Cheeger–Colding Theory

Teaching Assistant Session (1)

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1 Gromov's precompactness theorem

Theorem 1. *Let $\mathcal{C}$ be a collection of compact metric spaces equipped with Gromov–Hausdorff distance. For $r > 0$, define, using open balls in X,*

$$\mathrm{Cov}_X(r) := \min\{m : B_r(x_1), \dots, B_r(x_m) \ r\text{–balls in } X \text{ that covers } X\}.$$

Suppose that there exist $D < \infty$ and a function $N : (0,1) \to \mathbb{N}$ such that

$$\mathrm{diam}(X) \le D, \qquad \mathrm{Cov}_X(r) \le N(r) \quad \text{for every } X \in \mathcal{C} \text{ and } 0 < r < 1. \tag{1.1}$$

Then $\mathcal{C}$ is precompact in the Gromov–Hausdorff metric: every sequence in $\mathcal{C}$ has a subsequence converging to a compact metric space.

Proof. Let $\mathcal{M}$ denote the set of isometry classes of nonempty compact metric spaces, equipped with d_{GH}.

Step 1: Uniform finite ε–nets. Fix $0 < \varepsilon < 1$ and $X \in \mathcal{C}$. By condition (1.1) there exists $N = N(\varepsilon)$ and $F_X := \{x_1, \dots, x_N\} \subset X$ such that

$$X \subset \bigcup_{i=1}^{N} B_\varepsilon(x_i), \qquad d_{\mathrm{GH}}(X, F_X) \le \varepsilon. \tag{1.2}$$

The last inequality uses the inclusion of F_X into X as a common metric realization.

Step 2: bounded distance matrices have finitely many approximations. For each $X \in \mathcal{C}$, label the points of F_X and define $f : \mathcal{C} \to \mathbb{R}^{\binom{N}{2}}$, where $\mathbb{R}^{\binom{N}{2}}$ is equipped with ℓ^∞ norm,

$$f(X) = \big(d_X(x_i, x_j)\big)_{1 \le i < j \le N}.$$

Since $\mathrm{diam}(X) \le D$, the image set $f(\mathcal{C})$ is a bounded subset of the Euclidean space $\mathbb{R}^{\binom{N}{2}}$, and hence it is totally bounded. Choose $\{X_1, \cdots, X_K\} \subset \mathcal{C}$ so that it is an ε–net of $f(\mathcal{C})$. For any $X \in \mathcal{C}$, pick some X_α so that $||F_X - F_{X_\alpha}||_{\ell^\infty} = d_{\mathrm{GH}}(F_X, F_{X_\alpha}) < \varepsilon$, it then follows from (1.2) and triangle inequality that

$$d_{\mathrm{GH}}(X, X_\alpha) \le d_{\mathrm{GH}}(X, F_X) + d_{\mathrm{GH}}(F_X, F_{X_\alpha}) + d_{\mathrm{GH}}(F_{X_\alpha}, X_\alpha) \le 3\varepsilon.$$

Thus $\mathcal{C}$ is totally bounded in $(\mathcal{M}, d_{\mathrm{GH}})$. This metric space is complete, so the closure of $\mathcal{C}$ is compact. This proves the asserted subsequential convergence.

□

2 An elliptic mean-value estimate

Theorem 2 (Mean-value inequality for subharmonic functions)**.** *Let (M^n, g) be a complete Riemannian manifold without boundary, let $p \in M$, and suppose*

$$\mathrm{Ric}_g \geq -(n-1)\lambda g, \qquad \lambda \geq 0.$$

Let $0 < r \leq 1$, and let u be Lipschitz on $B_{2r}(p)$ and satisfy $-\Delta_g u \leq 0$ weakly there. Here $\Delta_g = \mathrm{div}_g \nabla$, and the weak inequality means

$$\int_{B_{2r}(p)} \langle \nabla u, \nabla \psi \rangle \, \mathrm{d}V_g \leq 0 \quad \textit{for every } 0 \leq \psi \in C_c^\infty(B_{2r}(p)).$$

Then there is a constant $C(n, \lambda)$ such that

$$\sup_{B_r(p)} u_+ \leq C(n,\lambda) \fint_{B_{2r}(p)} u_+ \, \mathrm{d}V_g, \qquad u_+ := \max\{u, 0\}, \tag{2.1}$$

where $\displaystyle\fint_E f \, \mathrm{d}V_g := \mathrm{Vol}_g(E)^{-1} \int_E f \, \mathrm{d}V_g$.

Proof. We first treat $n \geq 3$. Replace g by $\widetilde{g} = r^{-2} g$. Then $B^g_{2r}(p) = B^{\widetilde{g}}_2(p)$, $\Delta_{\widetilde{g}} = r^2 \Delta_g$, and

$$\mathrm{Ric}_{\widetilde{g}} \geq -(n-1)\lambda r^2 \widetilde{g} \geq -(n-1)\lambda \widetilde{g}.$$

Normalized averages are unchanged by this scaling. It therefore suffices to prove the result for $r = 1$; we again write the metric as g. Throughout this part write $B_t = B_t(p)$, set

$$v = u_+, \qquad \chi = \frac{n}{n-2} > 1, \qquad \mathrm{d}\mu = \frac{\mathrm{d}V_g}{\mathrm{Vol}_g(B_2)},$$

and use this *same* measure μ on every smaller ball.

Step 1: the normalized Sobolev input. The local Neumann Sobolev inequality, the triangle inequality, and Cauchy–Schwarz imply

$$\left(\int_{B_2} |f|^{2\chi} \, \mathrm{d}\mu \right)^{1/\chi} \leq C_S(n,\lambda) \int_{B_2} \left(|\nabla f|^2 + |f|^2 \right) \mathrm{d}\mu \tag{2.2}$$

for compactly supported Lipschitz f in B_2. Indeed, apply the Neumann inequality to $f - f_{B_2}$, where $f_{B_2} = \int_{B_2} f \, \mathrm{d}\mu$, and use $|f_{B_2}| \leq \|f\|_{L^2(B_2,\mu)}$. Using radius 2 rather than 1 is harmless: rescaling g to $g/4$ changes the Ricci parameter to 4λ, so the constant still depends only on n and λ.

Step 2: a Caccioppoli inequality. Let $q \geq 2$ and $0 \leq \phi \leq 1$ be Lipschitz with compact support in B_2. Test the weak inequality for u with $\psi = \phi^2 v^{q-1}$. These tests are admissible by smooth approximation and the Sobolev chain rule. Since $\nabla v = \nabla u$ almost everywhere on $\{u > 0\}$ and $\nabla v = 0$ almost everywhere on $\{u \leq 0\}$,

$$\begin{aligned}(q-1) \int_{B_2} \phi^2 v^{q-2} |\nabla v|^2 \, \mathrm{d}\mu &\leq -2 \int_{B_2} \phi v^{q-1} \langle \nabla \phi, \nabla v \rangle \, \mathrm{d}\mu \\ &\leq \frac{q-1}{2} \int_{B_2} \phi^2 v^{q-2} |\nabla v|^2 \, \mathrm{d}\mu + \frac{2}{q-1} \int_{B_2} |\nabla \phi|^2 v^q \, \mathrm{d}\mu.\end{aligned}$$

Thus

$$\int_{B_2} \phi^2 v^{q-2} |\nabla v|^2 \, \mathrm{d}\mu \leq \frac{4}{(q-1)^2} \int_{B_2} |\nabla \phi|^2 v^q \, \mathrm{d}\mu.$$

Applying the chain rule once more gives

$$\int_{B_2} \phi^2 |\nabla(v^{q/2})|^2 \, \mathrm{d}\mu \le \frac{q^2}{(q-1)^2} \int_{B_2} |\nabla\phi|^2 v^q \, \mathrm{d}\mu \le 4 \int_{B_2} |\nabla\phi|^2 v^q \, \mathrm{d}\mu.$$

Consequently,

$$\int_{B_2} |\nabla(\phi v^{q/2})|^2 \, \mathrm{d}\mu \le 10 \int_{B_2} |\nabla\phi|^2 v^q \, \mathrm{d}\mu. \tag{2.3}$$

The constants here are uniform for $q \ge 2$.

Step 3: Moser iteration from L^2 to L^∞. For $1 \le R < S \le 3/2$, choose a distance cutoff ϕ that equals 1 on B_R, vanishes outside B_S, and satisfies $|\nabla\phi| \le 2/(S-R)$. Its support is compactly contained in B_2. Apply (2.2) to $f = \phi v^{q/2}$ and use (2.3). Since $S - R \le 1$, the zeroth-order term can be included in the same bound, giving

$$\|v\|_{L^{q\chi}(B_R,\mu)} \le \left(\frac{C(n,\lambda)}{(S-R)^2} \right)^{1/q} \|v\|_{L^q(B_S,\mu)}. \tag{2.4}$$

For fixed $R < S$, set

$$q_j = 2\chi^j, \qquad \rho_j = R + (S-R)2^{-j}, \qquad j = 0, 1, \ldots.$$

Use (2.4) with outer radius ρ_j, inner radius ρ_{j+1}, and exponent q_j. After m steps,

$$\|v\|_{L^{q_m}(B_{\rho_m},\mu)} \le \prod_{j=0}^{m-1} \left(\frac{C(n,\lambda) 4^{j+1}}{(S-R)^2} \right)^{1/q_j} \|v\|_{L^2(B_S,\mu)}.$$

The product converges because

$$\sum_{j=0}^{\infty} \frac{1}{q_j} = \frac{\chi}{2(\chi-1)} = \frac{n}{4}, \qquad \sum_{j=0}^{\infty} \frac{j+1}{q_j} < \infty.$$

Restricting the left-hand side to B_R and letting $m \to \infty$, we obtain

$$H(R) := \sup_{B_R} v \le C(n,\lambda)(S-R)^{-a} \|v\|_{L^2(B_S,\mu)}, \qquad a = \frac{n}{2}. \tag{2.5}$$

The usual limit of L^q norms yields an essential supremum, which is the ordinary supremum because v is continuous.

Step 4: the L^1 endpoint by absorption. Put

$$A = \int_{B_2} v \, \mathrm{d}\mu = \fint_{B_2} v \, \mathrm{d}V_g.$$

On B_S we have $v^2 \le H(S)v$, so (2.5) implies

$$H(R) \le C(n,\lambda)(S-R)^{-a} H(S)^{1/2} A^{1/2}.$$

Young's inequality therefore gives, for every $0 < \theta < 1$,

$$H(R) \le \theta H(S) + C(n,\lambda,\theta)(S-R)^{-2a} A. \tag{2.6}$$

Choose

$$R_j = \frac{3}{2} - 2^{-j-1}, \qquad \theta = 2^{-2a-1}.$$

Then $R_0 = 1$, $R_j \uparrow 3/2$, and $R_{j+1} - R_j = 2^{-j-2}$. Iterating (2.6) gives

$$H(1) \le \theta^m H(R_m) + C(n,\lambda,\theta) 2^{4a} A \sum_{j=0}^{m-1} (\theta 2^{2a})^j.$$

The first term tends to zero: by completeness, $\overline{B_{3/2}}$ is a compact subset of B_2, and v is bounded there. The geometric series converges since $\theta 2^{2a} = 1/2$. It follows that

$$\sup_{B_1} u_+ \le C(n,\lambda) \fint_{B_2} u_+ \, \mathrm{d}V_g.$$

Scaling back proves (2.1) for $n \ge 3$.

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