Problem Sheet: Lectures 1–2

Alexander O. Ivanov's minicourse — Side 1: Lecture 1

Lecture 1: Hausdorff and Gromov–Hausdorff distances.

Distances and Hausdorff distance

A *distance* may take the value ∞ and is symmetric with $\rho(x,x)=0$. Adding the triangle inequality gives a generalized pseudometric; positive definiteness gives a generalized metric. For nonempty $A, B \subset X$,

$$d_H(A,B) = \max\Big\{\sup_{a\in A} |aB|, \sup_{b\in B} |Ab|\Big\}.$$

Equivalently, $d_H(A,B)$ is the least radius for which each set lies in the corresponding closed neighbourhood of the other.

1. On $\mathbb{R}$, classify each function as precisely as possible: a distance, generalized pseudometric, generalized metric, or metric:

$$|x-y|, \qquad |x^2-y^2|, \qquad |x-y|^2, \qquad \frac{|x-y|}{1+|x-y|}.$$

For every failed property, give an explicit counterexample.

2. Compute $d_H(A,B)$ and identify points realizing the directed suprema:
(a) $A=[0,1]\cup[2,3]$, $B=[0,3]\subset\mathbb{R}$;
(b) A is the closed unit disk and B its boundary circle;
(c) $A=\{0,\frac{3}{2},5\}$ and $B=\{\frac{1}{2},2,3,\frac{11}{2}\}\subset\mathbb{R}$.

Correspondences and distortion

A correspondence $R\subset X\times Y$ has surjective projections. Its distortion is

$$\operatorname{dist} R = \sup\Big\{\big||xx'|-|yy'|\big| : (x,y),(x',y')\in R\Big\},$$

$$d_{\mathrm{GH}}(X,Y) = \tfrac{1}{2}\inf_{R\in\mathcal{R}(X,Y)} \operatorname{dist} R.$$

If $\tau\subset\sigma$, then $\operatorname{dist}\tau\le\operatorname{dist}\sigma$, and $\operatorname{dist}(\tau\circ\sigma)\le\operatorname{dist}\sigma+\operatorname{dist}\tau$.

3. Let $X=\{0,1\}$ and $Y=\{0,2\}$ with the induced Euclidean metrics. Compute the distortion of $R=\{(0,0),(1,2)\}$. Then construct two relations for which

$$\operatorname{dist}(\tau\circ\sigma) < \operatorname{dist}\sigma+\operatorname{dist}\tau.$$

Elementary GH estimates

For bounded metric spaces,

$$2d_{\mathrm{GH}}(\Delta_1,X)=\operatorname{diam}X, \qquad |\operatorname{diam}X-\operatorname{diam}Y|\le 2d_{\mathrm{GH}}(X,Y),$$

and

$$2d_{\mathrm{GH}}(\lambda X,\mu X)=|\lambda-\mu|\operatorname{diam}X, \qquad d_{\mathrm{GH}}(\lambda X,\lambda Y)=\lambda d_{\mathrm{GH}}(X,Y).$$

The full relation $X\times Y$ gives $2d_{\mathrm{GH}}(X,Y)\le\max\{\operatorname{diam}X,\operatorname{diam}Y\}$.

4. Let X_a and X_b be two-point spaces whose nonzero distances are $a,b>0$. Compute $d_{\mathrm{GH}}(X_a,X_b)$ and exhibit an optimal correspondence. Use the answer to identify all equality cases among the elementary diameter bounds for this family.

5. Prove that
(a) for a bounded metric space X, and an unbounded metric space Y, $d_{\mathrm{GH}}(X,Y)=\infty$;
(b) $d_{\mathrm{GH}}(\mathbb{R},\mathbb{Q})=0$.

Irreducible correspondences

A correspondence is *irreducible* if none of its proper subrelations is a correspondence. Such a relation has block form

$$R=\bigcup_{i\in I}(X_i\times Y_i), \qquad \min\{\#X_i,\#Y_i\}=1,$$

for partitions $X=\bigsqcup_i X_i$ and $Y=\bigsqcup_i Y_i$. It is enough to minimize distortion over irreducible correspondences.

6. Let X,Y be three-point metric spaces. Write their side lengths in nondecreasing order as $\rho_1\le\rho_2\le\rho_3$ and $\nu_1\le\nu_2\le\nu_3$.
(a) List and count all irreducible correspondences between them.
(b) Prove that

$$2d_{\mathrm{GH}}(X,Y)=\max_{1\le i\le 3}|\rho_i-\nu_i|.$$

Suggested workflow

For a finite GH computation: first restrict to irreducible correspondences, classify their block types, compute each distortion, and only then minimize. Diameter inequalities provide quick lower bounds and a check on the result.

Difficult problems

1. Show that, for an arbitrary subset X of the Euclidean $\mathbb{R}^n$,

$$d_{\mathrm{GH}}(X,\mathbb{R}^n)<\infty \iff d_H(X,\mathbb{R}^n)<\infty.$$

2. Let X and Y be linear normed spaces. Prove that (a) $d_{\mathrm{GH}}(X,Y)<\infty\Longrightarrow d_{\mathrm{GH}}(X,Y)=0$; (b) if $\dim X<\infty$, $d_{\mathrm{GH}}(X,Y)<\infty\Longrightarrow X$ and Y are linearly isometric; (c) the result of (*b*) holds for arbitrary Banach spaces X and Y.

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Alexander O. Ivanov's minicourse — Side 2: Lecture 2

Lecture 2: Properties of the Gromov–Hausdorff distance.

Continuity and covering data

The diameter is 2-Lipschitz:

$$|\operatorname{diam} X - \operatorname{diam} Y| \le 2d_{\mathrm{GH}}(X,Y).$$

The density $\operatorname{dens}(X)$ is the least cardinality of an everywhere dense subset. The covering number $N_\varepsilon(X)$ is the least size of a finite ε-net, with value ∞ if no such net exists. Under GH convergence,

$$\operatorname{dens}(X) \le \liminf_n \operatorname{dens}(X_n), \qquad N_\varepsilon(X) \le \liminf_n N_\varepsilon(X_n).$$

1. Compute $N_\varepsilon(X)$ for (a) $X=[0,1]$; (b) the n-point simplex Δ_n; (c) the Euclidean unit circle. State carefully what happens at the critical radii.

2. Prove the scaling rules

$$\operatorname{dens}(\lambda X) = \operatorname{dens}(X), \qquad N_\varepsilon(\lambda X) = N_{\varepsilon/\lambda}(X).$$

Apply them to $\lambda\Delta_n$ and the circle of radius λ, and compute $\lim_{\varepsilon\to 0+} \varepsilon N_\varepsilon(\lambda S^1)$.

Distances to simplexes

For every bounded X,

$$d_{\mathrm{GH}}(X,\Delta_1) = \tfrac{1}{2}\operatorname{diam} X.$$

If $\#X < n$, then

$$2d_{\mathrm{GH}}(X,\Delta_n) = \max\{1, \operatorname{diam} X - 1\}.$$

If $\#X = n < \infty$ and $\varepsilon(X) = \min\{|xy| : x \ne y\}$, then

$$2d_{\mathrm{GH}}(X,\Delta_n) = \max\{1-\varepsilon(X), \operatorname{diam} X - 1\}.$$

3. Compute $d_{\mathrm{GH}}(X,\Delta_1)$ when $X=[0,1]$, $X=\Delta_n$, and when X is a Euclidean circle or sphere of radius R. Explain why an isometric embedding $\Delta_1 \hookrightarrow \Delta_n$ does not imply zero GH distance.

Suggested workflow

To prove GH convergence, first look for a common realization or a natural correspondence. To disprove continuity of a numerical invariant, compare it with the semicontinuity inequalities and remember that GH convergence forgets labels and completion.

Limits of compact spaces

If compact spaces X_n converge to X in GH distance, then X is totally bounded, its completion $\overline{X}$ is compact, and $d_{\mathrm{GH}}(X,\overline{X})=0$. Completeness itself is not preserved because GH is a pseudometric on arbitrary metric spaces.

4. Come up with a sequence $X_n \xrightarrow[GH]{} \Delta_1$ such that the finite numbers $\operatorname{dens}(X_n)$ are unbounded. Then explain why the constant sequence $[0,1]$ also converges to the noncomplete space $(0,1)$.

5. (a) Let X_n be a Euclidean sphere S^2 with the removed ball of radius $\frac{1}{n}$, equipped with the corresponding intrinsic metric. Show that $\lim_{n\to\infty} d_{\mathrm{GH}}(X_n, S^2) = 0$.

(b) Show that, for a one-dimensional sphere S^1 the similar statement is false.

Uniform convergence of metrics

For metrics ρ_k, ρ on one fixed set X,

$$\|\rho_k - \rho\|_\infty = \sup_{x,y\in X} |\rho_k(x,y) - \rho(x,y)|$$

implies

$$d_{\mathrm{GH}}\big((X,\rho_k),(X,\rho)\big) \le \tfrac{1}{2}\|\rho_k - \rho\|_\infty.$$

The converse fails: GH convergence is insensitive to relabeling points.

6. Show that the converse of this Theorem fails: GH-convergence does not force uniform convergence of metrics on a fixed set. Let $X=\mathbb{Z}$ with $d(x,y)=|x-y|$, and for each $k\ge 2$ let π_k be the transposition of 0 and k and put $d_k(x,y)=|\pi_k(x)-\pi_k(y)|$.

(a) Verify that each d_k is a metric and that (X,d_k) is isometric to (X,d); hence $X_k \to X_\infty$ in d_{GH}.

(b) Compute $d_k(0,1)$ and conclude that d_k does not converge to d even pointwise. Explain the meaning of this phenomenon ("GH-convergence is label-free").

Difficult problems

1. Show that, if X is compact, and, for subsets $X_n \subset X$ we have $\lim_{n\to\infty} d_{\mathrm{GH}}(X_n, X) = 0$, then $\lim_{n\to\infty} d_H(X_n, X) = 0$.

2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of compact length metric spaces that GH-converges to a circle S^1 with the natural intrinsic metric. Show that for sufficiently large $n\in\mathbb{N}$, the spaces X_n cannot be simply connected. What if X_n are not length spaces?