Cheeger–Colding Theory

Teaching Assistant Session (2)

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Part 1: Cheeger-Colding's almost splitting theorem

Theorem 1. Let (M, g, p) be a complete pointed Riemannian n-manifold without boundary, with $1 \leq k \leq n$.

$$\forall \varepsilon > 0 \quad \exists \delta_0 = \delta_0(n, \varepsilon) > 0.$$

- **Assume**

$$\mathrm{Ric}_g \geq -(n-1)\delta g, \qquad 0 < \delta \leq \delta_0$$

$$d_{\mathrm{GH}}\Big(\overline{B}_{\delta^{-1}}(p) \subset M, \overline{B}_{\delta^{-1}}((0^k, \bar{x})) \subset \mathbb{R}^k \times X\Big) \leq \delta, \quad (0^k, \bar{x}) \in \mathbb{R}^k \times X$$

- **Then**

$$\exists\, u : B_1(p) \longrightarrow \mathbb{R}^k \quad \text{a } (k, \varepsilon)\text{-splitting map}$$

Here X is a metric space, with the product metric on $\mathbb{R}^k \times X$.

In this morning, professor Jiang shows that the converse implication is also true. This kind of equivalence is the so called Cheeger-Colding's almost splitting theorem.

Most important tools in the study of Ricci curvature

Theorem. For any smooth function $u \in C^\infty(M)$, we have

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u \rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

Corollary. If $\mathrm{Ric} \geq (n-1)Kg$, $K \in \mathbb{R}$ we have

$$\begin{aligned}\frac{1}{2}\Delta|\nabla u|^2 &\geq |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u \rangle + (n-1)K|\nabla u|^2 \\ &\geq \frac{1}{n}(\Delta u)^2 + \langle \nabla u, \nabla \Delta u \rangle + (n-1)K|\nabla u|^2.\end{aligned}$$

It is very useful to use the tools of harmonic functions!

Elliptic estimates: L^1 controls L^∞

Maximum principle: Suppose $\mathrm{Ric} \geq -(n-1)$, if u is a Lipschitz function on M so that $\Delta u \geq -h$ in $B_2(p)$. Then

$$\sup_{B_1(p)} u \leq C(n) \sup_{B_2(p)} h + \sup_{\partial B_2(p)} u.$$

Local regularity: harmonic coordinites

A smooth local coordiniates $u = (x_1, \cdots, x_n)$ on M is called harmonic coordinites if $\Delta_g x_i = 0$ for all i.

By Bochner formula, in local harmonic coordinates, the metric g_{ij} satisfies an semi-linear elliptic PDEs:

$$\Delta g_{ij} = -2\mathrm{Ric}_{ij} + Q_{ij}(g, \partial g).$$

The following (k, ε)–splitting maps resembles harmonic coordinates when $k = n$.

Definition: almost splitting map

A map $u : B_r(p) \subset M \to \mathbb{R}^k$ is a (k, ε)-splitting map if it satisfies the following four conditions.

- **1. Harmonic functions:**
$$\Delta u_i = 0 \qquad (1 \le i \le k)$$
- **2. Unit gradient bounds:**
$$\sup_{B_r(p)} |\nabla u_i| \le 1 + \varepsilon$$
- **3. Almost orthonormal gradients:**
$$\fint_{B_r(p)} \sum_{1 \le i,j \le k} |\langle \nabla u_i, \nabla u_j \rangle - \delta_{ij}| \, dV_g \le \varepsilon$$
- **4. Almost linear:**
$$r^2 \fint_{B_r(p)} \sum_i |\nabla^2 u_i|^2 \, dV_g \le \varepsilon.$$

In the setting of $\mathrm{Ric} \ge -(n-1)\delta$, condition 2 follows a priori from conditions 3 and 4 together with Bochner formula and elliptic estimates if δ and ε are small enough.

The Buseman functions and harmonic replacements

Assume the hypotheses of Theorem 1. Via GH condition, choose points corresponding to the positive and negative Euclidean coordinate directions in the product, and define the signed distance functions:

- **1. Coordinate poles:**

$$\text{choose } R = \delta^{-1}, \qquad q_i^{\pm} \in M \approx (\pm Re_i, \bar{x}) \in \mathbb{R}^k \times X \text{ via GH condition}$$

- **2. Distance coordinates (Buseman functions):**

$$b_i^+(x) = d(x, q_i^+) - d(p, q_i^+),$$
$$b_i^-(x) = d(p, q_i^-) - d(x, q_i^-),$$

- **3. Harmonic replacement:**

$$\Delta u_i = 0 \text{ in } B_{10}(p), \qquad u_i = b_i^+ \text{ on } \partial B_{10}(p)$$

We will show that $u = (u_1, \ldots, u_k) : B_1(p) \to \mathbb{R}^k$ is the desired (k, ε)–splitting map.

The diagonal poles and Claim 1

For each pair of different coordinate directions, choose the diagonal pole and its distance function. By a direct contradiction argument, the notes obtain the following claim.

- **Diagonal directions:**

$$q_{ij}^{+} \sim \left(\frac{R(e_i + e_j)}{\sqrt{2}}, \bar{x} \right), \qquad b_{ij}^{+}(x) = d(x, q_{ij}^{+}) - d(p, q_{ij}^{+})$$

- **Claim 1: linear relations**

$$\sup_{B_{10}} \left| b_{ij}^{+} - \frac{b_i^{+} + b_j^{+}}{\sqrt{2}} \right| \leq \Psi(\delta \mid n)$$

$$\sup_{B_{10}} |b_i^{+} - b_i^{-}| \leq \Psi(\delta \mid n)$$

The opposite signed distances approximate the same coordinate; the diagonal distance approximates their normalized sum. Here $\Psi(\delta \mid n)$ means a positive number depending on δ and n such that for each fixed n, $\Psi(\delta \mid n) \to 0$ as $\delta \to 0$.

Claims 2, 3, and 4

The rest of the proof follows the following 4 claims. Claim 2 uses the elliptic estimates. Claim 3 follows from Claim 2, and Claim 4 combines the first 3 claims.

- **Claim 2: uniform harmonic approximation**

$$\sup_{B_9} |b_i^+ - u_i| \leq \Psi(\delta \mid n)$$

- **Claim 3: almost unit gradient lengths**

$$\fint_{B_8} \left||\nabla u_i|^2 - 1\right| dV_g \leq \Psi(\delta \mid n)$$

- **Claim 4: almost orthogonality**

$$\fint_{B_8} |\langle \nabla u_i, \nabla u_j \rangle| \, dV_g \leq \Psi(\delta \mid n) \qquad (i < j)$$

Once Claims 3 and 4 hold, Bochner's formula gives the remaining gradient and Hessian estimates.

Proof of Claim 2: the upper comparison

By Laplacian comparison, the distance function has the following distributional upper bound on the ball of radius 10. Subtract it from the harmonic replacement.

- **Laplacian comparison (imagine $\mathrm{Ric} \geq 0$):**
$$\Delta b_i^+ \leq h, \qquad h = C(n)\sqrt{\delta} \text{ in } B_{10}$$
- **Subtract the harmonic function:**
$$-\Delta(u_i - b_i^+) \leq h, \qquad u_i - b_i^+ = 0 \text{ on } \partial B_{10}$$
- **Apply maximal principle (elliptic estimates)**
$$\sup_{B_9}(u_i - b_i^+) \leq C(n)h$$

Proof of Claim 2: the lower comparison

For the opposite distance function, Laplacian comparison gives the inequality below. On the boundary, Claim 1 controls its difference from the prescribed boundary value of u_i.

- **The opposite distance function: (the second inequality follows from Claim 1)**

$$-\Delta(b_i^- - u_i) \le h, \qquad \sup_{\partial B_{10}} (b_i^- - u_i) \le \Psi(\delta \mid n)$$

- **Apply the same maximum estimate:**

$$\sup_{B_9}(b_i^- - u_i) \le \Psi(\delta \mid n) + C(n)h = \Psi(\delta \mid n)$$

- **Combine with Claim 1:**

$$\sup_{B_9} |u_i - b_i^+| \le \Psi(\delta \mid n)$$

Apply the same maximum principle estimate again. Combining this lower comparison with the previous upper comparison and Claim 1 proves Claim 2.

Proof of Claim 3: the gradient estimate

Let ϕ be a cutoff supported in $B_9(p)$ and $\phi = 1$ in $B_8(p)$. Set $w = u_i - b_i^+$, and choose $\tau = \Psi(\delta \mid n)$ to bound its absolute value using Claim 2.

- **Choose the cutoff and a nonnegative test:**

$$-\Delta w \le h, \quad |w| \le \tau, \quad (w+\tau)\phi^2 \ge 0, \quad |\nabla \phi| \le C(n)$$

- **Test the weak differential inequality:**

$$\int_{B_9} (-\Delta w)(w+\tau)\phi^2 \, dV_g \le h \int_{B_9} (w+\tau)\phi^2 \, dV_g$$

- **Integration by parts and Cauchy inequality:**

$$⨍_{B_9} \phi^2 |\nabla w|^2 \, dV_g \le C ⨍_{B_9} |\nabla \phi|^2 (w+\tau)^2 \, dV_g + Ch\tau,$$

$$⨍_{B_8} |\nabla (u_i - b_i^+)|^2 \, dV_g \le \Psi(\delta \mid n).$$

Use the nonnegative test function above. Integration by parts and Cauchy's inequality give the energy bound, exactly as in the notes.

Proof of Claim 3: the gradient estimate

The interior harmonic gradient estimate bounds the gradients of u_i. Since $|\nabla b_i| = 1$ almost everywhere, the energy estimate compares the squared gradient length with 1.

- **Interior harmonic gradient estimate:**

$$\sup_{B_8} |\nabla u_i| \leq C(n) \sup_{B_9} |u_i| \leq C(n)$$

- **Compare with the unit distance gradient:**

$$\begin{aligned} \fint_{B_8} \left| |\nabla u_i|^2 - 1 \right| dV_g &\leq \fint_{B_8} |\nabla u_i - \nabla b_i^+| \, |\nabla u_i + \nabla b_i^+| \, dV_g \\ &\leq C(n) \left(\fint_{B_8} |\nabla (u_i - b_i^+)|^2 \, dV_g \right)^{1/2} \leq \Psi(\delta \mid n). \end{aligned}$$

This proves Claim 3. The comparison uses the same difference of squares and Cauchy–Schwarz estimate as the original proof.

Proof of Claim 4: the diagonal combination

By Claims 1 and 2, the normalized sum of the 2 harmonic coordinates is uniformly close to the diagonal distance function on B_9.

- **Claims 1 and 2 give:**

$$\sup_{B_9} \left| \frac{u_i + u_j}{\sqrt{2}} - b_{ij}^+ \right| \leq \Psi(\delta \mid n)$$

- **Apply the argument of Claim 3:**

$$⨍_{B_8} \left| \frac{|\nabla(u_i + u_j)|^2}{2} - 1 \right| dV_g \leq \Psi(\delta \mid n)$$

The normalized sum is harmonic, and the diagonal distance has the same 1-sided Laplacian bound. Apply the argument of Claim 3 to this pair of functions.

Proof of Claim 4: polarization

Expand the squared gradient of the sum. The mixed term has a plus sign. Claim 3 already controls the 2 diagonal terms, while the previous estimate controls the whole sum.

- **Expand the gradient of the sum:**

$$⨍_{B_8} \left| |\nabla u_i|^2 + |\nabla u_j|^2 + 2\langle \nabla u_i, \nabla u_j \rangle - 2 \right| dV_g \leq \Psi(\delta \mid n)$$

- **Use Claim 3 and the triangle inequality:**

$$\begin{aligned} 2⨍_{B_8} |\langle \nabla u_i, \nabla u_j \rangle| \, dV_g \leq & ⨍_{B_8} \left| |\nabla u_i|^2 - 1 \right| dV_g + ⨍_{B_8} \left| |\nabla u_j|^2 - 1 \right| dV_g \\ & + ⨍_{B_8} \left| |\nabla (u_i + u_j)|^2 - 2 \right| dV_g \leq \Psi(\delta \mid n). \end{aligned}$$

The triangle inequality now proves Claim 4. We have checked the averaged Gram-matrix condition.

Finishing the proof: cutoff and Bochner formula

Let ϕ be the good cutoff, supported in B_8 and equal to 1 on B_6. The harmonic gradient bound and Bochner's formula give the following inequalities.

- **Good cutoff function**

$$\phi \equiv 1 \text{ on } B_6, \quad \operatorname{supp}\phi \subset B_8, \quad |\nabla\phi| + |\Delta\phi| \leq C(n)$$

- **Bochner formula:**

$$\begin{aligned}\Delta(|\nabla u_i|^2 - 1) &= 2|\nabla^2 u_i|^2 + 2\,\mathrm{Ric}(\nabla u_i, \nabla u_i) \\ &\geq 2|\nabla^2 u_i|^2 - C(n)\delta \geq -C(n)\delta.\end{aligned}$$

The cutoff will be used for the Hessian estimate. First, the almost subharmonic inequality improves the gradient bound.

The Hessian estimate

Finally, multiply the full Bochner inequality by the cutoff from and integrate by parts. Subtracting the constant 1 leaves only the gradient-length error against the cutoff Laplacian.

- **Multiply by the cutoff and integrate:**

$$\fint_{B_8} \phi |\nabla^2 u_i|^2 \, dV_g \le C(n)\delta + 2 \fint_{B_8} |\Delta \phi| \, \big| |\nabla u_i|^2 - 1 \big| \, dV_g$$

$$\le C(n)\delta + C(n) \fint_{B_8} \big| |\nabla u_i|^2 - 1 \big| \, dV_g \le \Psi(\delta \mid n),$$

- **Since the cutoff equals 1 on the inner ball:**

$$\fint_{B_6} |\nabla^2 u_i|^2 \, dV_g \le \Psi(\delta \mid n)$$

Claim 3 bounds the last integral, and the cutoff equals 1 on the ball of radius 6. Volume comparison changes only a dimensional constant when we pass between these fixed balls.

Conclusion of Theorem 1

We have checked that $u = (u_1, \dots, u_k)$, restricted to the unit ball, is a (k, Ψ)-splitting map.

- **Harmonicity and the Gram matrix:**

$$\Delta u_i = 0, \qquad \fint_{B_1} |\langle \nabla u_i, \nabla u_j \rangle - \delta_{ij}| \, dV_g \leq \Psi(\delta \mid n)$$

- **Hessian estimate:**

$$\fint_{B_1} \sum_i |\nabla^2 u_i|^2 \, dV_g \leq \Psi(\delta \mid n).$$

Use volume comparison to restrict the averaged bounds to the unit ball, and sum the Hessian bounds over the components. Choose δ so that $\Psi(\delta \mid n) \leq \varepsilon$.

Theorem: small excess implies almost splitting

Theorem 2. Let (M, g, p) be a complete pointed Riemannian n-manifold without boundary.

$$\forall \varepsilon > 0 \quad \exists \delta_0 = \delta_0(n, \varepsilon) > 0.$$

- **Assume**

$$\mathrm{Ric}_g \geq -(n-1)\delta g, \qquad 0 < \delta \leq \delta_0$$

There exist $q_1, q_2 \in M$ such that

$$d(p, q_1),\ d(p, q_2) \geq \delta^{-1},$$

$$e(p) := d(p, q_1) + d(p, q_2) - d(q_1, q_2) \leq \delta,$$

- **Then**

there exists a pointed proper length space $(Y, \bar{y})$ such that

$$d_{\mathrm{GH}}\Big(\overline{B}_1(p), \overline{B}_1((0, \bar{y})) \subset \mathbb{R} \times Y\Big) \leq \varepsilon$$

Corollary. Let X be a Ricci limit space with nonnegative Ricci curvature, if X contains a line, then X splits off $\mathbb{R}$.

Basic properties of excess function

Write $q_+ = q_1$ and $q_- = q_2$. The excess is nonnegative and 2-Lipschitz. It describes in what extent the distance triangle equality fails.

- **Basic properties 1: positivity and a uniform Lipschitz bound.**

$$e(x) = d(x, q_+) + d(x, q_-) - d(q_+, q_-) \geq 0, \qquad |\nabla e| \leq 2, \qquad e(p) \leq \delta$$

- **Basic properties 2: local smallness**

Under the condition of Theorem 2, using heat flow method we can prove

$$\sup_{B_{100}(p)} e(x) \leq \Psi(\delta|n).$$

Signed distances and harmonic replacement

Use the signed distances and harmonic replacement.

- **Define the coordinates: normalize both signed distances at p.**

$$b^+(x) = d(x, q_+) - d(p, q_+), \qquad b^-(x) = d(p, q_-) - d(x, q_-)$$

- **their difference is small, with the correct normalization.**

$$b^+(x) - b^-(x) = e(x) - e(p), \qquad \|b^+ - b^-\|_{L^\infty(B_{20})} \leq \Psi(\delta \mid n)$$

- **Replace harmonically: the same small Laplacian bounds remain available.**

$$\Delta h = 0 \text{ in } B_{20}, \quad h = b^+ \text{ on } \partial B_{20}, \qquad \Delta b^+ \leq \eta, \quad -\Delta b^- \leq \eta$$

The boundary condition is weak: $h - b^+$ belongs to $H_0^1(B_{20})$. The normalized difference is $e - e(p)$.

Uniform approximation of the harmonic replacement

Apply the standard elliptic estimates:

- **Upper comparison: the harmonic and distance data agree on the boundary.**
$$-\Delta(h - b^+) \le \eta, \qquad \sup_{B_{15}}(h - b^+) \le C(n)\eta$$
- **Lower comparison: the boundary error is at most the initial excess.**
$$-\Delta(b^- - h) \le \eta, \quad b^- - h \le e(p) \text{ on } \partial B_{20}, \quad \sup_{B_{15}}(b^- - h) \le e(p) + C(n)\eta$$
- **Combine: the initial excess cancels in the normalized difference.**
$$\sup_{B_{15}} |h - b^+| \le \sup_{B_{20}} e + C(n)\eta \le \Psi(\delta \mid n)$$

Since $b^- - b^+ = e(p) - e(x)$, the boundary error is at most $e(p)$. Adding the differences cancels that initial excess.

Almost unit gradient

Choose ϕ supported in B_{15}, equal to 1 on B_{13}, with bounded gradient. Set $w = h - b^+$.

- **Goal: compare the harmonic gradient length with 1.**

$$\fint_{B_{12}} \left||\nabla h|^2 - 1\right| dV_g \leq \Psi(\delta \mid n)$$

- **Nonnegative test: use $(w+\tau)\phi^2$ in the weak inequality.**

$$-\Delta w \leq \eta, \quad |w| \leq \tau = \Psi(\delta \mid n), \quad -\int_{B_{15}} \langle \Delta w, (w+\tau)\phi^2 \rangle \leq \eta \int_{B_{15}} (w+\tau)\phi^2 \, dV_g$$

- **Integrate by parts: absorb the cross term by Cauchy's inequality.**

$$\fint_{B_{13}} |\nabla (h - b^+)|^2 \, dV_g \leq C(n)(\eta\tau + \tau^2) \leq \Psi(\delta \mid n)$$

The shifted test is nonnegative. Integrate by parts and absorb the cross term by Cauchy's inequality.

The harmonic gradient is almost unit on average

The harmonic gradient is uniformly bounded, while the distance gradient has length 1 almost everywhere.

- **Uniform gradient bound: harmonicity and bounded boundary data give control.**

$$\sup_{B_{13}} |\nabla h| \le C(n) \sup_{B_{15}} |h| \le C(n)$$

- **Difference of squares: apply Cauchy–Schwarz to the energy error.**

$$\fint_{B_{13}} \left| |\nabla h|^2 - 1 \right| dV_g \le \fint_{B_{13}} |\nabla h - \nabla b^+| \, |\nabla h + \nabla b^+| \, dV_g$$

$$\le C(n) \left(\fint_{B_{13}} |\nabla (h - b^+)|^2 \, dV_g \right)^{1/2} \le \Psi(\delta \mid n).$$

Restricting the average to B_{12} changes only a dimensional constant. This proves the target estimate.

Bochner produces a 1-dimensional splitting map

Use the mean-value estimate and integrate Bochner's inequality against a good cutoff.

- **Bochner: the harmonic gradient bound controls the Ricci term.**

$$F = |\nabla h|^2 - 1, \qquad \Delta F \geq 2|\nabla^2 h|^2 - C(n)\delta$$

- **Mean value: improve average error to a pointwise stretching bound.**

$$\sup_{B_{10}} |\nabla h| \leq 1 + \Psi(\delta \mid n)$$

- **Cutoff integration: obtain the scale-corrected Hessian energy.**

$$⨍_{B_{10}} \left||\nabla h|^2 - 1\right| dV_g + 10^2 ⨍_{B_{10}} |\nabla^2 h|^2 \, dV_g \leq \Psi(\delta \mid n)$$

The curvature error is $C(n)\delta$. The radius-squared factor 100 is absorbed into Ψ. Thus h is a $(1, \Psi)$-splitting map on B_{10}.

Apply almost splitting Theorem

- **Choose a common error: $\Psi(\delta|n)$ controls curvature and all splitting conditions.**

$$h \text{ is } (1, \Psi(\delta \mid n))\text{-splitting on } B_{10}, \text{ where } \Psi(\delta|n) \longrightarrow 0.$$

- **A splitting map gives a product approximation.**

$$\mathrm{Ric}_g \geq -(n-1)\delta g, \qquad h \text{ is } (1, \Psi)\text{-splitting on } B_{10}$$

- **Conclude: small excess between 2 distant endpoints forces almost splitting.**

$$d_{\mathrm{GH}}\Big(\overline{B}_1(p), \overline{B}_1((0, \bar{y})) \subset \mathbb{R} \times Y\Big) \leq \varepsilon$$