, dz.$$

Integrating in x and over the half-interval gives

$$\int_{A_1} \int_{A_2} \int_{1/2}^{1} F(z_t)\, dt\, dy\, dx \leq 2^{n-1} |A_1| \int_{B_r} F.$$

- **For $t \leq 1/2$, fix y and change x.**

$$dx = (1-t)^{-n}\, dz, \qquad \text{similar argument gives} \quad \int_{A_1 \times A_2} \int_0^{1/2} F(z_t)\, dt \leq 2^{n-1} |A_2| \int_{B_r} F.$$

- **Add both above and use $|x - y| \leq 2r$.**

$$\Longrightarrow \quad \int_{A_1 \times A_2} \int_{[x,y]} F \leq 2^n r \big(|A_1| + |A_2|\big) \int_{B_r} F. \quad \square$$

A consequence of the segment inequality

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \qquad 0 \leq Kr^2 \leq A, \qquad F \geq 0, \quad \lambda > 0.$$

Set $I_F(x,y) := \int_{\gamma_{xy}} F$.

- **Then:** With product volume on $B_r \times B_r$,

$$\frac{\mathrm{Vol}\{(x,y) \in B_r \times B_r : I_F(x,y) > \lambda\}}{(\mathrm{Vol}\, B_r)^2} \leq \frac{1}{\lambda} \fint_{B_r \times B_r} I_F \leq \frac{C(n,A)r}{\lambda} \fint_{B_{2r}} F.$$

- A small volume average gives small line integrals for most pairs.
- **Lecture 2:** choose good endpoints close to the pair being compared, and estimate GH.

Application: Neumann Poincaré

Proposition 1.5: Weak Neumann Poincaré inequality

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \quad 0 \leq Kr^2 \leq A, \quad 1 \leq q < \infty,$$

$$u \in C^\infty(B_{2r}), \qquad u_{B_r} := \fint_{B_r} u.$$

Then

$$\implies \quad \fint_{B_r} |u - u_{B_r}|^q \leq C(n,q,A) r^q \fint_{B_{2r}} |\nabla u|^q.$$

- "Neumann-type": subtract the mean; no boundary condition.
- "Weak Poincaré": the ball on the right is enlarged.

Proof of Proposition 1.5

Under the hypotheses of Proposition 1.5.

- **Integrate along a unit-speed geodesic; apply Hölder.** For $x, y \in B_r$, choose $\gamma(0) = x$, $\gamma(\ell) = y$, $|\dot\gamma| = 1$, $\ell = d(x,y) \le 2r$; then $\gamma \subset B_{2r}$.

$$(u \circ \gamma)' = \langle \nabla u, \dot\gamma \rangle, \qquad u(y) - u(x) = \int_0^\ell \langle \nabla u, \dot\gamma \rangle \, ds,$$

$$|u(y) - u(x)|^q \le \left(\int_0^\ell |\nabla u|(\gamma(s)) \, ds \right)^q \le \ell^{q-1} \int_\gamma |\nabla u|^q \le (2r)^{q-1} \int_\gamma |\nabla u|^q.$$

- **Average in y, apply Jensen, then average in x.**

$$|u(x) - u_{B_r}|^q = \left| \fint_{B_r} (u(x) - u(y)) \, dV_y \right|^q \le \fint_{B_r} |u(x) - u(y)|^q \, dV_y,$$

$$\fint_{B_r} |u - u_{B_r}|^q \le \fint_{B_r \times B_r} |u(x) - u(y)|^q.$$

- **Hence the segment inequality gives the claimed bound.**

$$\begin{aligned} \fint_{B_r} |u - u_{B_r}|^q &\le (2r)^{q-1} \fint_{B_r \times B_r} \int_{\gamma_{xy}} |\nabla u|^q \\ &\le C(n,q) r^q \fint_{B_{2r}} |\nabla u|^q. \end{aligned}$$

Application: Dirichlet Poincaré inequality

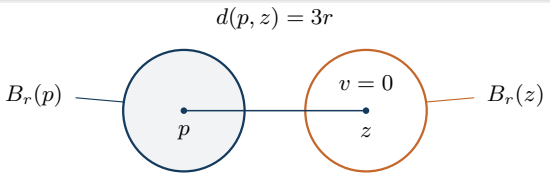
Proposition 1.6: Dirichlet Poincaré inequality

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \quad 0 \leq Kr^2 \leq A, \quad B_{4r}(p) \setminus \overline{B}_{3r}(p) \neq \varnothing,$$
$$1 \leq q < \infty, \qquad v \in C_0^\infty(B_r(p)),$$

Then

$$\Longrightarrow \quad \fint_{B_r(p)} |v|^q \leq C(n,q,A) r^q \fint_{B_r(p)} |\nabla v|^q.$$



The same bound holds for $v \in W_0^{1,q}(B_r(p))$, by density.

Proof sketch of Proposition 1.6

Under Proposition 1.6, extend v by zero outside $B_r(p)$.

- **Choose an exterior ball; compare its volume.**

$$d(p,z) = 3r \quad \Longrightarrow \quad v|_{B_r(z)} = 0, \qquad \operatorname{Vol} B_r(z) \asymp_n \operatorname{Vol} B_r(p).$$

- **Use the Neumann argument with the second endpoint outside.** For $x \in B_r(p)$ and $y \in B_r(z)$,

$$|v(x)|^q = |v(x) - v(y)|^q \le (5r)^{q-1} \int_{\gamma_{xy}} |\nabla v|^q.$$

- **Average in both endpoints; apply the segment inequality.** Volume comparison and the zero extension give

$$\Longrightarrow \quad \fint_{B_r(p)} |v|^q \le C(n,q) r^q \fint_{B_r(p)} |\nabla v|^q.$$

Theorem 1.7: Sobolev–Poincaré inequality

Theorem 1.7(Hajłasz–Koskela): Sobolev–Poincaré inequality

Assume that a complete Riemannian manifold N satisfies, for $D, P > 0$, $\lambda \geq 1$, every $x \in N$, $0 < s \leq R_0$, and $u \in \mathrm{Lip}_{\mathrm{loc}}(N)$,

$$\text{Doubling:} \qquad \mathrm{Vol}\, B_{2s}(x) \leq D\, \mathrm{Vol}\, B_s(x),$$

$$\text{Weak Poincaré:} \qquad \fint_{B_s(x)} |u - u_{B_s(x)}| \leq Ps \left(\fint_{B_{\lambda s}(x)} |\nabla u|^2 \right)^{1/2}.$$

Then there exist $C > 0$ and $\chi > 1$, depending only on D, P, λ, such that, for $0 < r \leq R_0/(10\lambda)$,

$$\Longrightarrow \quad \left(\fint_{B_r(x)} |u - u_{B_r(x)}|^{2\chi} \right)^{1/(2\chi)} \leq Cr \left(\fint_{B_r(x)} |\nabla u|^2 \right)^{1/2}.$$

Strong Sobolev–Poincaré: both integrals are over the same ball, the above holds for metric measure lenght spaces.

Hajłasz–Koskela, *Sobolev met Poincaré* (2000),

Tools for analysis: Sobolev inequality

Theorem 1.8: Sobolev–Poincaré inequality

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \qquad 0 \leq Kr^2 \leq A, \qquad v \in W_0^{1,2}(B_r).$$

Set $\chi = n/(n-2)$ if $n \geq 3$, and $\chi = 2$ if $n = 2$.

Then the local Sobolev inequality is

$$\Longrightarrow \quad \left(\fint_{B_r} |v|^{2\chi} \right)^{1/\chi} \leq C(n, A) \left(r^2 \fint_{B_r} |\nabla v|^2 + \fint_{B_r} |v|^2 \right).$$

- Use the volume-growth Sobolev refinement and zero extension.
- If $B_{4r} \setminus \overline{B}_{3r} \neq \varnothing$, then Proposition 1.6 absorbs the final L^2 term.
- The normalized estimate requires no absolute volume lower bound.

Recall: Mean-value estimates in Euclidean

For a ball contained in a Euclidean domain:

- **Harmonic functions:**

$$\Delta u = 0 \quad \Longrightarrow \quad u(x) = \fint_{B_s(x)} u.$$

- **Nonnegative subharmonic functions:**

$$w \geq 0, \Delta w \geq 0 \quad \Longrightarrow \quad w(x) \leq \fint_{B_r(x)} w \quad \Longrightarrow \quad \sup_{B_r(p)} w \leq 2^n \fint_{B_{2r}(p)} w.$$

- **Manifold analogue for $\Delta w \geq 0$, $w \geq 0$:**

$$\sup_{B_r} w \overset{?}{\leq} C \fint_{B_{2r}} w.$$

Mean-value inequality

Proposition 1.9: mean-value inequality

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \qquad 0 \leq Kr^2 \leq A.$$

Let w be a nonnegative weak subsolution:

$$a \geq 0, \qquad w \geq 0, \qquad w \in W^{1,2}_{\mathrm{loc}}(B_{2r}), \qquad -\Delta w \leq ar^{-2}w.$$

Then:

$$\Longrightarrow \quad \operatorname*{ess\,sup}_{B_r} w \leq C(n,a,A) \fint_{B_{2r}} w.$$

Simple Trick: If $-\Delta w \leq L^2 w$, let $u(x,t) = e^{-Lt}w(x)$, we have

$$-\Delta_{x,t} u = -\Delta u - \partial_t \partial_t u = -e^{Lt}\Delta w - L^2 e^{Lt} w(x) = -e^{Lt}(\Delta w + L^2 w) \leq 0.$$

Proof sketch: energy $\to$ Sobolev $\to$ exponent iteration $\to$ the L^1 mean.

For continuous w, the essential supremum is the ordinary supremum.

Proof sketch of 1.9: the energy step

Input: $\mathrm{Ric} \geq -(n-1)Kg$, $0 \leq Kr^2 \leq A$, $w \geq 0$, $-\Delta w \leq ar^{-2}w$.

- **1. Test with a cutoff times a power.** For $p \geq 2$ and $r \leq \rho < R \leq 2r$, choose

$$\phi = \zeta^2 w^{p-1}, \qquad \zeta \in \mathrm{Lip}_c(B_R), \quad \zeta = 1 \text{ on } B_\rho, \quad |\nabla \zeta| \leq \frac{C}{R-\rho}.$$

  Integration by parts gives

$$(p-1)\int \zeta^2 w^{p-2}|\nabla w|^2 \leq 2\int \zeta w^{p-1}|\nabla \zeta||\nabla w| + ar^{-2}\int \zeta^2 w^p.$$

- **Using Cauchy inequality.** Expanding $\nabla(\zeta w^{p/2})$, we obtain

$$\Longrightarrow \quad \int |\nabla(\zeta w^{p/2})|^2 \leq Cp^2 \int \left(|\nabla \zeta|^2 + ar^{-2}\zeta^2\right) w^p.$$

  Set $\|w\|_{p,B} := (\fint_B w^p)^{1/p}$.

- **2. Sobolev and energy raise p to $p\chi$.** For $r \leq \rho < R \leq 2r$,

$$\Longrightarrow \quad \|w\|_{p\chi, B_\rho} \leq \left[Cp^2\left(\frac{r^2}{(R-\rho)^2} + a + 1\right)\right]^{1/p} \|w\|_{p, B_R}.$$

  Note: $\lim_{q\to\infty} ||w||_{q,B} = ||w||_{\infty, B}$.

Proof sketch of 1.9: iterate exponents

- **3. Iterate on nested balls.** Choose
$$p_j = 2\chi^j \to \infty, \qquad R_j = r + r2^{-j} \downarrow r.$$
The product of iteration factors is finite, since
$$\sum_{j\geq 0} \frac{\log[Cp_j^2(4^{j+1}+a+1)]}{p_j} < \infty.$$

- **4. Interpolation and absorption on nested balls:**
$$\operatorname*{ess\,sup}_{B_r} w \leq C\left(\fint_{B_{2r}} w^2\right)^{1/2} \quad \Longrightarrow \quad \operatorname*{ess\,sup}_{B_r} w \leq C \fint_{B_{2r}} w.$$
Here $C = C(n, a)$; this last step is a standard local PDE argument.

Proposition 1.10: local elliptic estimates

Assume $\mathrm{Ric} \geq -(n-1)Kg$ and $0 \leq Kr^2 \leq A$, then the following holds

Harmonic functions. If $u \in C^\infty(B_{2r})$, $\Delta u = 0$, and $c \in \mathbb{R}$, then

$$\implies \quad \sup_{B_r} |\nabla u|^2 \leq C(n,A) \fint_{B_{3r/2}} |\nabla u|^2 \leq C(n,A) r^{-2} \fint_{B_{2r}} |u-c|^2.$$

Poisson subsolutions. If $u \in W^{1,2}_{\mathrm{loc}}(B_{2r})$, $-\Delta u \leq f$ weakly in B_{2r}, and $f_+ \in L^\infty(B_{2r})$, then

$$\implies \quad \operatorname*{ess\,sup}_{B_r} u \leq C(n,A) \fint_{B_{2r}} u_+ + C(n,A) r^2 \|f_+\|_{L^\infty(B_{2r})}.$$

Here $u_+ := \max\{u, 0\}$ and $f_+ := \max\{f, 0\}$.

The above is a direct consequence of mean value inequality.

Proof of 1.10:

Assume $\mathrm{Ric} \geq -(n-1)Kg$, $0 \leq Kr^2 \leq A$, $\Delta u = 0$ in B_{2r}, and $c \in \mathbb{R}$.

- **1. Test the harmonic equation.** Choose

$$\phi = \eta^2(u-c), \quad \eta = 1 \text{ on } B_{3r/2}, \quad \operatorname{supp}\eta \subset B_{2r}, \quad |\nabla\eta| \leq C/r.$$

Integration by parts and Cauchy inequality give

$$\begin{aligned}\int \eta^2 |\nabla u|^2 &= -2\int \eta(u-c)\langle \nabla\eta, \nabla u\rangle \\ &\leq \tfrac{1}{2}\int \eta^2|\nabla u|^2 + 2\int |\nabla\eta|^2 |u-c|^2.\end{aligned}$$

- **By volume doubling.**

$$\begin{aligned}&\int_{B_{3r/2}} |\nabla u|^2 \leq Cr^{-2}\int_{B_{2r}} |u-c|^2, \\ \Longrightarrow \quad &⨍_{B_{3r/2}} |\nabla u|^2 \leq Cr^{-2} ⨍_{B_{2r}} |u-c|^2.\end{aligned}$$

Proof of 1.10: Bochner and mean values

Set $w := |\nabla u|^2$.

- **2. Bochner gives a nonnegative subsolution.**

  $$\tfrac{1}{2}\Delta w = |\operatorname{Hess} u|^2 + \operatorname{Ric}(\nabla u, \nabla u) \geq -(n-1)Kw.$$

  Hence $-\Delta w \leq 2(n-1)r^{-2}w$.

- **3. Apply Proposition 1.9 and the energy bound.**

  $$\Longrightarrow \quad \sup_{B_r} |\nabla u|^2 \leq C \fint_{B_{3r/2}} |\nabla u|^2 \leq Cr^{-2} \fint_{B_{2r}} |u-c|^2.$$

  Use the same mean-value proof with the fixed radius ratio $2/3$.

Proof of 1.10: setting up the Poisson estimate

Assume

$$\mathrm{Ric} \geq -(n-1)Kg, \quad 0 \leq Kr^2 \leq A, \quad u \in W^{1,2}_{\mathrm{loc}}(B_{2r}), \quad -\Delta u \leq f.$$

- **Set:** $u_+ := \max\{u, 0\}, \quad f_+ := \max\{f, 0\}, \quad F := \|f_+\|_{L^\infty(B_{2r})} < \infty$. Then
$$-\Delta u_+ \leq f_+ \leq F.$$
- Set
$$w := u_+ + r^2 F \geq 0, \qquad \text{then } -\Delta w \leq F \leq r^{-2} w.$$
- **3. Apply Proposition 1.9 with $a = 1$:**
$$\operatorname*{ess\,sup}_{B_r} u \leq \operatorname*{ess\,sup}_{B_r} w \leq C \fint_{B_{2r}} w = C \fint_{B_{2r}} u_+ + Cr^2 F.$$

If $\Delta u = f \in L^\infty$, apply both signs: $\operatorname{ess\,sup}_{B_r} |u| \leq C \fint_{B_{2r}} |u| + Cr^2 \|f\|_{L^\infty(B_{2r})}$.

Hessian L^2 estimate

Lemma: Hessian L^2 estimate

Assume $\mathrm{Ric} \geq -(n-1)Kg$, $0 \leq Kr^2 \leq A$, and $\Delta u = 0$ in B_{2r}. **Then**

$$\implies \quad r^2 \fint_{B_r} |\operatorname{Hess} u|^2 \leq C(n, A) \fint_{B_{3r/2}} |\nabla u|^2.$$

Proof.

- **1. Choose a cutoff:** $\eta = 1$ on B_r, $\operatorname{supp} \eta \subset B_{3r/2}$, $|\nabla \eta| \leq C/r$.
- **2. Integrate Bochner against η^2 and use Cauchy:**

$$\begin{aligned} \int \eta^2 |\operatorname{Hess} u|^2 &\leq 2 \int \eta |\nabla \eta| |\operatorname{Hess} u| |\nabla u| + (n-1)K \int \eta^2 |\nabla u|^2 \\ &\leq \tfrac{1}{2} \int \eta^2 |\operatorname{Hess} u|^2 + Cr^{-2} \int_{B_{3r/2}} |\nabla u|^2. \end{aligned}$$

- **3. By volume doubling.** This proves bounded Hessian energy; smallness needs an additional geometric input.

Note: In the application, we can get small L^2-Hessian estimate.

Application of 1.4: Hessian line integrals

Assume $\mathrm{Ric} \geq -(n-1)Kg$, $0 \leq Kr^2 \leq 1$, and $u \in C^2(B_{2r})$. For unit-speed minimizing γ_{xy}, $x, y \in B_r$, define

$$\delta := r \left(\fint_{B_{2r}} |\operatorname{Hess} u|^2 \right)^{1/2}, \qquad I(x,y) := \int_{\gamma_{xy}} |\operatorname{Hess} u|.$$

- **1. Segment inequality and Cauchy–Schwarz:**

$$\Longrightarrow \quad \fint_{B_r \times B_r} I \leq Cr \fint_{B_{2r}} |\operatorname{Hess} u| \leq C\delta.$$

- **2. Markov's inequality controls the bad pairs:**

$$\frac{\mathrm{Vol}\{(x,y) \in B_r \times B_r : I(x,y) > \tau\}}{(\mathrm{Vol}\, B_r)^2} \leq \frac{C\delta}{\tau} \qquad (\tau > 0).$$

If $0 < \delta < 1$, choose $\tau = \sqrt{\delta}$; the bad fraction is at most $C\sqrt{\delta}$.

Almost affine along a geodesic

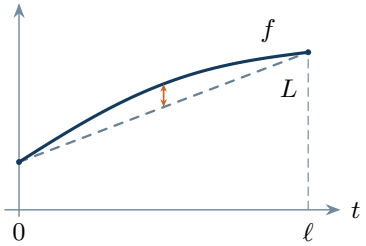
Assume $u \in C^2(B_{2r})$ and $\gamma : [0, \ell] \to B_{2r}$ is unit-speed minimizing, with

$$0 < \ell \le 2r, \qquad \int_\gamma |\operatorname{Hess} u| \le \varepsilon.$$

Set $f := u \circ \gamma$. Define its endpoint chord:

$$L(t) := \left(1 - \frac{t}{\ell}\right) f(0) + \frac{t}{\ell} f(\ell).$$

Then u is almost affine along this geodesic:



$$\Longrightarrow \quad |f(t) - L(t)| \le \min\{t, \ell - t\}\varepsilon \le r\varepsilon \qquad (0 \le t \le \ell).$$

The remaining calculation is one-dimensional and needs no Ricci hypothesis.

Proof

Input: $f = u \circ \gamma$, its endpoint chord L, $\int_\gamma |\operatorname{Hess} u| \le \varepsilon$, and $0 < \ell \le 2r$.

- **1. Compare the derivative with its average.**

$$f'' = \operatorname{Hess} u(\dot\gamma, \dot\gamma), \qquad L' = \frac{1}{\ell}\int_0^\ell f'(s)\, ds.$$

Hence

$$\Longrightarrow \quad |f'(t) - L'| \le \int_0^\ell |f''(s)|\, ds \le \varepsilon.$$

- **2. Integrate from the nearer endpoint.** Since $f - L = 0$ at both endpoints,

$$\Longrightarrow \quad |f(t) - L(t)| \le \min\{t, \ell - t\}\varepsilon \le r\varepsilon.$$

Next: construct small-Hessian functions, then use first variation to obtain distance estimates and GH approximations.