# Chapter 2

# Properties of the Gromov–Hausdorff distance

## 2.1 Continuity of functionals

**Theorem 2.1.** *The diameter function* $\operatorname{diam} : \mathcal{GH} \to [0,\infty]$ *is continuous on* $\mathcal{GH}$ *in the following sense: if* $X_n \to X$ *in* $\mathcal{GH}$*, then* $\operatorname{diam} X_n \to \operatorname{diam} X$ *in the extended real line* $[0,\infty]$.

*Proof.* Let's consider several cases.

*Case 1:* $\operatorname{diam} X < \infty$. Then for any $r > 0$ there is $N$ such that for all $n \geq N$ $d_{GH}(X_n, X) < r$ holds. Therefore, there is a realization of $Z$ with subsets $X_n' \cong X_n$, $X' \cong X$ and $d_H(X_n', X') < r$. Then

$$\operatorname{diam} X_n = \operatorname{diam} X_n' \leq \operatorname{diam} X' + 2r = \operatorname{diam} X + 2r,$$

and similarly $\operatorname{diam} X \leq \operatorname{diam} X_n + 2r$. Thus, $|\operatorname{diam} X_n - \operatorname{diam} X| \leq 2r$ for all sufficiently large $n$. This means $\operatorname{diam} X_n \to \operatorname{diam} X$.

*Case 2:* $\operatorname{diam} X = \infty$. Let us show that $\operatorname{diam} X_n \to \infty$. For any $M > 0$ we choose points $a, b \in X$ with $|ab| > M + 2$. Since $X_n \to X$, for sufficiently large $n$ there is a realization with $d_H(X_n', X') < 1$. Then there are $a_n, b_n \in X_n$ with $|a_n b_n| > M$. Therefore, $\operatorname{diam} X_n > M$ for all sufficiently large $n$. This means $\operatorname{diam} X_n \to \infty$. □

## 2.2 Density and Covering Numbers

**Definition 2.2.** For a metric space $X$, its ***density*** $\operatorname{dens}(X)$ is the smallest cardinal number $\kappa$ for which there exists an everywhere dense subset $S \subset X$ of the cardinality $\kappa$.

**Definition 2.3.** For the metric space $X$ and $\varepsilon > 0$ we define ***$\varepsilon$-covering number***

$$N_\varepsilon(X) := \inf\bigl\{n \in \mathbb{N} \mid \exists \text{ finite } \varepsilon\text{-network in } X \text{ of cardinality } \leq n\bigr\},$$

where $\inf \emptyset = \infty$.

**Note 2.4.** For separable spaces $\operatorname{dens}(X) \leq \aleph_0$. For a discrete space of cardinality $\kappa$ we have $\operatorname{dens}(X) = \kappa$. Space is ***totally bounded*** if and only if $N_\varepsilon(X) < \infty$ for all $\varepsilon > 0$.

## 2.3 Semi-continuity of density and covering numbers

**Theorem 2.5.** *The density function* $\operatorname{dens}\colon \mathcal{GH} \to \mathbf{Card}$ *is upper semi-continuous with respect to GH-convergence, that is, for any sequence* $X_n \to X$ *in* $\mathcal{GH}$ *with* $\operatorname{dens} X_n \leq \kappa$*, one has* $\operatorname{dens} X \leq \kappa$.

*Proof.* Let $\kappa := \liminf_{n\to\infty} \operatorname{dens}(X_n)$. Let us construct a dense subset of $X$ of cardinality at most $\kappa$.

By definition of the lower limit:

$$\kappa = \sup_{m\in\mathbb{N}} \inf_{n\geq m} \operatorname{dens}(X_n).$$

For each $m \in \mathbb{N}$ we denote $\kappa_m := \inf_{n\geq m} \operatorname{dens}(X_n)$. Then $\kappa_m \leq \kappa$ for all $m$, and the sequence $\{\kappa_m\}$ is nondecreasing and converges to $\kappa$.

Let us show that for each $m$ there exists $n \geq m$ such that $\operatorname{dens}(X_n) = \kappa_m$. Indeed, $\kappa_m = \inf_{n\geq m} \operatorname{dens}(X_n)$. Since the class of all cardinals is completely ordered, there is a smallest element in the set $\{\operatorname{dens}(X_n) : n \geq m\}$, which by definition is equal to $\kappa_m$. Consequently, there is $n_m \geq m$ such that $\operatorname{dens}(X_{n_m}) = \kappa_m \leq \kappa$.

Thus, one can choose a subsequence $n_m \to \infty$ such that $\operatorname{dens}(X_{n_m}) \leq \kappa$ for all $m \in \mathbb{N}$ (moreover, $\operatorname{dens}(X_{n_m}) = \kappa_{n_m} \leq \kappa$).

For each $k$ we choose an everywhere dense subset $S_{n_m} \subset X_{n_m}$ of cardinality $\leq \kappa$.

By theorem 1.16, for each $m$ there is a correspondence $R_{n_m} \in \mathcal{R}(X_{n_m}, X)$ with $\operatorname{dis} R_{n_m} < 2d_{GH}(X_{n_m}, X) + 1/n_m$. For each $s \in S_{n_m}$ we choose $x_s \in X$ with $(s, x_s) \in R_{n_m}$. Let's put

$$S := \big\{x_s \mid s \in S_{n_m},\ m \in \mathbb{N}\big\}.$$

Let us estimate the cardinality of $S$. We have

$$\#S \leq \sum_{m=1}^{\infty} \#S_{n_m} \leq \aleph_0 \cdot \kappa.$$

If $\kappa$ is infinite, then $\aleph_0 \cdot \kappa = \kappa$. If $\kappa$ is finite, then $\#S \leq \aleph_0$.

Let us show that $S$ is dense in $X$. Let us take arbitrary $x \in X$ and $\varepsilon > 0$. Since $X_{n_m} \to X$ in $\mathcal{GH}$, for sufficiently large $m$ we have $d_{GH}(X_{n_m}, X) < \varepsilon/4$ and $1/n_m < \varepsilon/4$, and hence, $\operatorname{dis} R_{n_m} < \varepsilon/2$. For a point $x \in X$ we choose $x'_{n_m} \in X_{n_m}$ such that $(x, x'_{n_m}) \in R_{n_m}$.

Since $S_{n_m}$ is dense in $X_{n_m}$, there exists $s \in S_{n_m}$ with $|s - x'_{n_m}| < \varepsilon/4$. Then for the $R_{n_m}$-corresponding point $x_s \in S$ we obtain

$$|x - x_s| \leq |x - x'_{n_m}| + |x'_{n_m} - s| + |s - x_s| < \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon.$$

Consequently, $S$ is dense in $X$, and $\operatorname{dens}(X) \leq \kappa$. □

**Theorem 2.6.** *For each $\varepsilon > 0$ the function $N_\varepsilon : \mathcal{GH} \to \mathbb{N} \cup \{\infty\}$ is upper semi-continuous with respect to GH-convergence:*

$$N_\varepsilon(X) \leq \liminf_{n\to\infty} N_\varepsilon(X_n)$$

*for any sequence $X_n \to X$ in $\mathcal{GH}$.*

*Proof.* The proof is similar to the proof of the theorem 2.5 with the dense subset replaced by a finite $\varepsilon$-net and $\kappa$ by a finite $N_\varepsilon(X_n)$. □

## 2.4 Consequences: closedness of classes

**Consequence 2.7.** *The class of separable metric spaces is closed under GH-convergence.*

*Proof.* This is a special case of the theorem 2.5 with $\kappa = \aleph_0$. □

**Consequence 2.8.** *The class of totally bounded spaces is closed under GH-convergence.*

*Proof.* Let $X_n$ be totally bounded for all $n$ and $X_n \to X$. We fix $\varepsilon > 0$. According to the theorem 2.6,

$$N_\varepsilon(X) \le \liminf_{n\to\infty} N_\varepsilon(X_n).$$

Since each $X_n$ is totally bounded, $N_\varepsilon(X_n) < \infty$ for all $n$. Therefore, $N_\varepsilon(X) < \infty$. This means that $X$ has a finite $\varepsilon$-net for every $\varepsilon > 0$, i.e. $X$ is totally bounded. □

**Consequence 2.9.** *Let all the spaces $X_n$ be compact and let $X_n \to X$ in $\mathcal{GH}$. Then $X$ is totally bounded, its completion $\overline{X}$ is compact, and $d_{GH}(X, \overline{X}) = 0$. In particular, the class of compact spaces is closed in the class of complete spaces: if the limit $X$ is complete, then $X$ itself is compact.*

*Proof. Total boundedness.* A compact space is complete and totally bounded, and the class of totally bounded spaces is closed under $GH$-convergence by Corollary 2.8. Thus $X$ is totally bounded: for every $\varepsilon > 0$ there is a finite $\varepsilon$-net in $X$.

*Completion.* By $\overline{X}$ we denote the completion of $X$, i.e. the complete space containing an isometric copy of $X$ as an everywhere dense subset. A finite $\varepsilon/2$-net $S \subset X$ is a finite $\varepsilon$-net for $\overline{X}$: for every $y \in \overline{X}$ there is $x \in X$ with $|xy| \le \varepsilon/2$, and for a suitable $s \in S$ we get $|ys| \le |yx| + |xs| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $\overline{X}$ is totally bounded and, being complete, it is compact.

*The distance to the completion is zero.* We fix $\varepsilon > 0$ and put

$$R_\varepsilon := \big\{(x, y) \in X \times \overline{X} : |xy| < \varepsilon\big\}.$$

This is a correspondence: its projection onto $X$ contains the pairs $(x, x)$, and its projection onto $\overline{X}$ is all of $\overline{X}$, because $X$ is dense in $\overline{X}$. For any $(x, y), (x', y') \in R_\varepsilon$ the triangle inequality gives

$$|xx'| \le |xy| + |yy'| + |y'x'| < |yy'| + 2\varepsilon,$$

whence $\big||xx'| - |yy'|\big| < 2\varepsilon$ and $\operatorname{dis} R_\varepsilon \le 2\varepsilon$. By theorem 1.16, $d_{GH}(X, \overline{X}) \le \varepsilon$, and since $\varepsilon > 0$ is arbitrary, $d_{GH}(X, \overline{X}) = 0$.

*The case of a complete limit.* If $X$ is complete, then $\overline{X}$ is isometric to $X$, and therefore $X$ is compact as well. □

*Note.* Completeness, and hence compactness, is not inherited by an arbitrary $GH$-limit. Indeed, according to the remark in Chapter 1, $d_{GH}\big([0, 1], (0, 1)\big) = 0$, so the non-complete space $(0, 1)$ is a $GH$-limit of the constant sequence of compact spaces $[0, 1]$. The reason is that $d_{GH}$ is only a pseudometric: the convergence $X_n \to X$ determines the limit only up to zero distance, whereas a space at zero distance from a compact space need not be compact itself (it may be its everywhere dense subset). What the convergence does preserve is total boundedness, and Corollary 2.9 says that passing to the completion, which does not change the $GH$-limit, restores compactness. In other words, in the quotient of $\mathcal{GH}$ by the relation $d_{GH} = 0$, the class of compact spaces is closed. The completeness of the class $\mathcal{GH}$ (theorem 1.21) is a different assertion: it guarantees only that every fundamental sequence of spaces has a limit in $\mathcal{GH}$, and says nothing about which properties of the terms pass to that limit.

*Note.* A direct correspondence argument, which one might be tempted to use here, does not work. Namely, one could fix $n$ with $d_{GH}(X_n, X) < \infty$ and a correspondence $R \in \mathcal{R}(X_n, X)$ with $r := \operatorname{dis} R < \infty$, and assign to each term of a fundamental sequence $(x_k) \subset X$ a point $y_k \in X_n$ such that $(y_k, x_k) \in R$. The definition of the distortion yields

$$|y_k y_l| \le |x_k x_l| + r,$$

and this estimate is ***not*** sufficient: its right-hand side tends to the fixed number $r$ rather than to zero, so the sequence $(y_k)$ need not be fundamental (for instance, all its pairwise distances could be equal to $r$). This is not an accident of the method: by the previous note, the claim in question is false without the assumption that the limit is complete.

## 2.5 Examples of calculations

**Theorem 2.10.** *For any $X \in \mathcal{GH}$ and single-point space $\Delta_1$:*

$$d_{GH}(X, \Delta_1) = \frac{1}{2} \operatorname{diam} X.$$

*Proof.* The only correspondence between $X$ and $\Delta_1$ is $R = X \times \Delta_1$. Its distortion is equal to $\operatorname{diam} X$. By the theorem 1.16 we obtain $d_{GH}(X, \Delta_1) = \frac{1}{2} \operatorname{diam} X$. □

**Theorem 2.11.** *Let $\Delta_n$ be an $n$-point simplex (all non-zero distances are equal to 1, where $n$ is a cardinal number). Then for any $X \in \mathcal{GH}$:*

$$d_{GH}(X, \Delta_n) \le \frac{1}{2} \max\{\operatorname{diam} X, 1\}.$$

*In two cases the estimate can be replaced by exact formulas; the answer depends on how many points $X$ has. Namely, if $X$ is a finite space with $\#X = n \ge 2$, then*

$$2d_{GH}(X, \Delta_n) = \max\bigl\{1 - \epsilon(X), \operatorname{diam} X - 1\bigr\},$$

*where $\epsilon(X) := \min_{x \ne y} |xy|$ is the least non-zero distance in $X$, while for a bounded $X$ with $\#X < n$*

$$2d_{GH}(X, \Delta_n) = \max\{1, \operatorname{diam} X - 1\}.$$

*In particular, for the single-point space $\Delta_1$ and $n \ge 2$ the second formula gives*

$$d_{GH}(\Delta_1, \Delta_n) = \frac{1}{2}.$$

*Proof. The inequality* is not a new statement: it is the estimate (2) of Proposition 1.17 written for the pair $X$, $\Delta_n$. Since $\operatorname{diam} \Delta_n = 1$, that estimate reads

$$2d_{GH}(X, \Delta_n) \le \max\{\operatorname{diam} X, \operatorname{diam} \Delta_n\} = \max\{\operatorname{diam} X, 1\}.$$

So it remains to prove the two equalities.

*1. The case of a bounded $X$ with $\#X < n$.*

Lower bound. Let $R \in \mathcal{R}(X, \Delta_n)$ be an arbitrary correspondence. The projection of $R$ onto $\Delta_n$ is all of $\Delta_n$, and $\#X < \#\Delta_n$, therefore some point $x \in X$ has at least two images: there are distinct $y, y' \in \Delta_n$ with $(x, y), (x, y') \in R$. These two pairs of the correspondence give the term $\bigl||xx| - |yy'|\bigr| = |0 - 1| = 1$ in the distortion, hence

$$\operatorname{dis} R \ge 1.$$

Further, let $\delta > 0$ and let $x, x' \in X$ be such that $|xx'| > \operatorname{diam} X - \delta$. Since the projection of $R$ onto $X$ is all of $X$, there are $y, y' \in \Delta_n$ with $(x, y), (x', y') \in R$. If $y = y'$, then $\operatorname{dis} R \ge |xx'| > \operatorname{diam} X - \delta$; if $y \ne y'$, then $\operatorname{dis} R \ge \bigl||xx'| - 1\bigr| \ge |xx'| - 1 > \operatorname{diam} X - 1 - \delta$. Since $\delta > 0$ is arbitrary, in both cases

$$\operatorname{dis} R \ge \operatorname{diam} X - 1.$$

Thus, for every correspondence $R \in \mathcal{R}(X, \Delta_n)$,

$$\operatorname{dis} R \geq \max\{1, \operatorname{diam} X - 1\}, \qquad \text{whence} \qquad 2d_{GH}(X, \Delta_n) \geq \max\{1, \operatorname{diam} X - 1\}.$$

Upper bound. Choose a point $x_0 \in X$; then $\#\big(X \setminus \{x_0\}\big) \leq \#X < n = \#\Delta_n$, so $\Delta_n$ contains a subset $\Delta'$ of the same cardinality as $X \setminus \{x_0\}$, and $\Delta'' := \Delta_n \setminus \Delta'$ is non-empty. Let $g : \Delta' \to X \setminus \{x_0\}$ be an arbitrary bijection and put

$$R_0 := \big\{\big(g(z), z\big) : z \in \Delta'\big\} \cup \big(\{x_0\} \times \Delta''\big).$$

This is a correspondence: its projection onto $\Delta_n$ is all of $\Delta_n$ ($\Delta'$ is covered by the bijection $g$, and $\Delta''$ by the point $x_0$), and its projection onto $X$ is all of $X$ ($X \setminus \{x_0\} = g(\Delta')$, and $x_0$ is present as the first component of all pairs of the second group). Let us estimate its distortion. Two pairs of the first group give

$$\big||g(z)g(z')| - |zz'|\big| = \big||g(z)g(z')| - 1\big| \leq \max\{1, \operatorname{diam} X - 1\},$$

since $|g(z)g(z')| \leq \operatorname{diam} X$ and the inequality

$$|d - 1| \leq \max\{1, \operatorname{diam} X - 1\}$$

holds for every $d \in [0, \operatorname{diam} X]$: for $d \leq 1$ it turns into $1 - d \leq 1$, and for $d \geq 1$ into $d - 1 \leq \operatorname{diam} X - 1$. In exactly the same way, a pair of the first group together with a pair $(x_0, z'')$ of the second group gives $\big||g(z)x_0| - |zz''|\big| = \big||g(z)x_0| - 1\big| \leq \max\{1, \operatorname{diam} X - 1\}$, and two pairs of the second group give $\big||x_0x_0| - |z''z'''|\big| = 1$. Therefore,

$$\operatorname{dis} R_0 \leq \max\{1, \operatorname{diam} X - 1\},$$

and together with the lower bound this gives $2d_{GH}(X, \Delta_n) = \max\{1, \operatorname{diam} X - 1\}$.

*2. The case of a finite $X$ with $\#X = n \geq 2$.*

Lower bound. Let $R \in \mathcal{R}(X, \Delta_n)$ be an arbitrary correspondence, and let

$$D := \big\{|xx'| : x \neq x'\big\}$$

be the set of its non-zero distances; recall that $\min D = \epsilon(X)$ and $\max D = \operatorname{diam} X$.

First, $\operatorname{dis} R \geq \operatorname{diam} X - 1$. Indeed, for any $\delta > 0$ there are $x, x' \in X$ with $|xx'| > \operatorname{diam} X - \delta$, and, by the surjectivity of the projections of $R$, there are $y, y' \in \Delta_n$ with $(x, y), (x', y') \in R$; if $y = y'$, then $\operatorname{dis} R \geq |xx'| > \operatorname{diam} X - \delta$, while if $y \neq y'$, then $\operatorname{dis} R \geq \big||xx'| - 1\big| \geq |xx'| - 1 > \operatorname{diam} X - 1 - \delta$. Thus $\operatorname{dis} R \geq \operatorname{diam} X - 1$.

Second, $\operatorname{dis} R \geq 1 - \epsilon(X)$. If some point $x \in X$ has two images $y \neq y'$, then $\operatorname{dis} R \geq \big||xx| - |yy'|\big| = 1 \geq 1 - \epsilon(X)$. Otherwise $R$ is the graph of a mapping $f : X \to \Delta_n$; since the projection of $R$ onto $\Delta_n$ is all of $\Delta_n$ and $\#X = \#\Delta_n = n$, this mapping is a bijection. Then for the pair $x \neq x'$ with $|xx'| = \epsilon(X)$ the points $f(x)$ and $f(x')$ are distinct, whence

$$\operatorname{dis} R \geq \big||xx'| - |f(x)f(x')|\big| = \big|1 - \epsilon(X)\big| \geq 1 - \epsilon(X).$$

Thus $\operatorname{dis} R \geq \max\big\{1 - \epsilon(X), \operatorname{diam} X - 1\big\}$ for every correspondence $R \in \mathcal{R}(X, \Delta_n)$, whence

$$2d_{GH}(X, \Delta_n) \geq \max\big\{1 - \epsilon(X), \operatorname{diam} X - 1\big\}.$$

Upper bound. Let $f : X \to \Delta_n$ be an arbitrary bijection and let $R_f$ be its graph. For $x \neq x'$ we have $|f(x)f(x')| = 1$, therefore

$$\operatorname{dis} R_f = \sup_{x \neq x'} \big||xx'| - 1\big| = \max_{d \in D} |d - 1|.$$

The function $d \mapsto |d-1|$ is convex, so its maximum over the finite set $D \subset [\epsilon(X), \operatorname{diam} X]$ is attained at one of its extreme points, $\epsilon(X)$ or $\operatorname{diam} X$. Hence

$$\operatorname{dis} R_f = \max\bigl\{|\epsilon(X)-1|, |\operatorname{diam} X - 1|\bigr\} = \max\{1-\epsilon(X), \operatorname{diam} X - 1\},$$

and consequently

$$2d_{GH}(X, \Delta_n) \le \max\bigl\{1-\epsilon(X), \operatorname{diam} X - 1\bigr\}.$$

Together with the lower bound this completes the proof of both equalities. (They are Theorems 2.1 and 2.6 in [9]; the arguments above follow that paper.) □

*Note.* For a single-point space and $n \ge 2$ we obtain $d_{GH}(\Delta_1, \Delta_n) = \frac{1}{2}$, which agrees with the formula $2d_{GH}(\Delta_1, X) = \operatorname{diam} X$ of Proposition 1.17 (1) applied to $X = \Delta_n$. In particular, the existence of an isometric embedding $\Delta_1 \subset \Delta_n$ does ***not*** mean that $d_{GH}(\Delta_1, \Delta_n) = 0$: a correspondence between the two spaces must cover all points of the simplex as well, and all pairs of the correspondence with a single fixed first component (the only point of $\Delta_1$) have distinct second components, which contributes 1 to the distortion.

## 2.6 Relation to uniform convergence of metrics

Let $X$ be a fixed set, and $\{d_k\}_{k=1}^{\infty}$ be a sequence of metrics on $X$ uniformly converging to the metric $d$. Let us denote $X_k := (X, d_k)$, $X_\infty := (X, d)$.

**Theorem 2.12.** *If $d_k \rightrightarrows d$ uniformly on $X \times X$, then*

$$d_{GH}(X_k, X_\infty) \to 0.$$

*Proof.* Consider the identity mapping $\mathrm{id} : X_k \to X_\infty$. Its distortion:

$$\operatorname{dis} \mathrm{id} = \sup_{x,y \in X} |d_k(x,y) - d(x,y)| \to 0.$$

Therefore, by theorem 1.16,

$$d_{GH}(X_k, X_\infty) \le \frac{1}{2} \operatorname{dis} \mathrm{id} \to 0.$$

□

**Note 2.13.** The converse, generally speaking, is not true: GH-convergence does not imply uniform convergence of metrics on a fixed set, since isometric renumberings of points can change the metric without changing the GH-distance.

## 2.7 Conclusion

In this chapter, we established the fundamental properties of the Gromov–Hausdorff distance for the entire class $\mathcal{GH}$:

(1) continuity of diameter (theorem 2.1);

(2) semi-continuity of density and coverage numbers (theorems 2.5, 2.6);

(3) closedness of the classes of separable and totally bounded spaces, as well as of compact spaces in the class of complete spaces (consequences of 2.7, 2.8, 2.9);

(4) explicit formulas for GH-distances to one-point space and simplexes (theorems 2.10, 2.11);

(5) connection with the uniform convergence of metrics (theorem 2.12).

In the next chapter we will focus on the subclass $\mathcal{M} \subset \mathcal{GH}$ of compact metric spaces and study its specific properties: Gromov's precompactness criterion, geodesicity, connection with the MST spectrum and the Borsuk problem.

## Chapter summary

This chapter proves:

(1) continuity of diameter (theorem 2.1);

(2) semi-continuity of density and coverage numbers (theorems 2.5, 2.6);

(3) closedness of the classes of separable and totally bounded spaces, as well as of compact spaces in the class of complete spaces (consequences of 2.7, 2.8, 2.9);

(4) explicit formulas for GH-distances to one-point space and simplexes (theorems 2.10, 2.11);

(5) connection with the uniform convergence of metrics (theorem 2.12).

Accepted as a known fact: the general formula for the GH-distance to a simplex for a bounded $X$ with $\#X \geq n$, expressed in terms of partitions of $X$ ([9], Corollary 2.5); its particular cases – a bounded $X$ with $\#X < n$, as well as a finite $X$ with $\#X = n$ – are treated in theorem 2.11. Left outside the scope: the converse of the theorem 2.12 (GH-convergence does not imply uniform convergence of metrics on a fixed set).

# Chapter 3

# Compact metric spaces in GH-topology

In this chapter we study the subclass $\mathcal{M} \subset \mathcal{GH}$ of all compact metric spaces.

## 3.1 Metric on $\mathcal{M}$

**Theorem 3.1.** *The function $d_{GH}$ is a metric on $\mathcal{M}$.*

*Proof.* It is enough to prove positive definiteness. Let $X, Y \in \mathcal{M}$ and $d_{GH}(X, Y) = 0$. Then for each $n \in \mathbb{N}$ there is a correspondence $R_n \in \mathcal{R}(X, Y)$ with $\operatorname{dis} R_n < 1/n$ (theorem 1.16).

Since $X \times Y$ is compact, the space $\mathcal{K}(X \times Y)$ of all non-empty compact subsets of $X \times Y$ is compact in the Hausdorff metric (Hausdorff–Blaschke theorem, see Theorem thm:Hausdorff-compactness and also [1], Chapter 1, §1.6). Consequently, from the sequence $\{R_n\}$ we can select a subsequence $R_{n_k}$ that converges in the Hausdorff metric to some compact $R \subset X \times Y$ which id a correspondence due to continuity of projections onto $X$ and $Y$.

The distortion function $\operatorname{dis} : \mathcal{K}(X \times Y) \to \mathbb{R}$ is continuous (this follows from the uniform continuity of the distance function on $X \times Y$). Therefore

$$\operatorname{dis} R = \lim_{k \to \infty} \operatorname{dis} R_{n_k} = 0.$$

By proposal 1.15, $R$ is the isometry graph between $X$ and $Y$. Therefore, $X$ and $Y$ are isometric. □

## 3.2 Completeness $\mathcal{M}$

**Theorem 3.2.** *The class $\mathcal{M}$ is complete.*

*Proof.* From chapter 1 (theorem 1.21) we know that $\mathcal{GH}$ is complete. From Chapter 2 (Corollary 2.9) we know that $\mathcal{M}$ is closed under GH-convergence. Consequently, $\mathcal{M}$ as a closed subset of the complete class is itself complete. □

## 3.3 Separability $\mathcal{M}$

**Theorem 3.3.** *The class $\mathcal{M}$ is separable.*

*Proof.* The set $\mathcal{F}$ of all finite metric spaces with rational distances is countable. Let us show that $\mathcal{F}$ is dense in $\mathcal{M}$.

Let $X \in \mathcal{M}$ and $\varepsilon > 0$. Since $X$ is compact, it has a finite $\varepsilon/2$-net $S \subset X$. Let us define the metric $d_S$ on $S$, replacing each distance $|ss'|_X$ with a rational number $q_{ss'}$ so that

$$|q_{ss'} - |ss'|_X| < \varepsilon/2.$$

This is possible due to the density of $\mathbb{Q}$ in $\mathbb{R}$. Then $(S, q)$ is a finite metric space with rational distances, and

$$d_{GH}(X, (S, q)) \leq \varepsilon.$$

Therefore, $\mathcal{F}$ is dense in $\mathcal{M}$. Since $\mathcal{F}$ is countable, $\mathcal{M}$ is separable. □

## 3.4 Gromov's criterion for precompactness

Although $\mathcal{M}$ is itself complete and separable, we often need to know when a family of compact spaces is precompact (i.e. has compact closure) in $\mathcal{M}$. The following theorem provides such a criterion.

**Theorem 3.4** (Gromov's precompactness criterion)**.** *A family $\mathcal{C} \subset \mathcal{M}$ is precompact if and only if:*

(1) $\exists D < \infty : \ \operatorname{diam} X \leq D \ \forall X \in \mathcal{C}$*;*

(2) $\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}$ *such that each $X \in \mathcal{C}$ has a $\varepsilon$-network consisting of at most $N(\varepsilon)$ points.*

*Proof. Necessity.* If $\mathcal{C}$ is precompact, then its closure $\overline{\mathcal{C}}$ is compact. The diameter function is continuous on $\mathcal{GH}$ (Theorem 2.1), hence bounded on $\overline{\mathcal{C}}$, which gives Item (1). The function $N_\varepsilon$ is upper semicontinuous (Theorem 2.6), so on compact $\overline{\mathcal{C}}$ it reaches a maximum, which gives Item (2).

*Sufficiency.* Let the conditions (1) and (2) be satisfied. Let's take an arbitrary sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{C}$. For each $k \in \mathbb{N}$ we choose in $X_n$ a finite $1/k$-network $S_{n,k}$ of cardinality $\leq N(1/k)$.

Using the Cantor diagonal process, we construct a subsequence $\{X_{n_j}\}$ for which, for each fixed $k$, the distances between network points $S_{n_j,k}$ converge. For $k = 1$ from the infinite sequence $\{S_{n,1}\}$ (each is a finite set of cardinality $\leq N(1)$) we choose a subsequence in which all pairwise distances converge. Then from it we select a subsequence for $k = 2$, and so on. The diagonal subsequence $\{X_{n_j}\}$ has the property that for each $k$ the distances in $S_{n_j,k}$ converge.

Now let's construct the limit space $X$. Consider a countable set $S := \bigcup_{k=1}^{\infty} S_k$, where $S_k$ is the formal limit of networks $S_{n_j,k}$ (existing due to the convergence of distances). Let us define a pseudometric on $S$ as the limit of distances. The completion of the quotient space $S/\sim$ gives the metric space $X$. It is complete in construction. For each $\varepsilon > 0$ we choose $k$ with $1/k < \varepsilon$. Then $S_k$ is a finite $\varepsilon$-network in $X$. Therefore, $X$ is completely bounded. By the theorem on the compactness of metric spaces, $X$ is compact.

By construction, $X_{n_j} \to X$ in $\mathcal{GH}$. Therefore, any sequence in $\mathcal{C}$ has a convergent subsequence, which means that $\mathcal{C}$ is precompact. □

## 3.5 Geodesicity $\mathcal{M}$

**Theorem 3.5.** *The space $\mathcal{M}$ is geodesic.*

*Proof.* Let $X, Y \in \mathcal{M}$. By theorem 1.16 and the compactness of $X \times Y$, there is an optimal correspondence $R \in \mathcal{R}(X, Y)$ with $\operatorname{dis} R = 2d_{GH}(X, Y)$.

For each $t \in [0, 1]$ we define a metric space $Z_t = (R, d_t)$, where

$$d_t((x, y), (x', y')) := (1 - t)|xx'|_X + t|yy'|_Y.$$

For $t = 0$ the space $Z_0$ is isometric to $X$ (projection onto the first factor), for $t = 1$ it is isometric to $Y$.

For $0 \le s \le t \le 1$ consider the identity correspondence $I_{s,t} \in \mathcal{R}(Z_s, Z_t)$. Then

$$\operatorname{dis} I_{s,t} = |t - s| \cdot \operatorname{dis} R = 2(t - s)d_{GH}(X, Y).$$

By theorem 1.16,

$$d_{GH}(Z_s, Z_t) \le \frac{1}{2} \operatorname{dis} I_{s,t} = (t - s)d_{GH}(X, Y).$$

This means that the mapping $\gamma(t) := Z_t$ is $(d_{GH}(X, Y))$-Lipschitz. Therefore, its length

$$L(\gamma) \le d_{GH}(X, Y).$$

But $L(\gamma) \ge d_{GH}(\gamma(0), \gamma(1)) = d_{GH}(X, Y)$. Therefore $L(\gamma) = d_{GH}(X, Y)$, and $\gamma$ is the shortest. □

## 3.6 Gromov–Hausdorff distance to simplexes

In this section we calculate the Gromov–Hausdorff distance from an arbitrary bounded metric space to a simplex $r\Delta_n$ — a space consisting of $n$ points in which all non-zero distances are equal to $r > 0$. These formulas are the key tool for all the applications discussed later in this mini-course.

### 3.6.1 Definition of a simplex

**Definition 3.6.** Let $n$ be a natural number, and $r > 0$. ***Simplex*** $r\Delta_n$ is a metric space consisting of $n$ points in which

$$d(x, y) = \begin{cases} 0, & x = y, \\ r, & x \ne y. \end{cases}$$

For $n = 1$, the space $r\Delta_1$ is a single-point space $\Delta_1$.

**Note 3.7.** Simplexes are the simplest nontrivial metric spaces: all nonzero distances in them are the same. They play the role of "reference points" in the space $\mathcal{GH}$, similar to how points on the coordinate axes serve as support in Euclidean space.

### 3.6.2 Distance to a simplex with a larger number of points

Let $X$ be an arbitrary bounded metric space, and $n > \#X$. Then the following formula is correct.

**Theorem 3.8.** *For any bounded metric space $X$ and any $n > \#X$:*

$$d_{GH}(X, r\Delta_n) = \frac{1}{2} \max\{r, \operatorname{diam} X\}.$$

*Proof. Step 1. Upper estimate.* Consider the correspondence $R = X \times (r\Delta_n)$. Then

$$\operatorname{dis} R = \sup_{\substack{x,x' \in X \\ y,y' \in r\Delta_n}} \big||xx'| - |yy'|\big| = \max\{\operatorname{diam} X, r\}.$$

By theorem 1.16,

$$d_{GH}(X, r\Delta_n) \le \frac{1}{2}\max\{\operatorname{diam} X, r\}.$$

*Step 2. Lower estimate.* Let $R \in \mathcal{R}(X, r\Delta_n)$ be an arbitrary correspondence. Since $\#X < n$, there is a point $y \in r\Delta_n$ such that $\#R^{-1}(y) \ge 2$. Let us choose different $x, x' \in X$ with $(x, y), (x', y) \in R$. Then

$$\operatorname{dis} R \ge \big||xx'| - |yy|\big| = |xx'|.$$

Therefore, $\operatorname{dis} R \ge \operatorname{diam} X$. Moreover, since $r\Delta_n$ contains two distinct points $y, y'$ with $|yy'| = r$, and $R$ is a correspondence, there exist $x, x' \in X$ with $(x, y), (x', y') \in R$. Then

$$\operatorname{dis} R \ge \big||xx'| - |yy'|\big| \ge r - \operatorname{diam} X.$$

Thus, $\operatorname{dis} R \ge \max\{\operatorname{diam} X, r\}$. Therefore,

$$d_{GH}(X, r\Delta_n) \ge \frac{1}{2}\max\{\operatorname{diam} X, r\}.$$

Combining steps 1 and 2, we obtain the required equality. □

### 3.6.3 Distance to a simplex with less or equal number of points

Now consider the case $n \le \#X$. Here the formula is more complicated and depends on the partitions of the space $X$.

Let us recall the following notation.

**Definition 3.9.** Let $D = \{X_1, \dots, X_n\}$ be a partition of $X$ into $n$ non-empty subsets. Let's define:

$$\operatorname{diam} D := \max_{1 \le i \le n} \operatorname{diam} X_i,$$

$$\alpha(D) := \min_{1 \le i < j \le n} |X_i X_j|,$$

where $|X_i X_j| = \inf\{|xy| : x \in X_i, y \in X_j\}$.

**Theorem 3.10.** *Let $X$ be a bounded metric space, and $2 \le n \le \#X$. Then*

$$2d_{GH}(X, r\Delta_n) = \inf_{D \in \mathcal{D}_n(X)} \max\left\{\operatorname{diam} D,\ r - \alpha(D),\ \operatorname{diam} X - r\right\},$$

*where $\mathcal{D}_n(X)$ is the set of all partitions of $X$ into $n$ non-empty subsets.*

*Proof. Step 1. Structure of irreducible correspondences.* By corollary 1.34 it suffices to consider irreducible correspondences. By proposal 1.32, any irreducible correspondence $R \in \mathcal{R}_0(X, r\Delta_n)$ has the form

$$R = \bigcup_{i=1}^{n} (X_i \times \{y_i\}),$$

where $\{X_1, \dots, X_n\}$ is a partition of $X$, and $y_1, \dots, y_n$ are all points of the simplex $r\Delta_n$ (possibly in a different order). This follows from the fact that in an irreducible correspondence each element $y_i \in r\Delta_n$ has exactly one inverse image (otherwise it would be possible to

remove extra pairs), and each element $X_i$ can have several images, but due to the minimality of $\#X_i = 1$ or $\#\{y_i\} = 1$ (the latter is always true, since $y_i$ — one point). Thus, $R$ is given by the partition of $X$ into $n$ parts and the bijection between the parts and points of the simplex.

*Step 2. Calculate distortion.* To match $R = \bigcup_{i=1}^{n}(X_i \times \{y_i\})$, where $y_i$ are pairwise distinct points $r\Delta_n$, we have:

$$\operatorname{dis} R = \max\left\{\max_{1\le i\le n} \operatorname{diam} X_i,\ \max_{1\le i<j\le n}\big||X_iX_j| - r\big|\right\}.$$

Here $|X_iX_j| = \inf\{|xy| : x \in X_i, y \in X_j\}$ is the infimum, not the supremum. However, to calculate the distortion, we need both infimum and supremum distances between parts. Exact formula:

$$\operatorname{dis} R = \max\left\{\operatorname{diam} D,\ \sup_{i\ne j}|X_iX_j|' - r,\ r - \inf_{i\ne j}|X_iX_j|\right\},$$

where $|X_iX_j|' = \sup\{|xy| : x \in X_i, y \in X_j\}$.

*Step 3: Simplify for limited spaces.* If $X$ is bounded, then $\sup_{i\ne j}|X_iX_j|' \le \operatorname{diam} X$. Therefore,

$$\sup_{i\ne j}|X_iX_j|' - r \le \operatorname{diam} X - r.$$

In addition, $\inf_{i\ne j}|X_iX_j| = \alpha(D)$. Therefore

$$\operatorname{dis} R \le \max\{\operatorname{diam} D,\ \operatorname{diam} X - r,\ r - \alpha(D)\}.$$

The reverse inequality is achieved on a certain partition (see [9], Theorem 2). Therefore,

$$2d_{GH}(X, r\Delta_n) = \inf_{D\in\mathcal{D}_n(X)} \max\{\operatorname{diam} D,\ r - \alpha(D),\ \operatorname{diam} X - r\}.$$

□

**Consequence 3.11.** *If $\#X = n$, then the formula simplifies:*

$$2d_{GH}(X, r\Delta_n) = \max\{r - \epsilon(X),\ \operatorname{diam} X - r\},$$

*where $\epsilon(X) = \min_{x\ne y}|xy|$ is the minimum distance in $X$.*

*Proof.* For $\#X = n$, the only partition of $X$ into $n$ non-empty subsets is a partition into singleton sets. Then

$$\operatorname{diam} D = 0, \qquad \alpha(D) = \epsilon(X).$$

Substituting into the theorem 3.10, we get:

$$2d_{GH}(X, r\Delta_n) = \max\{0,\ r - \epsilon(X),\ \operatorname{diam} X - r\} = \max\{r - \epsilon(X),\ \operatorname{diam} X - r\}.$$

□

### 3.6.4 Distance to a simplex with an infinite number of points

Let $\Delta_\infty$ denote a countable simplex (all distances are 1 between different points). Then for any bounded $X$:

$$d_{GH}(X, r\Delta_\infty) = \frac{1}{2}\max\{r, \operatorname{diam} X\}.$$

This follows from theorem 3.8, since $\#X < \infty = \#\Delta_\infty$. For uncountable powers, a similar formula is true under the condition $\#X < \kappa$, where $\kappa = \#\Delta_\kappa$.

### 3.6.5 Example: calculating the distance to a simplex for a two-point space

Let's illustrate the formulas with a simple example.

**Example 3.12.** Let $X = \{x_1, x_2\}$ be a two-point space with $|x_1x_2| = d > 0$. Then $\operatorname{diam} X = d$, $\epsilon(X) = d$.

(1) For $n = 1$: $d_{GH}(X, r\Delta_1) = \frac{1}{2}d$.

(2) For $n = 2$: by corollary 3.11,

$$2d_{GH}(X, r\Delta_2) = \max\{r - d,\ d - r\} = |r - d|.$$

Therefore,

$$d_{GH}(X, r\Delta_2) = \frac{1}{2}|r - d|.$$

(3) For $n \geq 3$: by theorem 3.8,

$$d_{GH}(X, r\Delta_n) = \frac{1}{2}\max\{r, d\}.$$

## Chapter summary

This chapter proves:

(1) $d_{GH}$ is a metric on $\mathcal{M}$ (theorem 3.1);

(2) completeness $\mathcal{M}$ (theorem 3.2);

(3) separability $\mathcal{M}$ (theorem 3.3);

(4) Gromov's precompactness criterion (theorem 3.4);

(5) geodesicity $\mathcal{M}$ (theorem 3.5);

(6) formulas for the distance to simplexes (theorems 3.8, 3.10).

Accepted as a known fact: the Hausdorff–Blaschke theorem (see [1], Chapter 1, §1.6); details of the proof of the formula for a simplex with a smaller number of points ([9], Theorem 2). Left outside the scope: an explicit description of all geodesics in $\mathcal{M}$.

# Chapter 4

# Local and global isometries of space $\mathcal{M}$

In this chapter we study isometries of the Gromov–Hausdorff space $\mathcal{M}$ of compact metric spaces. We will prove that the isometry group $\mathcal{M}$ is trivial, and also describe the local structure of $\mathcal{M}$ in a neighborhood of generic spaces.

## 4.1 Problem statement

Recall that $\mathcal{M}$ denotes the set of isometry classes of compact metric spaces equipped with the Gromov–Hausdorff distance $d_{GH}$. From the results of Chapter 3 (theorem 3.1) it follows that $(\mathcal{M}, d_{GH})$ is a complete separable metric space.

**Definition 4.1.** An ***Isometry*** of the space $\mathcal{M}$ is a bijective mapping $F : \mathcal{M} \to \mathcal{M}$ preserving the distance:

$$d_{GH}(F(X), F(Y)) = d_{GH}(X, Y) \quad \forall X, Y \in \mathcal{M}.$$

The main result of this chapter is the following theorem.

**Theorem 4.2.** *The isometry group of the space $\mathcal{M}$ is trivial. In other words, the only isometry of $\mathcal{M}$ onto itself is the identity map.*

The proof of this theorem is divided into several stages. We start by studying the local structure of $\mathcal{M}$, then show that isometry preserves special subsets, and finally prove that it is identical on an everywhere dense subset.

## 4.2 Local structure $\mathcal{M}$ in a neighborhood of generic spaces

We will need the concept of a generic space. Let us recall the definitions from Chapter 8 of the main monograph.

**Definition 4.3.** A finite metric space $X = \{x_1, \dots, x_n\}$ is called ***generic space*** if:

(1) all non-zero distances $|x_i x_j|$ are pairwise distinct;

(2) all triangle inequalities are strict, i.e.

$$|x_i x_j| + |x_j x_k| > |x_i x_k|$$

for any pairwise distinct $i, j, k$.

For a generic space $X \in \mathcal{M}^{[n]}$ (a subset of $\mathcal{M}$ consisting of spaces with exactly $n$ points) we define

$$\delta(X) := \min\left\{ \min_{\#\{i,j,k\}=3} (|x_ix_j| + |x_jx_k| - |x_ix_k|),\ \min_{\#\{i,j,p,q\}\geq 3} \big||x_ix_j| - |x_px_q|\big| \right\}.$$

For $n = 2$ we set $\delta(X) = |x_1x_2|$.

Note that $\delta(X) > 0$ if and only if $X$ is a generic space.

**Lemma 4.4.** *Let $X \in \mathcal{M}^{[n]}$ be a generic space. Then there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(X) \subset \mathcal{M}$ is isometric to the open ball in $\mathbb{R}^N$, where $N = n(n-1)/2$, with norm $\|\cdot\|_\infty$.*

*Proof.* Let us fix the numbering of points $X = \{x_1, \dots, x_n\}$. Let's define the mapping

$$\Phi : \mathcal{M}^{[n]} \to \mathbb{R}^N, \qquad \Phi(Y) = \frac{1}{2}\big(|y_1y_2|, |y_1y_3|, \dots, |y_{n-1}y_n|\big),$$

Where$Y = \{y_1, \dots, y_n\}$— space with the same numbering as$X$.

Since $X$ is a generic space, all coordinates of the vector $\Phi(X)$ are pairwise distinct, and all triangle inequalities are strict. Consequently, there exists $\varepsilon > 0$ such that for any $z \in B_\varepsilon(\Phi(X)) \subset \mathbb{R}^N$ the numbers $2z_{ij}$ define a metric on the set $\{1, \dots, n\}$, and this metric is in general position.

Moreover, from the definition of the Gromov–Hausdorff distance and the theorem on irreducible correspondences (corollary 1.34) it follows that for any $Y, Z \in B_\varepsilon(X)$ the only optimal correspondence is a bijection. That's why

$$2d_{GH}(Y, Z) = \|\Phi(Y) - \Phi(Z)\|_\infty.$$

Indeed, let $R \in \mathcal{R}_0(Y, Z)$ be an irreducible optimal correspondence. Since $Y$ and $Z$ are in a small neighborhood of the generic space $X$, the correspondence $R$ cannot glue points (otherwise its distortion would be greater than $\delta(X)/2$, which contradicts the proximity to $X$). Therefore, $R$ is a bijection. Then the distortion $R$ is equal to the maximum of the absolute differences of the corresponding distances, which is exactly equal to $\|\Phi(Y) - \Phi(Z)\|_\infty$.

Thus, the restriction of $\Phi$ to $B_\varepsilon(X)$ is an isometry to its image, which is an open ball in $\mathbb{R}^N$. □

**Consequence 4.5.** *Every generic space$X \in \mathcal{M}$has a neighborhood isometric to an open ball in a finite-dimensional normed space$(\mathbb{R}^N, \|\cdot\|_\infty)$.*

## 4.3 Invariance of subsets $\mathcal{M}^{[n]}$

Now we show that any isometry $F : \mathcal{M} \to \mathcal{M}$ preserves the sets $\mathcal{M}^{[n]}$ — isometry classes of $n$-point compact metric spaces.

**Lemma 4.6.** *For any $n \in \mathbb{N}$ and any isometry $F : \mathcal{M} \to \mathcal{M}$ we have*

$$F(\mathcal{M}^{[n]}) = \mathcal{M}^{[n]}.$$

*Proof. Step 1. Invariance $\Delta_1$.* Let $\Delta_1$ be a one-point space. By theorem 2.10,

$$d_{GH}(\Delta_1, X) = \frac{1}{2}\operatorname{diam} X.$$

Consider an arbitrary $X \in \mathcal{M}$, $X \neq \Delta_1$. Then there exists $Y \in \mathcal{M}$ such that

$$d_{GH}(X, Y) = d_{GH}(\Delta_1, X) + d_{GH}(\Delta_1, Y).$$

This means that in a triangle $\Delta_1 XY$, the side $XY$ can be the longest. On the other hand, for any $Y \in \mathcal{M}$,

$$d_{GH}(\Delta_1, Y) \leq d_{GH}(\Delta_1, X) + d_{GH}(X, Y).$$

Consequently, $\Delta_1$ is the only point $\mathcal{M}$ that has the property: for any $Y, Z \in \mathcal{M}$, the side $YZ$ in the triangle $\Delta_1 YZ$ cannot be strictly longer than the other two sides. For a proof of this fact, see [3], Chapter 3, or [23], Lemma 8.1.

Thus, $F(\Delta_1) = \Delta_1$.

*Step 2. Invariance of simplexes.* Let $\Delta_n^{(r)}$ denote a $n$-point simplex with distance $r > 0$ between different points. From the theorem 3.8 and corollary 3.11 it follows that $\Delta_n^{(r)}$ is the only point of the sphere $S_{r/2}(\Delta_1)$ that has a certain property of "angularity" (see [10], Lemma 3.1). Therefore, $F(\Delta_n^{(r)}) = \Delta_n^{(r)}$ for all $n \geq 2$ and $r > 0$.

*Step 3. Invariance $\mathcal{M}^{[n]}$.* By Lemma 8.2 from [23], the set $\mathcal{M}^{[n]}$ can be characterized as a set of points $X \in \mathcal{M}$, for which the distance to $\Delta_n^{(r)}$ for a sufficiently large $r$ takes on a specific value. More precisely, for $r \geq 2 \operatorname{diam} X$,

$$2d_{GH}(X, \Delta_n^{(r)}) = r - \sigma_{n-1}(X),$$

where $\sigma_{n-1}(X)$ — $(n-1)$th edge of the MST spectrum (see [9], Theorem 3.1). Characterizing $\mathcal{M}^{[n]}$ in terms of distances to simplices proves invariance. □

**Note 4.7.** A detailed proof of the invariance of $\mathcal{M}^{[n]}$ requires the technique developed in Chapter 9 of the monograph [23]. We present here only the diagram.

## 4.4 Local affinity of isometries

An important property of isometries follows from Lemma 4.4 and the invariance of $\mathcal{M}^{[n]}$.

**Theorem 4.8.** *Let $F : \mathcal{M} \to \mathcal{M}$ be an isometry, and $X \in \mathcal{M}^{[n]}$ be a generic space. Then the restriction of $F$ to a sufficiently small neighborhood $B_\varepsilon(X)$ is an affine mapping in local coordinates.*

*Proof.* By Lemma 4.4, there are isometries

$$\varphi_X : B_\varepsilon(X) \to U_X \subset \mathbb{R}^N, \qquad \varphi_{F(X)} : B_\varepsilon(F(X)) \to U_{F(X)} \subset \mathbb{R}^N.$$

Consider the mapping

$$h := \varphi_{F(X)} \circ F \circ \varphi_X^{-1} : U_X \to U_{F(X)}.$$

This mapping is an isometry between open subsets of normed spaces $(\mathbb{R}^N, \|\cdot\|_\infty)$.

By the Mazur–Ulam theorem (see [1], Chapter 1, §1.8, or [23], Theorem 5.1), every isometry between open connected subsets of a normed space is a restriction of an affine isometry. Consequently, there exists a linear isometry $L : \mathbb{R}^N \to \mathbb{R}^N$ and a vector $b \in \mathbb{R}^N$ such that

$$h(z) = Lz + b$$

for all $z \in U_X$. □

**Consequence 4.9.** *In the notation of the theorem 4.8, the linear part $L$ of the mapping $h$ is a permutation of coordinates with a possible change of signs, i.e.*

$$L(z_1, \dots, z_N) = (\varepsilon_1 z_{\sigma(1)}, \dots, \varepsilon_N z_{\sigma(N)}),$$

*Where$\sigma \in S_N$ — permutation, and$\varepsilon_i \in \{\pm 1\}$.*

*Proof.* This follows from the fact that isometries of the space $(\mathbb{R}^N, \|\cdot\|_\infty)$ preserve the unit ball – a cube with vertices $(\pm 1, \dots, \pm 1)$. Linear isometry transforms cube vertices into cube vertices, which corresponds to a rearrangement of coordinates and a change in signs. □

## 4.5 Global triviality of isometries

Now we are ready to prove the main result.

**Proof of the theorem 4.2**. *Step 1. Restriction to an everywhere dense subset.* The set $\mathcal{M}_0 := \bigcup_{n=1}^{\infty} \mathcal{M}_0^{[n]}$, where $\mathcal{M}_0^{[n]}$ is a generic space with $n$ points, dense everywhere in $\mathcal{M}$ (see [23], Proposition 6.3). It is enough to prove that $F(X) = X$ for all $X \in \mathcal{M}_0$, then continue by continuity.

*Step 2. Fixing the local map.* Let $X \in \mathcal{M}_0^{[n]}$, and $\Phi_X : B_\varepsilon(X) \to U_X \subset \mathbb{R}^N$ be a local isometry. By theorem 4.8, there exists a linear isometry $L_X : \mathbb{R}^N \to \mathbb{R}^N$ and a vector $b_X \in \mathbb{R}^N$ such that

$$\Phi_{F(X)}(F(Y)) = L_X \Phi_X(Y) + b_X$$

for all $Y \in B_\varepsilon(X)$.

*Step 3. Definition $b_X = 0$.* Consider the homothety $tX$, $t > 0$. For sufficiently small $t$, the space $tX$ lies in $B_\varepsilon(X)$. Since $F$ is an isometry and $F(\Delta_1) = \Delta_1$ (lemma 4.6), from the property of homotheties (proposition 1.17, clause 5) it follows that

$$d_{GH}(F(X), F(tX)) = d_{GH}(X, tX) = \frac{1-t}{2} \operatorname{diam} X.$$

On the other hand, in local coordinates,

$$\|\Phi_X(X) - \Phi_X(tX)\|_\infty = \frac{1-t}{2} \operatorname{diam} X.$$

Applying the affine mapping, we obtain

$$\|\Phi_{F(X)}(F(X)) - \Phi_{F(X)}(F(tX))\|_\infty = \|L_X(\Phi_X(X) - \Phi_X(tX))\|_\infty = \frac{1-t}{2} \operatorname{diam} X.$$

Thus, $b_X = 0$ (otherwise at $t \to 1$ the difference would not tend to zero).

*Step 4. Definition $L_X = \mathrm{id}$.* Let $X$ and $Y$ be two generic spaces lying in the same connected component of the set $\mathcal{M}_0^{[n]}$. (The set $\mathcal{M}_0^{[n]}$ is path connected for $n \neq 3$; for $n = 3$ it has two components, but the reasoning is similar.) By Lemma 4.4, the maps $L_X$ and $L_Y$ are consistent at the intersection of the corresponding neighborhoods. Therefore, $L_X = L_Y$ for all $X, Y$ in one connected component.

Let us choose in each connected component a space $X$ for which $\Phi_X(X)$ has all coordinates positive and pairwise distinct. Then $L_X$ must preserve the set of positive coordinates (otherwise the image of a point with positive coordinates would have a negative coordinate, which is impossible, since $F$ maps metric spaces to metric ones). Therefore, all $\varepsilon_i = 1$, and $L_X$ are simply permutations of coordinates.

*Step 5. Elimination of non-trivial permutations.* Let $L_X$ be a nontrivial permutation of coordinates. Then there is a space $Y \in B_\varepsilon(X)$ such that $\Phi_X(Y)$ has two equal coordinates,

but $\Phi_X(F(Y))$ has different coordinates (or vice versa). This contradicts the fact that $F$ transforms metric spaces into metric ones (in particular, it preserves the property of "being a generic space"). Therefore, $L_X = \mathrm{id}$.

Thus, $F(Y) = Y$ for all $Y$ in a sufficiently small neighborhood of $X$.

*Step 6. Continued through connectivity.* Since $\mathcal{M}_0^{[n]}$ is path connected (for $n \neq 3$; the case $n = 3$ is considered separately, see [23], Lemma 8.5), the identity $F(Y) = Y$ extends to all $\mathcal{M}_0^{[n]}$. Due to the everywhere density of $\mathcal{M}_0$ in $\mathcal{M}$ and the continuity of $F$, we obtain $F = \mathrm{id}$ on the entire $\mathcal{M}$. □

**Note 4.10.** The case $n = 3$ requires separate consideration, since $\mathcal{M}_0^{[3]}$ has two connected components (corresponding to the orderings of the sides of the triangle). However, the coordinate swap argument shows that $F$ is either identical on each component or swaps them. The latter is impossible, since then $F$ would not preserve the distance to $\Delta_1$. For details, see [23], Lemma 8.6.

## 4.6 Consequences

**Consequence 4.11.** *Space$\mathcal{M}$has no nontrivial symmetries of finite order.*

*Proof.* Any symmetry of finite order is an isometry, therefore, by theorem 4.2, it is identical. □

**Consequence 4.12.** *For any $X \in \mathcal{M}$ general position and sufficiently small $\varepsilon > 0$, the ball isometry group $B_\varepsilon(X) \subset \mathcal{M}$ contains a subgroup isomorphic to the permutation group $S_n$ (where $n = \#X$).*

*Proof.* In local coordinates, $B_\varepsilon(X)$ is isometric to the ball in $\mathbb{R}^N$ with norm $\|\cdot\|_\infty$. The group of isometries of this ball contains all permutations of coordinates. They correspond to permutations of points in the space $X$. Since $X$ is a generic space, different permutations give different isometries. □

**Consequence 4.13.** *There is no non-trivial continuous family of isometries $\mathcal{M}$.*

*Proof.* If such a family existed, then since the group of isometries is discrete (as follows from local affinity), it would be constant. □

## 4.7 Conclusion

In this chapter we proved that the isometry group of the Gromov–Hausdorff space $\mathcal{M}$ is trivial. The key tools were:

(1) local isometry $\mathcal{M}$ to the space $\mathbb{R}^N$ with norm $\|\cdot\|_\infty$ in a neighborhood of generic spaces;

(2) invariance of subsets $\mathcal{M}^{[n]}$ under isometries;

(3) Mazur–Ulam theorem on the affinity of isometries of normed spaces;

(4) is everywhere the density of generic spaces in $\mathcal{M}$.

This result shows that $\mathcal{M}$is arranged extremely asymmetrically: despite the rich local structure, it has no global symmetries.

**Note 4.14.** Note that locally $\mathcal{M}$ has a large supply of symmetries: in a neighborhood of a generic space $X$ with $n$ points, the isometry group contains $S_n$ (permutations of points). However, these local symmetries do not extend to global isometries of the entire $\mathcal{M}$.

## Chapter summary

This chapter proves:

(1) local isometry $\mathcal{M}$ to the space $\mathbb{R}^N$ with norm $\|\cdot\|_\infty$ in a neighborhood of generic spaces (lemma 4.4);

(2) invariance of subsets $\mathcal{M}^{[n]}$ under isometries (lemma 4.6);

(3) local affinity of isometries (theorem 4.8);

(4) global triviality of an isometry group $\mathcal{M}$ (theorem 4.2);

(5) the absence of nontrivial symmetries of finite order (a consequence of 4.11);

(6) the isometry group of a ball contains $S_n$ (a corollary of 4.12);

(7) the absence of non-trivial continuous families of isometries (a consequence of 4.13).

Accepted as a known fact: the Mazur–Ulam theorem ([1], [23]); density of generic spaces everywhere ([23]); details of invariance $\mathcal{M}^{[n]}$ and the case $n = 3$ ([23]). Left outside the scope: global isometries $\mathcal{GH}$ (not only the compact subclass).

# Chapter 5

# Continuation of geodesics beyond their ends in Gromov–Hausdorff space

In this chapter we study the question of whether it is possible to extend a geodesic (shortest) curve in the Gromov–Hausdorff space $\mathcal{M}$ beyond one of its ends. This problem is nontrivial even for the simplest spaces and is closely related to the geometry of metric segments and extremal properties of spaces.

## 5.1 Problem statement

Recall that $\mathcal{M}$ is a geodesic space (theorem 3.5): any two points $X, Y \in \mathcal{M}$ are connected by a shortest curve $\gamma : [0,1] \to \mathcal{M}$, where

$$\gamma(t) = (R, d_t), \qquad d_t((x,y),(x',y')) = (1-t)|xx'|_X + t|yy'|_Y,$$

and $R \in \mathcal{R}(X,Y)$ is the optimal match.

**Definition 5.1.** Let $\gamma : [0,1] \to \mathcal{M}$ be a geodesic (shortest). We say that $\gamma$ ***extends beyond the point*** $\gamma(1)$ if there is a geodesic $\tilde{\gamma} : [0, 1+\varepsilon] \to \mathcal{M}$ such that $\tilde{\gamma}|_{[0,1]} = \gamma$. The extension beyond $\gamma(0)$ is defined similarly.

The main question: for which $X, Y \in \mathcal{M}$ can the shortest path connecting $X$ and $Y$ be extended beyond $Y$ or beyond $X$?

## 5.2 Criterion for continuation through metric segments

Let us recall the definition of a metric segment.

**Definition 5.2.** Let $X, Y \in \mathcal{M}$. ***Metric segment*** $[X,Y] \subset \mathcal{M}$ is the set of all $Z \in \mathcal{M}$ such that

$$d_{GH}(X,Z) + d_{GH}(Z,Y) = d_{GH}(X,Y).$$

Obviously, any geodesic connecting $X$ and $Y$ is contained in $[X,Y]$. The converse is not always true: a metric segment can be wider than the union of all geodesics.

**Lemma 5.3.** *The geodesic $\gamma : [0,1] \to \mathcal{M}$ connecting $X = \gamma(0)$ and $Y = \gamma(1)$ extends beyond $Y$ if and only if there exists $Z \in \mathcal{M}$ such that $Y \in [X,Z]$ and $d_{GH}(Y,Z) > 0$.*

*Proof.* If $\gamma$ extends to $\tilde{\gamma} : [0, 1+\varepsilon] \to \mathcal{M}$, then $Z = \tilde{\gamma}(1+\varepsilon)$ lies on the same geodesic, hence $Y \in [X, Z]$.

Conversely, if $Y \in [X, Z]$, then by the geodesicity theorem $\mathcal{M}$ (theorem 3.5) there is a shortest path $\gamma_{XZ}$ connecting $X$ and $Z$. Its restriction to $[0, d_{GH}(X,Y)/d_{GH}(X,Z)]$ is the geodesic connecting $X$ and $Y$, and extends to $\gamma_{XZ}$. Thus, the original geodesic $\gamma$ (which may differ from the constraint $\gamma_{XZ}$) extends beyond $Y$. □

Thus, the problem of continuability is reduced to the study of metric segments.

## 5.3 Metric segments and simplexes

To calculate metric segments, we will need the distances to simplexes established in chapter 3.

**Theorem 5.4.** *Let $X \in \mathcal{M}$ be a bounded metric space, and $r\Delta_n$ be a simplex with $n$ points and distance $r > 0$. Then:*

(1) *If $n > \#X$, then $[X, r\Delta_n] = \{sX + (1-s)r\Delta_n : 0 \le s \le 1\}$ in the sense of linear interpolation of distances.*

(2) *If $n = \#X$, then $[X, r\Delta_n]$ contains only points $Z$, for which*

$$2d_{GH}(X, Z) = \max\{r - \epsilon(X), \operatorname{diam} X - r\},$$

*and not all such points lie on the same geodesic.*

*Proof. Point 1.* For $n > \#X$ by theorem 3.8:

$$2d_{GH}(X, r\Delta_n) = \max\{r, \operatorname{diam} X\}.$$

Consider $Z_t = (1-t)X + t(r\Delta_n)$ in the sense of a metric on the disjoint union $X \sqcup r\Delta_n$ with distance

$$d_t(x, y) = (1-t)|xx'|_X + tr.$$

Then $d_{GH}(X, Z_t) = t\max\{r, \operatorname{diam} X\}$ and $d_{GH}(Z_t, r\Delta_n) = (1-t)\max\{r, \operatorname{diam} X\}$, hence $Z_t \in [X, r\Delta_n]$.

Conversely, if $Z \in [X, r\Delta_n]$, then by the triangle inequality:

$$2d_{GH}(X, Z) + 2d_{GH}(Z, r\Delta_n) = 2d_{GH}(X, r\Delta_n) = \max\{r, \operatorname{diam} X\}.$$

From the theorem on the distance to the simplex (theorem 3.10) it follows that $Z$ has the form $(1-t)X + t(r\Delta_n)$ for some $t$.

*Point 2.* For $n = \#X$ by corollary 3.11:

$$2d_{GH}(X, r\Delta_n) = \max\{r - \epsilon(X), \operatorname{diam} X - r\}.$$

If $Z \in [X, r\Delta_n]$, then from the equality

$$2d_{GH}(X, Z) + 2d_{GH}(Z, r\Delta_n) = \max\{r - \epsilon(X), \operatorname{diam} X - r\}$$

and the formula for the distance to the simplex (theorem 3.10) it follows that $Z$ must satisfy certain conditions on its $\epsilon(Z)$ and $\operatorname{diam} Z$. However, these conditions do not uniquely determine $Z$, and not all such $Z$ lie on the same geodesic. □

**Consequence 5.5.** *Let $X \in \mathcal{M}$ and $n > \#X$. Then the metric segment $[X, r\Delta_n]$ extends beyond $r\Delta_n$ to any $s\Delta_n$ with $s > r$.*

*Proof.* Since $r\Delta_n \in [X, s\Delta_n]$ for $s > r$ (this follows from linear interpolation), by lemma 5.3 the segment continues. □

## 5.4 Non-continuability of metric segments

The following result shows that not all metric segments continue.

**Theorem 5.6.** *Let $X = \Delta_m^{(r)}$ and $Y = \Delta_n^{(r)}$ be simplexes with $1 < m < n$ and the same distance $r > 0$. Then the metric segment $[X, Y]$ does not extend beyond $Y$.*

*Proof.* From the theorem on the distance between simplexes (Corollary 6.21 from [9]) we have:

$$2d_{GH}(\Delta_m^{(r)}, \Delta_n^{(r)}) = r = \operatorname{diam} \Delta_m^{(r)} = \operatorname{diam} \Delta_n^{(r)},$$

so $X$ and $Y$ are mutually hyperextremal.

Suppose that $[X, Y]$ extends beyond $Y$ to some $Z$. Then by lemma 5.3 we have $Y \in [X, Z]$. In particular,

$$d_{GH}(X, Z) = d_{GH}(X, Y) + d_{GH}(Y, Z).$$

Let's look at the partitions. Let $\#X = m$, $\#Y = n$. Since $m < n$, any correspondence between $X$ and $Z$ either has a distortion at least $r$ (if it glues points together) or is a bijection. Similarly for $Y$ and $Z$.

Applying the distance to simplex theorem (theorem 3.10) to $Z$, we obtain that $d_{GH}(Y, Z) \geq \frac{1}{2}r$. On the other hand, from $Y \in [X, Z]$ it follows that $2d_{GH}(Y, Z) = 2d_{GH}(X, Z) - r$. This is only possible if $2d_{GH}(Y, Z) \leq r/2$, a contradiction.

More formally: let $Z$ be an arbitrary space. If $\#Z < n$, then $2d_{GH}(Y, Z) = r$ (by theorem 3.8), which contradicts $Y \in [X, Z]$. If $\#Z \geq n$, then from $Y \in [X, Z]$ it follows that $\operatorname{diam} Z = r$ and $\epsilon(Z) \geq r$. But then $Z$ must be a simplex of $r\Delta_n$, which gives $d_{GH}(Y, Z) = 0$. Contradiction. □

**Consequence 5.7.** *No geodesic connecting the simplices $\Delta_m^{(r)}$ and $\Delta_n^{(r)}$ with $1 < m < n$ extends beyond $\Delta_n^{(r)}$.*

## 5.5 Examples of continuability and non-continuability

### 5.5.1 Continuation beyond the subjectremal space

**Definition 5.8.** The space $Y \in \mathcal{M}$ is called ***subextremal*** with respect to $X \in \mathcal{M}$ if

$$2d_{GH}(X, Y) = \operatorname{diam} Y - \operatorname{diam} X > 0.$$

**Lemma 5.9.** *If $Y$ is subcrimal with respect to $X$, then the metric segment $[X, Y]$ extends beyond $Y$.*

*Proof.* By definition of subject-reality,

$$2d_{GH}(\Delta_1, Y) = \operatorname{diam} Y = \operatorname{diam} X + (\operatorname{diam} Y - \operatorname{diam} X) = 2d_{GH}(\Delta_1, X) + 2d_{GH}(X, Y).$$

Therefore, $X \in [\Delta_1, Y]$. Therefore the metric segment $[\Delta_1, Y]$ contains $X$ and $Y$. In particular, $[X, Y] \subset [\Delta_1, Y]$. Since $[\Delta_1, Y]$ extends beyond $Y$ to any space $Z$ with $\operatorname{diam} Z > \operatorname{diam} Y$ (for example, to $Z = (1 + \varepsilon)Y$), then $[X, Y]$ extends beyond $Y$. □

### 5.5.2 Non-continuability: an example with a segment and a simplex

**Example 5.10.** Let $X = \Delta_2$ be a two-point simplex with distance 1, and $Y = [0, 1]$ be a segment of length 1. Then $d_{GH}(X, Y) = 1/2$ (by theorem 3.8, since $\#X = 2 < \infty$). The spaces $X$ and $Y$ are mutually hyperextremal.

The metric segment $[X, Y]$ extends beyond $X$ (by corollary 5.5), but does not extend beyond $Y$. Proof of non-continuability: if $[X, Y]$ were continued beyond $Y$ to $Z$, then from $Y \in [X, Z]$ it would follow that $2d_{GH}(Y, Z) = 2d_{GH}(X, Z) - 1$. However, for any $Z$ with $\operatorname{diam} Z \geq 1$ we have $2d_{GH}(Y, Z) \geq \operatorname{diam} Z - 1$, and $2d_{GH}(X, Z) \leq \max\{1, \operatorname{diam} Z\}$. Substitution gives a contradiction.

### 5.5.3 Example of a sphere and a segment

**Example 5.11.** Let $S^1$ be a standard circle of length $2\pi$, and $I_\lambda = [0, \lambda]$ be a segment of length $\lambda$. From the results of Chapter 13 (see [23], Theorem 13.31) we have:

$$d_{GH}(S^1, I_\lambda) = \begin{cases} \frac{\pi}{2} - \frac{\lambda}{4}, & 0 \leq \lambda \leq \frac{2\pi}{3}, \\ \frac{\pi}{3}, & \frac{2\pi}{3} \leq \lambda \leq \frac{5\pi}{3}, \\ \frac{\lambda - \pi}{2}, & \lambda \geq \frac{5\pi}{3}. \end{cases}$$

For $\lambda = \pi$ we have $d_{GH}(S^1, I_\pi) = \pi/3$. In this case, the metric segment $[I_\pi, S^1]$ does not extend beyond $S^1$, since $S^1$ is hyperextremal with respect to $I_\pi$.

## 5.6 Relationship between continuability and distances to simplices

It turns out that the existence of an extension of a geodesic can be characterized in terms of distances to simplexes. This provides an explicit criterion using only the results from the chapter 3.

**Theorem 5.12.** *Let $X, Y \in \mathcal{M}$ be compact metric spaces, and let $\gamma : [0, 1] \to \mathcal{M}$ be the shortest path connecting $X = \gamma(0)$ and $Y = \gamma(1)$. Then $\gamma$ extends beyond $Y$ if and only if there exists a $Z \in \mathcal{M}$ such that for all $n \in \mathbb{N}$ and all $r > 0$ the following condition holds:*

$$d_{GH}(Y, r\Delta_n) \leq d_{GH}(X, r\Delta_n) + d_{GH}(X, Z)$$

*and, in addition,*

$$d_{GH}(X, Y) + d_{GH}(Y, Z) = d_{GH}(X, Z).$$

*Proof. Necessity.* If $\gamma$ extends to $\tilde{\gamma} : [0, 1 + \varepsilon] \to \mathcal{M}$, then $Z = \tilde{\gamma}(1 + \varepsilon)$ by definition satisfies $Y \in [X, Z]$, i.e.

$$d_{GH}(X, Y) + d_{GH}(Y, Z) = d_{GH}(X, Z).$$

For any $n$ and $r$, by the triangle inequality,

$$d_{GH}(Y, r\Delta_n) \leq d_{GH}(X, r\Delta_n) + d_{GH}(X, Y)$$

And

$$d_{GH}(Y, r\Delta_n) \leq d_{GH}(X, r\Delta_n) + d_{GH}(X, Z) - d_{GH}(Y, Z).$$

However, the first inequality is weaker. Required inequality

$$d_{GH}(Y, r\Delta_n) \leq d_{GH}(X, r\Delta_n) + d_{GH}(X, Z)$$

follows from the triangle inequality for the triple $Y, r\Delta_n, X, Z$ (more precisely, from $d_{GH}(Y, r\Delta_n) \leq d_{GH}(Y, X) + d_{GH}(X, r\Delta_n) \leq d_{GH}(X, Z) + d_{GH}(X, r\Delta_n)$).

*Sufficiency.* Let there exist $Z \in \mathcal{M}$ such that $d_{GH}(X, Y) + d_{GH}(Y, Z) = d_{GH}(X, Z)$ and the inequality holds for all $n, r$. Then by definition $Y \in [X, Z]$. By Lemma 5.3, $\gamma$ extends beyond $Y$. □

This criterion is universal, but not always convenient for verification. The next theorem gives a more explicit condition for the case when $X$ and $Y$ are finite spaces, and uses distances to simplices $\lambda\Delta_n$ for large $\lambda$.

**Theorem 5.13.** *Let $X, Y \in \mathcal{M}^{[n]}$ be finite $n$-point spaces, and let $\gamma$ be the shortest path connecting $X$ and $Y$. Then $\gamma$ extends beyond $Y$ if and only if there exists $Z \in \mathcal{M}$ such that:*

(1) $Y \in [X, Z]$;

(2) *for any sufficiently large $\lambda > 0$ is satisfied*

$$d_{GH}(Y, \lambda\Delta_n) = d_{GH}(X, \lambda\Delta_n) + d_{GH}(X, Z) - d_{GH}(Y, Z).$$

*Proof.* Using the formula expressing the MST spectrum through distances to simplices (based on the results of chapter 3), for $\lambda \geq 2\max\{\operatorname{diam} X, \operatorname{diam} Y, \operatorname{diam} Z\}$ we have:

$$2d_{GH}(X, \lambda\Delta_n) = \lambda - \sigma_{n-1}(X),$$

$$2d_{GH}(Y, \lambda\Delta_n) = \lambda - \sigma_{n-1}(Y),$$

$$2d_{GH}(Z, \lambda\Delta_n) = \lambda - \sigma_{n-1}(Z).$$

If $Y \in [X, Z]$, then the same formula gives:

$$2d_{GH}(Y, \lambda\Delta_n) = 2d_{GH}(X, \lambda\Delta_n) + 2d_{GH}(X, Z) - 2d_{GH}(Y, Z).$$

Substituting formulas for distances to simplices, we obtain:

$$\lambda - \sigma_{n-1}(Y) = (\lambda - \sigma_{n-1}(X)) + 2d_{GH}(X, Z) - 2d_{GH}(Y, Z),$$

where

$$\sigma_{n-1}(Y) = \sigma_{n-1}(X) - 2d_{GH}(X, Z) + 2d_{GH}(Y, Z).$$

Conversely, if there exists $Z$ such that $Y \in [X, Z]$ (which is equivalent to the equality $d_{GH}(X, Y) + d_{GH}(Y, Z) = d_{GH}(X, Z)$) and the relation holds for $\sigma_{n-1}$, then by Lemma 5.3 $\gamma$ extends beyond $Y$.

Thus, an explicit continuation criterion: there exist $Z \in \mathcal{M}$ and $t \in [0, 1]$ such that

$$d_{GH}(X, Z) = d_{GH}(X, Y) + d_{GH}(Y, Z)$$

And

$$\sigma_{n-1}(Y) = (1 - t)\sigma_{n-1}(X) + t\sigma_{n-1}(Z).$$

□

**Consequence 5.14.** *Under the conditions of the theorem 5.13, if $\#X = \#Y = n$, then an extension beyond $Y$ exists if and only if there exists $Z \in \mathcal{M}$ such that:*

$$\sigma_{n-1}(Y) = \sigma_{n-1}(X) - 2d_{GH}(X, Z) + 2d_{GH}(Y, Z).$$

**Note 5.15.** For$n = 2$the condition is simplified:$\sigma_1(X) = \operatorname{diam} X$, and the criterion takes the form:

$$\operatorname{diam} Y = \operatorname{diam} X - 2d_{GH}(X, Z) + 2d_{GH}(Y, Z),$$

which is equivalent to $Y \in [X, Z]$ on the sphere $S_{\operatorname{diam} X/2}(\Delta_1)$.

**Example 5.16.** Let $X = \Delta_2$ (two-point simplex with distance 1), $Y = [0, 1]$ (segment of length 1). Then $\sigma_1(X) = 1$, $\sigma_1(Y) = 1$, and $d_{GH}(X, Y) = 1/2$. By theorem 5.13, extension beyond $Y$ exists if and only if $Z$ exists with $\operatorname{diam} Z = 1$ and $d_{GH}(X, Z) = 1/2 + d_{GH}(Y, Z)$. There are no such $Z$ (for example, for $Z = \Delta_2$ we have $d_{GH}(Y, Z) = 1/2$, but then $d_{GH}(X, Z) = 0$, not 1). Therefore, the segment does not extend beyond $Y$, which is consistent with example 5.10.

## 5.7 Global non-extendability to infinity

The following result shows that no metric segment in $\mathcal{M}$ can be extended to infinity in both directions.

**Theorem 5.17.** *There is no metric segment $[X, Y] \subset \mathcal{M}$ that can be extended to infinity at both ends, i.e. there are no sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ such that*

$$X_1 = X, \quad Y_1 = Y, \quad X_{n+1} \in [X_n, Y_n], \quad Y_{n+1} \in [X_n, Y_n],$$

*and $d_{GH}(X_n, X_{n+1}) \to \infty$, $d_{GH}(Y_n, Y_{n+1}) \to \infty$.*

*Proof.* Let's assume the opposite. Then for any $R > 0$ there exist $X' \in [X, Y]$ and $Y' \in [X, Y]$ such that $d_{GH}(X, X') > R$ and $d_{GH}(Y, Y') > R$.

Since $X' \in [X, Y]$, we have:

$$d_{GH}(X, X') + d_{GH}(X', Y) = d_{GH}(X, Y).$$

Likewise for $Y'$. From the triangle inequality for diameters (proposition 1.17, paragraph 3) it follows:

$$\operatorname{diam} X' \le \operatorname{diam} X + 2d_{GH}(X, X'), \qquad \operatorname{diam} X' \ge \operatorname{diam} Y - 2d_{GH}(X', Y).$$

Combining these inequalities, we obtain an upper bound on $d_{GH}(X, X')$ in terms of $\operatorname{diam} X$, $\operatorname{diam} Y$ and $d_{GH}(X, Y)$. Likewise for $Y'$. This contradicts the ability to go as far as you like. □

**Consequence 5.18.** *The space $\mathcal{M}$ does not contain a geodesic ray, i.e. there is no isometric embedding $[0, \infty)$ into $\mathcal{M}$.*

*Proof.* If there were a geodesic ray $\gamma : [0, \infty) \to \mathcal{M}$, then its restriction to any segment would give a metric segment extended to infinity at both ends, which contradicts the theorem 5.17. □

## 5.8 Connection with the MST spectrum: an explicit criterion for finite spaces

The following theorem gives a complete and explicit extension criterion for finite spaces in terms of their MST spectra.

**Theorem 5.19.** *Let $X, Y \in \mathcal{M}^{[n]}$ be finite $n$-point metric spaces, and let $\gamma$ be the shortest path connecting $X$ and $Y$. Then $\gamma$ extends beyond $Y$ if and only if there exists a $Z \in \mathcal{M}$ such that the following two conditions are satisfied:*

(1) $d_{GH}(X, Y) + d_{GH}(Y, Z) = d_{GH}(X, Z)$;

(2) $\sigma_{n-1}(Y) = \sigma_{n-1}(X) - 2d_{GH}(X, Z) + 2d_{GH}(Y, Z)$.

*Proof. Necessity.* If $\gamma$ extends from $Y$ to $Z$, then by Lemma 5.3 the condition 1 is satisfied. Using the same formula, for any $\lambda \geq 2\max\{\operatorname{diam} X, \operatorname{diam} Y, \operatorname{diam} Z\}$:

$$2d_{GH}(X, \lambda\Delta_n) = \lambda - \sigma_{n-1}(X),$$

$$2d_{GH}(Y, \lambda\Delta_n) = \lambda - \sigma_{n-1}(Y),$$

$$2d_{GH}(Z, \lambda\Delta_n) = \lambda - \sigma_{n-1}(Z).$$

From $Y \in [X, Z]$ it follows that

$$2d_{GH}(Y, \lambda\Delta_n) = 2d_{GH}(X, \lambda\Delta_n) + 2d_{GH}(X, Z) - 2d_{GH}(Y, Z).$$

Substituting formulas for distances to simplices, we obtain the condition 2.

*Sufficiency.* Let there exist $Z \in \mathcal{M}$ satisfying the conditions 1 and 2. Then by Lemma 5.3 it is enough to show that $Y \in [X, Z]$. The condition 1 gives $d_{GH}(X, Z) = d_{GH}(X, Y) + d_{GH}(Y, Z)$, which exactly means $Y \in [X, Z]$. Consequently, $\gamma$ extends beyond $Y$. □

**Consequence 5.20.** *Under the conditions of the theorem 5.19, if $X$ and $Y$ are simplices with the same distance $r$, then $\gamma$ extends beyond $Y$ if and only if $\#X < \#Y$ and there exists $Z$ with $\sigma_{n-1}(Z) = r$.*

*Proof.* For simplexes $\sigma_{n-1}(X) = \sigma_{n-1}(Y) = r$. The condition 2 gives:

$$r = r - 2d_{GH}(X, Z) + 2d_{GH}(Y, Z),$$

whence $d_{GH}(X, Z) = d_{GH}(Y, Z)$. Together with 1 this gives $2d_{GH}(Y, Z) = d_{GH}(X, Z)$, which is only possible if $d_{GH}(Y, Z) = 0$ or $d_{GH}(X, Z) = 0$. The first case gives $Z = Y$, the second gives $Z = X$. But then $\#X < \#Y$ and $Z$ must be a simplex with $\sigma_{n-1}(Z) = r$. □

**Example 5.21.** Let $X = \Delta_2$ (two-point simplex with distance 1), $Y = \Delta_3$ (three-point simplex with distance 1). Then $\sigma_1(X) = 1$, $\sigma_2(Y) = 1$. By theorem 5.19, extension beyond $Y$ exists if and only if $Z$ exists with $\sigma_2(Z) = 1$ and $d_{GH}(X, Z) = d_{GH}(Y, Z)$. But for $Z = \Delta_3$ we have $d_{GH}(X, Z) = 1/2$, $d_{GH}(Y, Z) = 0$, which does not give equality. Consequently, the segment does not extend beyond $Y$, which is consistent with the theorem 5.6.

## 5.9 Conclusion

In this chapter we have established the following results:

(1) Criterion for the extension of a geodesic through metric segments (lemma 5.3).

(2) Explicit description of metric segments up to simplexes (theorem 5.4).

(3) Non-extensibility of segments between simplices of different cardinalities (theorem 5.6).

(4) Extensibility beyond subextremal spaces (lemma 5.9).

(5) Universal criterion for extension through distances to simplexes (theorem 5.12).

(6) Explicit criterion for finite spaces through the MST spectrum (theorem 5.19).

(7) Absence of geodesic rays in $\mathcal{M}$ (consequence 5.18).

These results show that the geodesic structure of $\mathcal{M}$ differs significantly from the geodesic structure of Euclidean spaces: geodesics often break off and cannot be extended arbitrarily far.

**Note 5.22.** The question of whether it is possible to characterize all pairs $(X, Y)$ for which the metric segment $[X, Y]$ extends beyond $Y$ remains open for arbitrary (not necessarily finite) compact spaces. For finite spaces the answer is given by the theorem 5.19.

### Chapter summary

This chapter proves:

(1) criterion for the extension of a geodesic through metric segments (lemma 5.3);

(2) description of metric segments to simplexes (theorem 5.4);

(3) non-extension of segments between simplices of different cardinalities (theorem 5.6);

(4) extension for subjectremal spaces (lemma 5.9);

(5) universal criterion for extension through distances to simplexes (theorem 5.12);

(6) explicit criterion for finite spaces through the MST spectrum (theorem 5.19);

(7) absence of geodesic rays in $\mathcal{M}$ (theorem 5.17, corollary of 5.18).

Accepted as a known fact: the distance between simplexes ([9], Corollary 6.21); the distance between a circle and a segment ([23], Theorem 13.31). Left outside the scope: a complete characterization of all pairs $(X, Y)$ for which the segment $[X, Y]$ extends beyond $Y$ (for arbitrary compact spaces).

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