Seminar Problems for Lectures 3–4

Compact metric spaces in the Gromov–Hausdorff topology

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Task 3.1: Covering Numbers

Problem 1

Compute $N_\varepsilon(X)$ for

(a) $X = [0, 1]$;
(b) the n-point simplex Δ_n;
(c) the Euclidean unit circle.

State carefully what happens at the critical radii.

Task 3.1: Solution

Solution

For closed ε-balls,

$$N_\varepsilon([0,1]) = \max\left\{1, \left\lceil \frac{1}{2\varepsilon} \right\rceil \right\},$$

with critical radii $\varepsilon = \frac{1}{2m}$. For the n-point simplex,

$$N_\varepsilon(\Delta_n) = \begin{cases} n, & 0 < \varepsilon < 1, \\ 1, & \varepsilon \geq 1, \end{cases}$$

with critical radius $\varepsilon = 1$. For the Euclidean unit circle,

$$N_\varepsilon(S^1) = \min\left\{ m \geq 1 : 2\sin\frac{\pi}{2m} \leq \varepsilon \right\}.$$

The critical radii are $\varepsilon_m = 2\sin\frac{\pi}{2m}$; at $\varepsilon = \varepsilon_m$ one has $N_\varepsilon(S^1) = m$.

Task 3.2: Scaling Rules

Problem 2

Prove the scaling rules

$$\operatorname{dens}(\lambda X) = \operatorname{dens}(X), \qquad N_\varepsilon(\lambda X) = N_{\varepsilon/\lambda}(X).$$

Apply them to $\lambda\Delta_n$ and the circle of radius λ, and compute

$$\lim_{\varepsilon \to 0+} \varepsilon N_\varepsilon(\lambda S^1).$$

Task 3.2: Solution

Solution

Scaling $x \mapsto \lambda x$ preserves dense subsets and carries an ε-net to a $\lambda\varepsilon$-net. Hence, for $\lambda > 0$,

$$\operatorname{dens}(\lambda X) = \operatorname{dens}(X), \qquad N_\varepsilon(\lambda X) = N_{\varepsilon/\lambda}(X).$$

Thus

$$N_\varepsilon(\lambda \Delta_n) = \begin{cases} n, & 0 < \varepsilon < \lambda, \\ 1, & \varepsilon \geq \lambda, \end{cases} \qquad \operatorname{dens}(\lambda \Delta_n) = n.$$

For the circle,

$$N_\varepsilon(\lambda S^1) = \min\left\{ m \geq 1 : 2\lambda \sin\frac{\pi}{2m} \leq \varepsilon \right\} \sim \frac{\pi\lambda}{\varepsilon}.$$

Therefore

$$\lim_{\varepsilon \to 0+} \varepsilon N_\varepsilon(\lambda S^1) = \pi\lambda.$$

Task 3.3: Precompact Families

Problem 3

Decide which families are precompact in $\mathcal{M}$. For every negative answer, identify the failed condition of Gromov's criterion.

(a) all nonempty compact subsets of $[0, 1]$ with the induced metric;
(b) $\{\Delta_n : n \in \mathbb{N}\}$;
(c) $\{\lambda S^1 : 0 < \lambda \leq 1\}$, with the intrinsic metric;
(d) all compact X such that $\operatorname{diam} X \leq D$.

Task 3.3: Solution

Solution

(a) Precompact. Every $X \subset [0,1]$ has $\operatorname{diam} X \le 1$ and $N_\varepsilon(X) \le N_\varepsilon([0,1]) < \infty$.

(b) Not precompact. The diameters are bounded ($\operatorname{diam} \Delta_n = 1$), but $N_\varepsilon(\Delta_n) = n$ for every $0 < \varepsilon < 1$, so the uniform covering condition fails.

(c) Precompact. The intrinsic diameter is $\pi\lambda \le \pi$, and $N_\varepsilon(\lambda S^1) = N_{\varepsilon/\lambda}(S^1)$ is uniformly bounded for $\lambda \le 1$.

(d) Not precompact for $D > 0$. The diameter bound holds, but uniform covering numbers fail: $D\Delta_n$ has diameter D and $N_{D/2}(D\Delta_n) = n$.

Task 3.4: Grids and Lattices

Problem 4

(a) Let X_n be the square grid of mesh $1/n$ in $[0,1]^2$, equipped with the Euclidean metric. Prove that $X_n \to [0,1]^2$ in GH distance and give an explicit upper bound for $d_{\mathrm{GH}}(X_n, [0,1]^2)$.

(b) Prove that

$$d_{\mathrm{GH}}(\mathbb{R}^2, \mathbb{Z}^2) = d_H(\mathbb{R}^2, \mathbb{Z}^2).$$

Task 3.4: Solution

Solution

(a) The grid

$$X_n = \left\{ \left(\frac{i}{n}, \frac{j}{n} \right) : 0 \le i, j \le n \right\}$$

is a δ-net in $Q = [0,1]^2$ with

$$\delta = \frac{\sqrt{2}}{2n}.$$

Hence

$$d_{\mathrm{GH}}(X_n, Q) \le \delta = \frac{\sqrt{2}}{2n} \longrightarrow 0.$$

Task 4.1: Degenerating Sequences

Problem 5

Come up with

(a) a sequence $X_n \xrightarrow[GH]{} \Delta_1$ such that the finite numbers $\mathrm{dens}(X_n)$ are unbounded;

(b) a sequence of compact Riemannian manifolds of uniformly bounded curvature, GH-converging to a manifold of strictly lower dimension.

Task 4.1: Solution

Solution

(a) Take $X_n = \frac{1}{n}\Delta_n$. Then

$$\operatorname{dens}(X_n) = n, \qquad d_{\mathrm{GH}}(X_n, \Delta_1) = \frac{\operatorname{diam} X_n}{2} = \frac{1}{2n} \longrightarrow 0.$$

(b) Take the flat product tori

$$X_n = S^1 \times (S^1)_{1/n},$$

where $(S^1)_{1/n}$ is the circle of radius $1/n$. Their Gaussian curvature is identically 0, hence uniformly bounded. As $n \to \infty$ the second factor collapses and

$$X_n \xrightarrow[GH]{} S^1,$$

a manifold of dimension $1 < 2$.

Task 4.2: Punctured Spheres

Problem 6

(a) Let X_n be the Euclidean sphere S^2 with the removed ball of radius $1/n$, equipped with the corresponding intrinsic metric. Show that

$$\lim_{n\to\infty} d_{\mathrm{GH}}(X_n, S^2) = 0.$$

(b) Show that, for the one-dimensional sphere S^1, the similar statement is false.

Task 4.2: Solution

Solution

(a) The removed cap has diameter $O(1/n)$. Pair each removed point with a boundary point and use the identity elsewhere. The detour around the cap changes intrinsic distances by $O(1/n)$, so the corresponding relation has distortion $O(1/n)$. Hence

$$d_{\mathrm{GH}}(X_n, S^2) \leq \frac{C}{n} \longrightarrow 0.$$

(b) For S^1, removing an arc of length $2/n$ with the intrinsic metric leaves an interval of length

$$2\pi - \frac{2}{n}.$$

Its diameter tends to 2π, while $\operatorname{diam} S^1 = \pi$. Therefore

$$d_{\mathrm{GH}}(X_n, S^1) \geq \frac{1}{2} \left| \operatorname{diam} X_n - \operatorname{diam} S^1 \right| \longrightarrow \frac{\pi}{2} > 0,$$

so X_n does not converge to S^1 in GH distance.