Problem Sheet: Lectures 3-4

Alexander O. Ivanov's minicourse — Compact metric spaces in the Gromov–Hausdorff topology

Lectures 3-4: Compact metric spaces in the Gromov–Hausdorff topology.

Continuity and covering data

The diameter is 2-Lipschitz:

$$|\operatorname{diam} X - \operatorname{diam} Y| \le 2d_{\mathrm{GH}}(X,Y).$$

The density $\operatorname{dens}(X)$ is the least cardinality of an everywhere dense subset. The covering number $N_\varepsilon(X)$ is the least size of a finite ε-net, with value ∞ if no such net exists. Under GH convergence,

$$\operatorname{dens}(X) \le \liminf_n \operatorname{dens}(X_n), \qquad N_\varepsilon(X) \le \liminf_n N_\varepsilon(X_n).$$

1. Compute $N_\varepsilon(X)$ for (a) $X = [0,1]$; (b) the n-point simplex Δ_n; (c) the Euclidean unit circle. State carefully what happens at the critical radii.

2. Prove the scaling rules

$$\operatorname{dens}(\lambda X) = \operatorname{dens}(X), \qquad N_\varepsilon(\lambda X) = N_{\varepsilon/\lambda}(X).$$

Apply them to $\lambda\Delta_n$ and the circle of radius λ, and compute $\lim_{\varepsilon\to 0+} \varepsilon N_\varepsilon(\lambda S^1)$.

The compact GH space and precompactness

The class $\mathcal{M}$ of compact metric spaces up to isometry is a complete, separable, geodesic metric space with the distance d_{GH}. A family $\mathcal{C} \subset \mathcal{M}$ is precompact iff

$$\sup_{X\in\mathcal{C}} \operatorname{diam} X < \infty, \qquad \sup_{X\in\mathcal{C}} N_\varepsilon(X) < \infty \quad \text{for every } \varepsilon > 0,$$

where $N_\varepsilon(X)$ is the least size of an ε-net in X. If $T \subset X$ is a δ-net, then $d_{\mathrm{GH}}(X,T) \le \delta$.

3. Decide which families are precompact in $\mathcal{M}$. For every negative answer, identify the failed condition of Gromov's criterion.

(a) all nonempty compact subsets of $[0,1]$ with the induced metric;
(b) $\{\Delta_n : n \in \mathbb{N}\}$;
(c) $\{\lambda S^1 : 0 < \lambda \le 1\}$, with the intrinsic metric;
(d) all compact X such that $\operatorname{diam} X \le D$.

4. (a) Let X_n be the square grid of mesh $1/n$ in $[0,1]^2$, equipped with the Euclidean metric. Prove that $X_n \to [0,1]^2$ in GH distance and give an explicit upper bound for $d_{\mathrm{GH}}(X_n, [0,1]^2)$.

(b) Prove that $d_{\mathrm{GH}}(\mathbb{R}^2, \mathbb{Z}^2) = d_{\mathrm{H}}(\mathbb{R}^2, \mathbb{Z}^2)$.

Limits of compact spaces

If compact spaces X_n converge to X in GH distance, then X is totally bounded, its completion $\overline{X}$ is compact, and $d_{\mathrm{GH}}(X, \overline{X}) = 0$. Completeness itself is not preserved because GH is a pseudometric on arbitrary metric spaces.

5. Come up with

(a) a sequence $X_n \xrightarrow[GH]{} \Delta_1$ such that the finite numbers $\operatorname{dens}(X_n)$ are unbounded;
(b) a sequence of compact Riemannian manifolds of uniformly bounded curvature, GH-converging to a manifold of strictly lower dimension.

6. (a) Let X_n be a Euclidean sphere S^2 with the removed ball of radius $\frac{1}{n}$, equipped with the corresponding intrinsic metric. Show that $\lim_{n\to\infty} d_{\mathrm{GH}}(X_n, S^2) = 0$.

(b) Show that, for a one-dimensional sphere S^1 the similar statement is false.

Difficult problems

1. Show that for a converging sequence of Alexandrov spaces of curvature $\ge k$, its limit is also an Alexandrov space of curvature $\ge k$.

2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact length metric spaces that GH-converges to a circle S^1 with the natural intrinsic metric. Show that for sufficiently large $n \in \mathbb{N}$, the spaces X_n cannot be simply connected. What if X_n are not length spaces?