On the Existence of Minimal Parametric Networks

Galstyan A. Kh.
Research Associate

School of Mathematics, Harbin Institute of Technology

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The problem of finding minimal (that is, shortest in length) networks connecting a prescribed finite set of points (the boundary) in a metric space is a classical problem of the calculus of variations and geometric optimization, with numerous applications in science and engineering. Its best-known formulation is the Steiner problem, which asks for a shortest connected network (a metric graph) containing the boundary, possibly with extra vertices; these added vertices are called interior vertices.

Searching for such a minimal network can be extremely difficult because of the so-called "combinatorial explosion": as the number of boundary points grows, the number of possible ways to interconnect the original and added points grows very rapidly. To reduce the combinatorial complexity, other formulations of the minimal connection problem are considered. One possibility is to fix the tree structure, making it a parameter. This gives rise to the problem of finding a minimal parametric network.

The first question that arises in solving this problem is whether its formulation is correct in a specific metric space. Namely, is it necessarily solvable there? That is, for every boundary in the given metric space, does there exist a minimal parametric network joining this boundary?

The forthcoming lecture will be devoted to which spaces (and hyperspaces) are suitable for solving the problem of finding shortest networks, which topological and functional-analytic conditions are responsible for this.

Definition 1

Let $G = (V, E)$ be a connected graph, and let $\mathcal{A} \subset V$. Then G is said to join $\mathcal{A}$. The vertices in $\mathcal{A}$ are called boundary vertices, the set $\mathcal{A}$ itself is called the boundary of the graph, and the vertices in $V \setminus \mathcal{A}$ are called interior vertices.

Henceforth all graphs are assumed finite and connected.

Definition 2

A network of the type $G = (V, E)$ in an extended pseudometric space X is an arbitrary mapping $g\colon V \to X$. The graph G is also called the parameterizing graph of g.

The length of an edge $\{g(v_i), g(v_j)\}$ of the network g is defined as the distance $|g(v_i)\, g(v_j)|$, and the length of the network g is defined as the sum of the lengths of all its edges:

$$|g| := \sum_{\{v_i, v_j\} \in E} |g(v_i)\, g(v_j)|. \tag{1}$$

Note that since X is an extended pseudometric space, the length of an edge, and hence the length of a network in X, may be equal to ∞.

Let a boundary ∂G be a subset of an extended pseudometric space X. We shall assume that in this case all networks g satisfy the following condition: **the restriction of g to ∂G is the identity mapping**. Thus, each such network g is uniquely determined by the images of the interior vertices of the parametrizing graph G.

Definition 3

Graphs $G = (V, E)$ and $H = (W, F)$ are called isomorphic if there exists a bijective map $\phi\colon V \to W$ such that $\{u, v\} \in E$ if and only if $\{\phi(u), \phi(v)\} \in F$. Such a mapping ϕ is called an isomorphism of the graphs G and H.

Definition 4

Let a graph $G_1 = (V_1, E_1)$ parameterizing a network g_1 and a graph $G_2 = (V_2, E_2)$ parameterizing a network g_2 be given and $\mathcal{A} = \partial G_1 = \partial G_2$. These networks are said to be of the same type (parameter), say G_1, if there exists an isomorphism $\phi\colon V_1 \to V_2$ between the graphs G_1 and G_2 which is the identity on $\mathcal{A}$.

For a fixed boundary $\mathcal{A}$ in an extended pseudometric space X, consider a graph $G = (V, E)$ joining $\mathcal{A}$, that is $\partial G = \mathcal{A} \subset X$. Denote by $[G, \mathcal{A}]$ the set of all networks of this type G in the extended pseudometric space X. By the convention made above, all networks $g \in [G, \mathcal{A}]$ satisfy the condition $g\big|_{\partial G = \mathcal{A}} = \mathrm{id}$.

Definition 5

A network from $[G, \mathcal{A}]$ that has the smallest possible length among all networks in $[G, \mathcal{A}]$ is called a minimal parametric network of type G joining $\mathcal{A}$.

A minimal parametric network does not always exist. Nevertheless, the infimum of the quantities $|g|$ over all networks $g \in [G, \mathcal{A}]$ always exists.

Definition 6

The quantity

$$\mathrm{mpn}[G, \mathcal{A}] = \inf_{g \in [G, \mathcal{A}]} |g| \tag{2}$$

is called the length of the minimal parametric network of type G joining $\mathcal{A}$.

Definition 7

The length of the minimal Steiner tree joining a subset $\mathcal{A}$ is the value

$$\mathrm{smt}(\mathcal{A}) = \inf_{G\colon\, \partial G = \mathcal{A}} \mathrm{mpn}[G, \mathcal{A}]. \tag{3}$$

Definition 8

Let X be a extended pseudometric space, let a graph $G = (V, E)$ join $\mathcal{A} \subset X$, let $g \in [G, \mathcal{A}]$ be such that $|g| = \mathrm{smt}(\mathcal{A})$. If there does not exist a graph $G' = (V', E')$ joining $\mathcal{A}$ and a network $g' \in [G', \mathcal{A}]$ such that $|g'| = \mathrm{smt}(\mathcal{A})$ and $\#(V' \setminus \mathcal{A}) < \#(V \setminus \mathcal{A})$, then g is called a *minimal Steiner tree joining* $\mathcal{A}$.

Proposition 1

Let X be a extended pseudometric space, a graph G join $\mathcal{A} \subset X$ and $g \in [G, \mathcal{A}]$ be a minimal Steiner tree joining $\mathcal{A}$. Then any interior vertex of G has at least 3 adjacent vertices.

Proposition 2

Let X be a extended pseudometric space, a graph $G = (V, E)$ join $\mathcal{A} \subset X$, $\#\mathcal{A} = n$ and $g \in [G, \mathcal{A}]$ be a minimal Steiner tree joining $\mathcal{A}$. Then the graph G has at most $n - 2$ interior vertices.

Proposition 3

Let X be an extended pseudometric space, $\mathcal{A}$ be a finite subset of X and G be a graph joining $\mathcal{A}$. If there exist points $a, b \in \mathcal{A}$ such that $|a\,b| = \infty$, then $\mathrm{mpn}[G, \mathcal{A}] = \infty$.

Corollary 9

Let X be an extended pseudometric space and $\mathcal{A}$ be a finite subset of X. If there exist points $a, b \in \mathcal{A}$ such that $|a\,b| = \infty$, then $\mathrm{smt}(\mathcal{A}) = \infty$.

Proposition 4

Let X be an extended pseudometric space, $\mathcal{A}$ be a finite subset of X and G be a graph joining $\mathcal{A}$. If for any points $a, b \in \mathcal{A}$ we have $|a\,b| < \infty$, then $\mathrm{mpn}[G, \mathcal{A}] < \infty$.

Note that on an extended pseudometric space X one can naturally introduce a binary relation $\sim$, which is an equivalence relation: $x \sim y$ if and only if $|x\,y| < \infty$.

Definition 10

The equivalence class of the relation $\sim$ on X is called a finiteness class.

Remark 1

Thus, the problem of finding a minimal parametric network or a minimum Steiner tree can be nontrivial only within a finiteness class. Moreover, each finiteness class is a pseudometric space.

Let X be a topological space, and let τ be another topology on X. We denote by $\mathrm{Cl}_\tau A$ the closure in the topology τ. Note that we allow τ to coincide with an initial topology on X. Define

$$\mathcal{P}_{\mathrm{Cl}_\tau}(X) \quad := \quad \{A \subset X : A \text{ is nonempty and } \tau\text{-closed}\}.$$

Let X be a pseudometric space:

$$\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B}}(X) \quad := \quad \{A \subset X : A \text{ is nonempty, } \tau\text{-closed and bounded}\}.$$

Definition 11

The above families of sets $\mathcal{P}_*(X)$ endowed with Hausdorff distance are called *hyperspaces* over space X.

Note that if X is a pseudometric space then, generally speaking, a hyperspace $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$ is an extended pseudometric space and $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B}}(X)$ is one of finiteness classes on $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$. In what follows, when speaking about hyperspaces, we consider $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B}}(X)$.

RESEARCH QUESTION: Consider a pseudometric space X and a finiteness class $\mathcal{P}^f_{\mathrm{Cl}_\tau}(X)$ on the space $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$ that is different from $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B}}(X)$. Let τ be the pseudometric topology. For hyperspace $\mathcal{P}^f_{\mathrm{Cl}_\tau}(X)$, find conditions on X or on type of unbounded subsets, under which there exists a minimal parametric network of type G joining $\mathcal{A}$ for every finite collection of elements $\mathcal{A}$ in $\mathcal{P}^f_{\mathrm{Cl}_\tau}(X)$ and for every graph G.

Compactly Equipped Spaces

Definition 12

Let X be a topological space and $f\colon X \to \mathbb{R} \cup \{-\infty, \infty\}$. The function f is called *lower semicontinuous* at a point $x \in X$ if for any net $\{x_\alpha\}_{\alpha\in\Lambda} \subset X$ converging to x, we have

$$\liminf_{\alpha} f(x_\alpha) \geq f(x). \tag{4}$$

Accordingly, f is called *lower semicontinuous* on X if it is lower semicontinuous at every point of X.

Definition 13

A pseudometric space X is called *compactly equipped* if every closed ball B in X can be equipped with its own topology τ satisfying:

- (B, τ) is a compact topological space;
- the pseudometric of X is lower semicontinuous on $B \times B$ with respect to the product topology $\tau \times \tau$.

Theorem 14

Let X be a compactly equipped space. Then, for any finite set $\mathcal{A} \subset X$ and for any graph $G = (V, E)$ with boundary $\mathcal{A}$, there exists a minimal parametric network of type G joining $\mathcal{A}$.

Proof:

Let $m = \mathrm{mpn}[G, \mathcal{A}]$. For each $n \in \mathbb{N}$ choose $g_n \in [G, \mathcal{A}]$ such that $|g_n| < m + \frac{1}{n}$. Then the sequence of lengths $|g_n|$ is bounded by $m + 1$.

Fix a point $a \in \mathcal{A}$. Since the graph $G = (V, E)$ is connected, for any vertex $v \in V$ there exists a path $\{v_1 = v, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k = a\}$ connecting v to a. By the triangle inequality we get

$$\big|g_n(v)\, a\big| \le \sum_{i=1}^{k-1} \big|g_n(v_i)\, g_n(v_{i+1})\big| \le |g_n| \le m + 1 \tag{5}$$

for all n. Hence, all points $g_n(v)$ lie in the closed ball $B = B_{m+1}(a)$. By the definition of a compactly equipped space, on B there exists a compact topology τ such that the pseudometric of the space X is lower semicontinuous on $B \times B$ with respect to the topology $\tau \times \tau$.

Let $V_{\text{int}} = V \setminus \mathcal{A}$ be the set of interior vertices. Any network $g \in [G, \mathcal{A}]$ with images in B is uniquely determined by the restriction $g|_{V_{\text{int}}}$, since $g|_{\mathcal{A}} = \mathrm{id}$. Consider the space

$$Y = \prod_{v \in V_{\text{int}}} B, \tag{6}$$

endowed with the product topology τ_Y generated by τ. By Tychonoff's theorem, Y is compact.

Consider the sequence of image tuples $\big\{g_n(V_{\text{int}})\big\} \subset Y$. It is well known that in a compact space every net (in particular, every sequence) has a convergent subnet. Let $\big\{g_\alpha(V_{\text{int}})\big\}_{\alpha \in \Lambda}$ (provided by $g_\alpha(V_{\text{int}}) = g_{n(\alpha)}(V_{\text{int}})$) be a subnet converging in τ_Y to some point $\widetilde{g}(V_{\text{int}}) \in Y$. This means that for each vertex $v \in V_{\text{int}}$, the net $\big\{g_\alpha(v)\big\} \subset B$ converges to $\widetilde{g}(v) \in B$ in the topology τ. For boundary vertices $a \in \mathcal{A}$, set $\widetilde{g}(a) = a$. Thus, $\widetilde{g} \in [G, \mathcal{A}]$.

Estimate the length of $\widetilde{g}$. For any edge $\{u, v\} \in E$, we have $g_\alpha(u) \to \widetilde{g}(u)$ and $g_\alpha(v) \to \widetilde{g}(v)$ in the topology τ. By the lower semicontinuity of the pseudometric of X in the topology $\tau \times \tau$, we have

$$\big|\widetilde{g}(u)\,\widetilde{g}(v)\big| \le \liminf_\alpha \big|g_\alpha(u)\,g_\alpha(v)\big|. \tag{7}$$

Summing over all edges, we obtain

$$|\widetilde{g}| = \sum_{\{u,v\}\in E} \big|\widetilde{g}(u)\,\widetilde{g}(v)\big| \le \sum_{\{u,v\}\in E} \liminf_\alpha \big|g_\alpha(u)\,g_\alpha(v)\big| \le$$

$$\le \liminf_\alpha \sum_{\{u,v\}\in E} \big|g_\alpha(u)\,g_\alpha(v)\big| = \liminf_\alpha |g_\alpha|.$$

Since $\{g_\alpha\}_{\alpha\in\Lambda}$ is a subnet of the sequence $\{g_n\}$, for which $\lim\limits_{n\to\infty} |g_n| = m$, we have $\lim\limits_\alpha |g_\alpha| = m$. Consequently, $\liminf\limits_\alpha |g_\alpha| = m$, whence

$$\mathrm{mpn}[G, \mathcal{A}] \le |\widetilde{g}| \le m = \mathrm{mpn}[G, \mathcal{A}]. \tag{8}$$

Thus, $|\widetilde{g}| = \mathrm{mpn}[G, \mathcal{A}]$, which by definition means that $\widetilde{g}$ is a minimal parametric network of type G joining $\mathcal{A}$.

□

Corollary 15

Let X be a compactly equipped space. Then, for any finite set $\mathcal{A} \subset X$, there exists a minimal Steiner tree joining $\mathcal{A}$.

The notion of a compactly equipped space is meaningful. Namely, for example, it follows from Riesz's lemma that the closed unit ball of a normed space X is compact if and only if X is finite-dimensional. Moreover, the following criterion is well known.

Theorem 16 (Dixmier)

A normed space X is a dual space if and only if there exists a locally convex topology τ on X such that

- *each closed ball of X is τ-compact;*
- *the metric of X is lower semicontinuous on $X \times X$ with respect to the product topology $\tau \times \tau$.*

If X is dual, then the weak* topology can be taken as such a topology τ.

Corollary 17

A dual space is compactly equipped.

RESEARCH QUESTION: Are there any examples of non-dual compactly equipped normed spaces? In general, it would be interesting to classify compactly equipped spaces.

Indeed, the following proposition holds (we leave it without proof).

Proposition 5

A compactly equipped space is complete.

But there also exist incomplete metric spaces such that every finite subset can be joined by a minimal parametric network of arbitrary type. We shall formulate an example of such a space as a proposition.

Proposition 6

Let $X = \mathbb{Q}$ be the set of rational numbers with metric $|x\,y| = |x - y|$ for all $x, y \in \mathbb{Q}$. Then for any finite set $\mathcal{A} \subset \mathbb{Q}$ and any graph $G = (V, E)$ with boundary $\mathcal{A}$, there exists a minimal parametric network of type G joining $\mathcal{A}$.

Brief idea of proof:

- Reduce the problem of finding a minimal parametric network to a linear programming problem where the objective function corresponds to the length of the network;
- Show that there are points in space that satisfy the constraint system and the objective function is bounded from below;
- Use the fundamental linear programming theorem, which says that in such a situation there is an optimal basic feasible solution to the constraint system from the point of view of the objective function;
- Say that from Cramer's rule, any basic solution will be rational in our situation, since all coefficients of the system are rational, which completes the proof.

□

RESEARCH QUESTION: Derive the conditions for the existence of a minimal parametric network in incomplete pseudometric spaces.

In the article *B. B. Bednov, N. P. Strelkova, On the existence of shortest networks in Banach spaces*, Strelkova gave an example of a Banach space in which for any $n \geq 3$ there exists a collection $\mathcal{A} = \{x_1, \ldots, x_n\}$ for which no minimal Steiner tree joining this collection exists. In the same article, Bednov also gave a sufficient condition on a Banach space X for any finite collection in X and for any type to be joinable by a minimal parametric network of that type. This condition is precisely the existence of a norm-one projector $P\colon X^{**} \to X$.

So, a bidual is a dual space, which means that X^{**} is a compactly equipped space and so any finite set in it can be joined by a minimal parametric network. The projector of norm 1 on X^{**} allows us to obtain a minimal network in X itself. Thus, we can replace X^{**} by an arbitrary compactly equipped normed space Y.

Theorem 18

Let X be a Banach space, Y be a compactly equipped Banach space such that $X \subset Y$, and let there exist a projector $P\colon Y \to X$ of norm 1. Then, for any finite set $\mathcal{A} \subset X$ and for any graph $G = (V, E)$ with boundary $\mathcal{A}$, there exists a minimal parametric network of type G joining $\mathcal{A}$.

Brief idea of proof:

- In Y, there exists a minimal parametric network of type G joining $\mathcal{A} \subset X$.
- Apply the projector P to the image of the network. Since $\|P\| = 1$, the projection will not increase the length of the network.
- But $\operatorname{mpn}\big[G, \mathcal{A} \subset X\big] \geq \operatorname{mpn}\big[G, \mathcal{A} \subset Y\big] \Rightarrow$ we obtain a minimal parametric network in X.

□

RESEARCH QUESTION:

- How much broader is the class of Banach spaces X from the previous theorem (which can be embedded in a compactly equipped Banach space Y with a norm-1 projector) than the class of compactly equipped Banach spaces? Provide as many examples as possible and, in this way, compile a list of spaces where the problem of finding minimal parametric networks has been solved.
- Ideally, of course, we would like to answer the following question. Is it possible, based on counterexamples of spaces such as those studied by Strelkova, where the problem of finding the shortest network is unsolvable, and based on the resulting theory of compactly equipped spaces, to derive a criterion for the solvability of the problem of finding minimal parametric networks in Banach spaces?

Definition 19

Let (X, τ) be a topological space. The *Vietoris topology* is a topology on the set $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$ defined by a subbase consisting of two types of sets for each τ-open $U \subset X$:

- $U^- = \left\{M \in \mathcal{P}_{\mathrm{Cl}_\tau}(X) \colon M \cap U \neq \emptyset\right\}$;
- $U^+ = \left\{M \in \mathcal{P}_{\mathrm{Cl}_\tau}(X) \colon M \subset U\right\}$.

Thus, the subbase of the Vietoris topology is:

$$\mathcal{S} = \{U^- : U \in \tau\} \cup \{U^+ : U \in \tau\}. \tag{9}$$

The Vietoris topology on $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$ generated by τ will be denoted in this text by V_τ.

Theorem 20 (N. Strickland, "Topology")

Let (X, τ) be a compact Hausdorff topological space. Then $\left(\mathcal{P}_{\mathrm{Cl}_\tau}(X), V_\tau\right)$ is also a compact Hausdorff topological space.

Definition 21

Let (X, τ) be a Hausdorff topological space and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$.

- The *lower limit* of the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is the set of all points $x \in X$ such that for each neighborhood $U_x \in \tau$ of x, there exists an index $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, we have $A_\lambda \cap U_x \neq \emptyset$. The lower limit of the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is denoted by $\mathrm{Li}_\lambda\, A_\lambda$.
- The *upper limit* of the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is the set of all points $x \in X$ such that for each neighborhood $U_x \in \tau$ of x and for any index $\lambda_0 \in \Lambda$, there exists $\lambda \geq \lambda_0$ such that $A_\lambda \cap U_x \neq \emptyset$. The upper limit of the net $\{A_\lambda\}_{\lambda\in\Lambda}$ is denoted by $\mathrm{Ls}_\lambda\, A_\lambda$.
- The net $\{A_\lambda\}_{\lambda\in\Lambda}$ is said to converge in the *sense of Kuratowski* to a subset $A \subset X$ if $\mathrm{Li}_\lambda\, A_\lambda = \mathrm{Ls}_\lambda\, A_\lambda = A$. In this case, the set A is called the *Kuratowski limit* and is denoted by $\mathrm{Lim}_\lambda\, A_\lambda$.

Example 22

Let X be a metric space and let τ be the metric topology. Take two sequences: $\{A_i\}$ and $\{B_j\}$ in $\mathcal{P}_{\mathrm{Cl}_\tau}(X)$. The first converges in the Hausdorff metric to A, and the second converges in the Hausdorff metric to B. Consider the sequence $\{X_k\}$ that is the union of these two sequences: $X_{2k} = A_k$ and $X_{2k+1} = B_k$. Then $\mathrm{Li}_k\, X_k = \mathrm{Cl}_\tau(A) \cap \mathrm{Cl}_\tau(B)$ and $\mathrm{Ls}_k\, X_k = \mathrm{Cl}_\tau(A) \cup \mathrm{Cl}_\tau(B)$.

Theorem 23 (G. Beer, "Topologies on Closed and Closed Convex Sets")

Let (X, τ) be a compact Hausdorff topological space. A net $\{A_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{P}_{\mathrm{Cl}_\tau}(X)$ converges in the topology V_τ to $A \in \mathcal{P}_{\mathrm{Cl}_\tau}(X)$ if and only if $\mathrm{Lim}_\lambda A_\lambda = A$.

Theorem 24

Let X be a pseudometric space and let a closed ball B in X be equipped with its own topology τ such that:

- *(B, τ) is a compact Hausdorff topological space;*
- *the pseudometric of X is lower semicontinuous on $B \times B$ with respect to the product topology $\tau \times \tau$.*

Then the Hausdorff distance d_H induced by the pseudometric of X is lower semicontinuous on $\mathcal{P}_{\mathrm{Cl}_\tau}(B) \times \mathcal{P}_{\mathrm{Cl}_\tau}(B)$ in the topology $V_\tau \times V_\tau$.

Proof:

Let $\{(M_\alpha, N_\alpha)\}_{\alpha\in\Lambda}$ be a net in $\mathcal{P}_{\mathrm{Cl}_\tau}(B) \times \mathcal{P}_{\mathrm{Cl}_\tau}(B)$ converging to $(M, N) \in \mathcal{P}_{\mathrm{Cl}_\tau}(B) \times \mathcal{P}_{\mathrm{Cl}_\tau}(B)$ in the topology $V_\tau \times V_\tau$. We will show that $\liminf\limits_{\alpha} d_H(M_\alpha, N_\alpha) \geq d_H(M, N)$.

Assume that $\liminf\limits_{\alpha} d_H(M_\alpha, N_\alpha) < d_H(M, N)$. Since the pseudometric of X is bounded on B, then $d_H(M, N) < \infty$. So there exist $\varepsilon > 0$ and a subnet $\{(M_\beta, N_\beta)\}$ such that for every β we have $d_H(M_\beta, N_\beta) < d_H(M, N) - 2\varepsilon$. Hence,

$$\sup_{x\in M_\beta} |x\, N_\beta| < d_H(M, N) - 2\varepsilon \tag{10}$$

for every β.

Without loss of generality, assume that $d_H(M, N) = \sup_{x\in M} |x\, N|$ (otherwise, interchange M and N). By the definition of supremum, for the chosen ε, there exists $m \in M$ such that $d_H(M, N) - |m\, N| < \varepsilon$, that is,

$$|m\, N| > d_H(M, N) - \varepsilon. \tag{11}$$

Since τ is a compact Hausdorff topology and the net $\{M_\beta\}$ converges to M with respect to the topology V_τ, it follows from the theorem from Beer's book that $\mathrm{Lim}_\beta\, M_\beta = M$. So, for the point m, there exist points $m_\beta \in M_\beta$ such that the net $\{m_\beta\}$ converges to m with respect to the topology τ. By (10), we have

$$|m_\beta\, N_\beta| < d_H(M, N) - 2\varepsilon. \tag{12}$$

Since (B, τ) is compact and the N_β are τ-closed, the N_β are τ-compact. Together with the lower semicontinuity of the pseudometric of X on B in the topology $\tau \times \tau$, this yields the existence of points $n_\beta \in N_\beta$ such that

$$|m_\beta\, n_\beta| = |m_\beta\, N_\beta| < d_H(M, N) - 2\varepsilon. \tag{13}$$

By the compactness of (B, τ), the net $\{n_\beta\}$ has a subnet $\{n_\gamma\}$ which converges in τ to some $n \in B$. The net $\{N_\gamma\}$ converges to N in the topology V_τ, therefore $\mathrm{Lim}_\gamma\, N_\gamma = N$ by the theorem from Beer's book. Since $n_\gamma \in N_\gamma$, by the definition of the Kuratowski limit, we have $n \in N$. Thus, we have $(m_\gamma, n_\gamma) \to (m, n)$. From (13) and from the lower semicontinuity of the pseudometric of X on B in the topology $\tau \times \tau$, it follows that

$$|m\, n| \le \liminf_\gamma |m_\gamma\, n_\gamma| \le d_H(M, N) - 2\varepsilon. \tag{14}$$

Since $n \in N$, we have $|m\, N| \le |m\, n| \le d_H(M, N) - 2\varepsilon$. But this contradicts (11).

Hence, $\liminf_\alpha d_H(M_\alpha, N_\alpha) \ge d_H(M, N)$, and therefore d_H is lower semicontinuous on $\mathcal{P}_{\mathrm{Cl}_\tau}(B) \times \mathcal{P}_{\mathrm{Cl}_\tau}(B)$ in the topology $V_\tau \times V_\tau$.

□

Examples of Compactly Equipped Spaces

Definition 25

Let X be a pseudometric space and τ be a topology on X. Then X is called *compactly equipped with a topology* τ (or *τ-compactly equipped*) if the following conditions hold:

- each closed ball $B \subset X$ is τ-compact;
- for each closed ball $B \subset X$, the pseudometric of X is lower semicontinuous on $B \times B$ with respect to the product topology $\tau \times \tau$.

Theorem 26

Let X be a pseudometric space and let a closed ball B in X be equipped with its own topology τ such that:

- *(B, τ) is a compact Hausdorff topological space*;
- *the pseudometric of X is lower semicontinuous on $B \times B$ with respect to the product topology $\tau \times \tau$.*

Then $\mathcal{P}_{\mathrm{Cl}_\tau}(B)$ is a V_τ-compactly equipped (and hence, compactly equipped) space.

Brief idea of proof:

- The Hausdorff distance d_H is lower semicontinuous on $\mathcal{P}_{\mathrm{Cl}_\tau}(B) \times \mathcal{P}_{\mathrm{Cl}_\tau}(B)$ with respect to $V_\tau \times V_\tau \Rightarrow$ each closed ball $\mathbb{B} \subset \mathcal{P}_{\mathrm{Cl}_\tau}(B)$ is V_τ-closed.
- By the theorem from Strickland's book, $\left(\mathcal{P}_{\mathrm{Cl}_\tau}(B), V_\tau\right)$ is a compact space $\Rightarrow \mathbb{B}$ is V_τ-compact $\Rightarrow$ $\mathcal{P}_{\mathrm{Cl}_\tau}(B)$ is V_τ-compactly equipped.

Corollary 27

Let X be a dual space and τ be the weak* topology on X. Then $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B}}(X)$ is a V_τ-compactly equipped (and hence, compactly equipped) metric space.

Let X be a linear topological space:

$$\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{Conv}}(X) \quad := \quad \{A \subset X\colon A \text{ is nonempty, } \tau\text{-closed and convex}\}.$$

We leave the following lemma without proof.

Lemma 28

Let X be a dual space, B be a closed ball in X and τ be the weak* topology on X. Then $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{Conv}}(B)$ is V_τ-closed in $\mathcal{P}_{\mathrm{Cl}_\tau}(B)$.

Let X be a normed space:

$$\mathcal{P}_{\mathrm{Cl,B,Conv}}(X) \quad := \quad \{A \subset X : A \text{ is nonempty, closed, bounded and convex}\}.$$

$$\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B,Conv}}(X) \quad := \quad \{A \subset X : A \text{ is nonempty, } \tau\text{-closed, bounded and convex}\}.$$

Theorem 29 (Mazur)

Let X be a reflexive space, and let τ be the weak* topology on X. Then

$$\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B,Conv}}(X) = \mathcal{P}_{\mathrm{Cl,B,Conv}}(X).$$

Theorem 30

Let X be a reflexive space, and let τ be the weak* topology on X. Then $\mathcal{P}_{\mathrm{Cl,B,Conv}}(X)$ is a V_τ-compactly equipped (and hence, compactly equipped) metric space.

Brief idea of proof:

- Let $\mathbb{B}$ be a closed ball in $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B},\mathrm{Conv}}(X) \Rightarrow \mathbb{B}$ is a closed ball in $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{Conv}}(B)$ for some closed ball $B \subset X \Rightarrow \mathbb{B} = \mathbb{B}' \cap \mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{Conv}}(B)$ for some closed ball $\mathbb{B}' \subset \mathcal{P}_{\mathrm{Cl}_\tau}(B)$.
- $\mathcal{P}_{\mathrm{Cl}_\tau}(B)$ is V_τ-compactly equipped $\Rightarrow \mathbb{B}'$ is V_τ-compact.
- By Lemma 28, the space $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{Conv}}(B)$ is V_τ-closed $\Rightarrow \mathbb{B}$ is V_τ-compact in $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B},\mathrm{Conv}}(X)$.
- The Hausdorff distance d_H is lower semicontinuous on $\mathbb{B} \times \mathbb{B} \Rightarrow \mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B},\mathrm{Conv}}(X)$ is V_τ-compactly equipped.
- By Mazur's theorem, $\mathcal{P}_{\mathrm{Cl}_\tau,\mathrm{B},\mathrm{Conv}}(X) = \mathcal{P}_{\mathrm{Cl},\mathrm{B},\mathrm{Conv}}(X)$.

□

Thank you for your attention!