Seminar Problems for Lectures 1–2

Hausdorff and Gromov–Hausdorff Distances

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Hausdorff and Gromov–Hausdorff Distances

Task 1.1: Classifying Distances

Problem 1.1

On $\mathbb{R}$, classify each function as precisely as possible: a distance, generalized pseudometric, generalized metric, or metric:

$$|x-y|, \qquad |x^2-y^2|, \qquad |x-y|^2, \qquad \frac{|x-y|}{1+|x-y|}.$$

For every failed property, give an explicit counterexample.

Task 1.1: Solution

Solution

$|x-y|$ and $|x-y|/(1+|x-y|)$ are metrics. The function $|x^2-y^2|$ is a pseudometric but not a metric, since $d(1,-1)=0$. The square $|x-y|^2$ fails the triangle inequality, for example at $0, 1, 2$.

Task 1.1: Solution

The case $|x-y|/(1+|x-y|)$

The only nontrivial part is the triangle inequality.

$$a = \rho(x,y), \qquad b = \rho(y,z), \qquad c = \rho(x,z).$$

Since ρ is a metric, $c \le a + b$. The function

$$f(t) = \frac{t}{1+t}$$

is increasing for $t \ge 0$, so

$$d(x,z) = f(c) \le f(a+b).$$

Now for $a, b \ge 0$,

$$f(a+b) = \frac{a+b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b} = f(a) + f(b),$$

because the difference of the right-hand side and left-hand side is

$$\frac{ab(a+b+2)}{(1+a)(1+b)(1+a+b)} \ge 0.$$

Therefore,

$$d(x,z) \le d(x,y) + d(y,z).$$

So $d = \rho/(1+\rho)$ satisfies all metric axioms. In fact, it is a bounded metric with $d(x,y) < 1$ for all x, y.

Task 1.2: Hausdorff Distance

Problem 1.2

Compute $d_H(A, B)$ and identify points realizing the directed suprema:

(a) $A = [0, 1] \cup [2, 3]$ and $B = [0, 3] \subset \mathbb{R}$;

(b) A is the closed unit disk and B is its boundary circle;

(c)
$$A = \left\{0, \frac{3}{2}, 5\right\}, \qquad B = \left\{\frac{1}{2}, 2, 3, \frac{11}{2}\right\} \subset \mathbb{R}.$$

Task 1.2: Solution

Solution

(a) $d_H(A, B) = 1/2$, attained at $3/2$ in the missing gap.

(b) $d_H(A, B) = 1$, attained at the centre of the disk.

(c) $d_H(A, B) = 3/2$, with the directed supremum from B attained at 3.

Task 1.3: Distortion and Composition

Problem 1.3

Let $X = \{0, 1\}$ and $Y = \{0, 2\}$ with the induced Euclidean metrics. Compute the distortion of

$$R = \{(0, 0), (1, 2)\}.$$

Construct metric spaces and two relations for which

$$\operatorname{dist}(\tau \circ \sigma) < \operatorname{dist} \sigma + \operatorname{dist} \tau.$$

Task 1.3: Solution

Solution

For $R = \{(0,0),(1,2)\}$, the paired distances are 0 and 1 versus 0 and 2, hence $\operatorname{dist} R = 1$. Strict subadditivity follows by composing relations whose individual distortions are positive while the composite collapses some pairs; for example, take three copies of $\{0,1\}$ and relations $\sigma = \{(0,0),(1,0),(1,1)\}$ and $\tau = \{(0,0),(1,0),(1,1)\}$. Then $\operatorname{dist}\sigma = \operatorname{dist}\tau = 1$ while $\operatorname{dist}(\tau \circ \sigma) = 1 < 2$.

Task 1.4: Two-Point Spaces

Problem 1.4

Let X_a and X_b be two-point spaces whose nonzero distances are $a, b > 0$.

(a) Compute $d_{\mathrm{GH}}(X_a, X_b)$.

(b) Exhibit an optimal correspondence.

(c) Use the answer to identify all equality cases among the elementary diameter bounds for this family.

Task 1.4: Solution

Solution

The bijection $R = \{(0,0),(a,b)\}$ has distortion $|a-b|$, hence

$$d_{\mathrm{GH}}(X_a, X_b) = \frac{|a-b|}{2}.$$

Thus the diameter lower bound is always sharp; the elementary upper bound is not attained for $a, b > 0$.

Task 1.5: Zero and Infinite GH Distance

Problem 1.5

Prove that:

(a) if X is bounded and Y is unbounded, then

$$d_{\mathrm{GH}}(X, Y) = \infty;$$

(b)

$$d_{\mathrm{GH}}(\mathbb{R}, \mathbb{Q}) = 0.$$

Task 1.5: Solution

Solution

(a) A finite-distortion correspondence would force the diameters to differ by a finite amount, which is impossible when X is bounded and Y is unbounded.

(b) Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the relation $R_\varepsilon = \{(x, q) : |x - q| < \varepsilon\}$ has distortion at most 2ε. Hence $d_{\mathrm{GH}}(\mathbb{R}, \mathbb{Q}) = 0$.

Task 1.6: Three-Point Spaces

Problem 1.6

Let X, Y be three-point metric spaces. Write their side lengths in nondecreasing order as

$$\rho_1 \le \rho_2 \le \rho_3, \qquad \nu_1 \le \nu_2 \le \nu_3.$$

(a) List and count all irreducible correspondences between them. Separate the bijections from the non-bijective block type, and justify that no other type occurs.

(b) Prove that

$$2d_{\mathrm{GH}}(X, Y) = \max_{1 \le i \le 3} |\rho_i - \nu_i|.$$

Task 1.6: Solution

Solution

Every irreducible correspondence between three-point spaces is either a bijection (there are $3! = 6$) or has one singleton block on one side and a two-point block on the other. Matching side lengths in increasing order minimizes the maximum absolute error, so

$$2d_{\mathrm{GH}}(X, Y) = \max_i |\rho_i - \nu_i|.$$

Properties of the Gromov–Hausdorff Distance

Task 2.1: Covering Numbers

Problem 2.1

Compute $N_\varepsilon(X)$ for:

(a) $X = [0, 1]$;

(b) the n-point simplex Δ_n;

(c) the Euclidean unit circle.

State carefully what happens at the critical radii.

Task 2.1: Solution

Solution

For closed ε-balls,

$$N_\varepsilon([0,1]) = \max\left\{1, \left\lceil \frac{1}{2\varepsilon} \right\rceil\right\}, \qquad N_\varepsilon(\Delta_n) = \begin{cases} n, & 0 < \varepsilon < 1, \\ 1, & \varepsilon \geq 1. \end{cases}$$

For the Euclidean unit circle,

$$N_\varepsilon(S^1) = \min\left\{m \geq 1 : 2\sin\frac{\pi}{2m} \leq \varepsilon\right\}.$$

Task 2.2: Scaling Rules

Problem 2.2

Prove the scaling rules

$$\operatorname{dens}(\lambda X) = \operatorname{dens}(X), \qquad N_\varepsilon(\lambda X) = N_{\varepsilon/\lambda}(X).$$

Apply them to $\lambda\Delta_n$ and the circle of radius λ, and compute

$$\lim_{\varepsilon \to 0+} \varepsilon N_\varepsilon(\lambda S^1).$$

Task 2.2: Solution

Solution

Scaling preserves dense subsets and carries ε/λ-nets to ε-nets. Therefore

$$\operatorname{dens}(\lambda X) = \operatorname{dens}(X), \qquad N_\varepsilon(\lambda X) = N_{\varepsilon/\lambda}(X).$$

Consequently, $\lim_{\varepsilon \to 0+} \varepsilon N_\varepsilon(\lambda S^1) = \pi\lambda$.

Task 2.3: Distance to a Point

Problem 2.3

Compute $d_{\mathrm{GH}}(X, \Delta_1)$ when:

(a) $X = [0, 1]$;

(b) $X = \Delta_n$;

(c) X is a Euclidean circle or sphere of radius R.

Explain why an isometric embedding $\Delta_1 \hookrightarrow \Delta_n$ does not imply zero GH distance.

Task 2.3: Solution

Solution

In general,

$$d_{\mathrm{GH}}(X, \Delta_1) = \frac{1}{2}\,\mathrm{diam}(X).$$

Thus the values are $1/2$ for $[0,1]$, $1/2$ for Δ_n ($n \geq 2$), and R for a Euclidean circle or sphere of radius R. An isometric embedding only identifies a subspace; GH distance compares the entire spaces.

Task 2.4: Semicontinuity and Completion

Problem 2.4

(a) Give a sequence $X_n \to \Delta_1$ such that the finite numbers $\operatorname{dens}(X_n)$ are unbounded.

(b) Explain why the constant sequence $[0,1]$ also converges to the noncomplete space $(0,1)$.

Task 2.4: Solution

Solution

Take $X_n = (1/n)\Delta_n$. Then $\operatorname{dens}(X_n) = n$ while

$$d_{\mathrm{GH}}(X_n, \Delta_1) = \frac{1}{2n} \longrightarrow 0.$$

Also $d_{\mathrm{GH}}([0,1],(0,1)) = 0$ because $(0,1)$ is dense in its completion $[0,1]$.

Task 2.5: Punctured Spheres

Problem 2.5

(a) Let X_n be the Euclidean sphere S^2 with a ball of radius $1/n$ removed, equipped with the corresponding intrinsic metric. Show that

$$\lim_{n\to\infty} d_{\mathrm{GH}}(X_n, S^2) = 0.$$

(b) Show that the analogous statement for the one-dimensional sphere S^1 is false.

Task 2.5: Solution

Solution

For the punctured S^2, the removed cap has diameter tending to zero. The natural correspondence with S^2 therefore has distortion tending to zero, so $d_{\mathrm{GH}}(X_n, S^2) \to 0$. For S^1, removing an arc produces an interval with its intrinsic metric; the lost shortcut prevents convergence to the circle.

Task 2.6: GH Convergence Is Label-Free

Problem 2.6

Let $X = \mathbb{Z}$ with $d(x, y) = |x - y|$. For each $k \geq 2$, let π_k be the transposition of 0 and k, and put

$$d_k(x, y) = |\pi_k(x) - \pi_k(y)|.$$

(a) Verify that each d_k is a metric and that (X, d_k) is isometric to (X, d); hence $X_k \to X$ in d_{GH}.

(b) Compute $d_k(0, 1)$ and conclude that d_k does not converge to d even pointwise. Explain the meaning of this phenomenon.

Task 2.6: Solution

Solution

Each d_k is the pullback of d by π_k, hence $(\mathbb{Z}, d_k) \cong (\mathbb{Z}, d)$ and their GH distance is zero. However,

$$d_k(0,1) = |k-1|,$$

so d_k does not converge pointwise to d. GH convergence is independent of point labels.

Discussion