Analogues of the Gromov–Hausdorff distance

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Introduction

Two approaches to defining the Gromov–Hausdorff distance have become classic.

I. The first "geometric – external" approach is based on an isometric embedding of the given metric spaces (X, ϱ_X) and (Y, ϱ_Y) into a "common" ambient metric space (Z, ϱ_Z) — $i\colon (X, \varrho_X) \to (Z, \varrho_Z)$ and $j\colon (Y, \varrho_Y) \to (Z, \varrho_Z)$.

Subsequently, the Gromov–Hausdorff distance $d_{GH}(X, Y)$ is taken to be the Hausdorff distance $d_H\big(i(X), j(Y)\big)$ between the images of the two metric spaces under consideration (the author hopes that the definition of the Hausdorff distance will be provided in many other presentations and will be clearly articulated in our report a little later).

II. The second "functional – internal" approach is based on considering pairs of mappings $f\colon (X, \varrho_X) \to (Y, \varrho_Y)$, $g\colon (Y, \varrho_Y) \to (X, \varrho_X)$ and assessing the distortion of distances under the mappings under consideration.

Approach I is based on first defining a distance — which need not be a metric and may, in particular, take the values 0 and ∞ on the space 2^Z of all non-empty subsets (or the space $\exp Z$ of all non-empty compact subsets) (or the space $\exp^c Z$ of all non-empty compact connected subsets) of a metric space (Z, ϱ_Z).

Approach II is based on first defining a distortion on the space of mappings Y^X of X into Y (or a specific subset thereof – $C(X, Y) = \{f \in Y^X : f \text{ is continuous } \}$) or on the supspace of set-valued mappings $\left(2^Y\right)^X \supset Y^X$.

In some cases, elements of both Approach I and Approach II are combined.

Gromov–Hausdorff distance d_{GH}

I. Let us begin by considering various distance functions on the set 2^Z ($\exp Z$) for a fixed (generalized pseudo)-metric space (Z, ϱ_Z): $X, Y \subset Z$.

1. $|XY| = \inf\{|xy| : x \in X, y \in Y\}$.
2. $D(X, Y) = \sup\{|xy| : x \in X, y \in Y\}$.

Distance functions I.1 and I.2 provide little information regarding set similarity.

3. $d_H(X, Y) = \max\{\sup\{|xY|, \sup\{|Xy| : x \in X, y \in Y\}\}$.

The distance function $d_H(X, Y)$ is called the Hausdorff distance and is a metric on the set $\exp Z$. The topology induced by the metric d_H coincides with the Vietoris topology, which is defined in purely topological terms for an arbitrary topological space Z. In particular, Hausdorff metrics on $\exp Z$ generated by different admissible metrics on Z induce the same topology. A metric ϱ_Z on a topological space (Z, τ_Z) is called admissible if $\tau_{\varrho_Z} = \tau_Z$.

The Gromov–Hausdorff distance $d_{GH}(X, Y)$ is the infimum (minimum in the case of compact spaces) of the Hausdorff distances $d_H(i(X), j(Y))$ over all isometric embeddings $i\colon (X, \varrho_X) \to (Z, \varrho_Z)$ and $j\colon (Y, \varrho_Y) \to (Z, \varrho_Z)$:

$$d_{GH}(X, Y) = \inf\big\{ d_H\big(i(X), j(Y)\big) : \\ i\colon X \to Z \text{ and } j\colon Y \to Z \text{ are isometries } \big\}.$$

Taking the infimum is necessary for the correctness of the introduced concept. This leads to independence from the chosen "external" parameters — the space Z and the embeddings i and j. **These parameters allow for variations.**

The directed metric $d_H^{\rightarrow}$

4. Sometimes it is useful to consider the asymmetric distance

$$d_H^{\rightarrow}(X, Y) = \sup\{|xY| : x \in X\}.$$

Obviously,

$$d_H(X, Y) = \max\{d_H^{\rightarrow}(X, Y),\ d_H^{\rightarrow}(Y, X)\}.$$

The directed Gromov–Hausdorff distance is defined analogously.

$$d_{GH}^{\rightarrow}(X, Y) = \inf\Big\{ d_H^{\rightarrow}\big(i(X), j(Y)\big) : \\ i : X \to Z \text{ and } j : Y \to Z \text{ are isometries } \Big\}.$$

Since the ordered distances from X to Y and from Y to X may be realized by different pairs of isometric embeddings, in the general case we can only **assert** the inequality

$$d_{GH}(X, Y) \geq \max\big\{ d_{GH}^{\rightarrow}(X, Y),\ d_{GH}^{\rightarrow}(Y, X)\big\}.$$

The equality is not true in general.

Let $X = \sqcup_{k=1}^{\infty} S^{2k-1}$ and $Y = \sqcup_{k=1}^{\infty} S^{2k}$, where the distance between points on different spheres is 2, and the distance between two points on the same sphere is the standard distance on the Euclidean sphere. It is easy to verify that $d_H^{\rightarrow}(X, Y) = d_H^{\rightarrow}(Y, X) = 0$. It follows from the results of S. Lim, F. Mémoli, and Z. Smith that $d_H(X, Y) \geq d_H(S^1, S^2) = \frac{\pi}{3}$.

It is clear that if a metric space X embeds isometrically into Y, then $d_H^{\rightarrow}(X, Y) = 0$. The converse is not true in general. For the unit interval $X = I = [0, 1]$ and any proper dense subset $Y \subset \overline{Y} = I$, the equality $d_H(X, Y) = 0$ holds, yet X does not embed isometrically into Y. For compact metric spaces X and Y, the equality $d_H^{\rightarrow}(X, Y) = 0$ is equivalent to the metric embeddability of X into Y.

Y. Dutrieux and G. Lancien posed the problem:

Problem

If two separable Banach spaces have the same compact (finite) subsets up to isometry, then are they isometrically embedded in each other?

S. Bogatyy and A. Tuzhilin showed that for finite-dimensional spaces the answer is positive.

Theorem

For a normed space V and a finite-dimensional normed space W, the following conditions are equivalent:
(1) *the spaces V and W are linearly isometric;*
(5′) $\dim W \leqslant \dim V$ *and there is an isometric embedding* $f\ B_V \to W$*;*
(10) *the spaces V and W have the same compact subsets up to isometry;*
(10′) $\dim W \leqslant \dim V$ *and each finite subset of V isometrically is embeddable in W.*

Dutrieux and Lancien's problem can be modified as follows.

Problem

Let two separable Banach spaces V and W are such that $d_H^{\rightarrow}(X, W) = d_H^{\rightarrow}(Y, V) = 0$ for any compact (finite) subsets $X \subset V$, $Y \subset W$. Are they isometrically embedded in each other? Is it true that $d_H^{\rightarrow}(V, W) = d_H^{\rightarrow}(W, V) = 0$?

The condition $d_H^{\rightarrow}(X, Y) < d$ is equivalent to the inclusion $X \subset O(Y, d)$, where $O(Y, d)$ denotes the open d-neighborhood of the set Y in Z, i.e., $O(Y, d) = \{z \in Z : |zY| < d\}$; the set $B(Y, d) = \{z \in Z : |zY| \leq d\}$ is called the closed d-neighborhood of the set $Y \subset Z$. It is clear that the inclusion $X \subset O(Y, d)$ is equivalent to the existence of a multivalued metric projection $p_d : X \to Y$ such that

$$|xp_d(x)| < d \text{ for every point } x \in X. \tag{1}$$

It is easy to verify that in condition (1), the inequality $|xp_d(x)| < d$ can be replaced by $D(x, p_d(x)) < d$ or $d_H(x, p_d(x)) < d$. The required set-valued metric projection $p_d : X \to Y$ can be defined constructively by the formula

$$p_d(x) = O(x, d) \cap Y = \{y : y \in Y,\ |xy| < d\}.$$

For subsequent purposes, the strict inequality can be replaced by a non-strict one:

$$\overline{p}_d(x) = B(x, d) \cap Y = \{y : y \in Y,\ |xy| \leq d\}.$$

The set-valued mappings p_d and $\overline{p}_d$ possess different continuity properties; therefore, both are important.

The Axiom of Choice allows one to construct a single-valued metric projection $\hat{p}_d : X \to Y$ from the mappings p_d and $\overline{p}_d$. A single-valued metric projection typically exhibits very poor continuity properties.

The conditions

$$|xp_d(x)| < d, \quad D(x, p_d(x)) < d, \quad d_H(x, p_d(x)) < d,$$

$$|x\overline{p}_d(x)| \leq d, \quad \ldots, \quad |x\hat{p}_d(x)| < d$$

can be naturally formulated in terms of distances on the spaces $\left(2^Y\right)^X$ and Y^X as the distance to the identity mapping. We will discuss this in greater detail later in Section II.

The continuous metric d_B

5. Here we consider continuous distance d_B on 2^Z introduced by the Karol Borsuk in 1954, which can naturally be regarded as a continuous analogue of the Hausdorff distance d_H.

For every $X, Y \in 2^Z$ we define $d_B(X, Y)$ as the lower bound of the numbers $d \geq 0$ such that there exists a continuous mapping f of X into Y and a continuous mapping g of Y into X satisfying the conditions

$$|xf(x)| \leq d, \ |yg(y)| \leq d \text{ for every } x \in X, \ y \in Y. \qquad (2)$$

Let us observe that fo every number d satisfying the inequalities (2) we have $d_H(X, Y) \leq d$. Hence

$$d_H(X, Y) \leq d_B(X, Y).$$

It follows that the identical mapping in 2^Z induces the continuous mapping of $2^Z_{d_B}$ onto $2^Z_{d_H}$. But the converse is not true. For instance, denoting by X_k the set constituted by the numbers $0, 1/k, \ldots, (k-1)/k, 1$, and by $I = [0, 1]$ the unite closed interval, we see at once that

$$\lim_{k\to\infty} d_H(X_k, I) = 0 \qquad \text{and} \qquad d_B(X_k, I) = \frac{1}{2}.$$

For a finite-dimensional metric space X the following conditions are equivalent:

1) $X \in ANR$;

2) X is locally contractible.

For a metric space $X \in ANR$ the following conditions are equivalent:

1) $X \in AR$;

2) X is contractible.

Properties of the metric d_B

Theorem

If $\dim X \geq n$, *then there exists an* $\alpha_n > 0$ *such that* $d_B(X, Y) < \alpha_n$ *implies* $\dim Y \geq n$.

The number α_n (or, more precisely, the supremum of such numbers) is called the n-dimensional Aleksandrov diameter. To be precise, the number α_n is related to the directed distance $d_B^{\rightarrow}(X, Y)$.

A classical result states that $\alpha_n\big(B^n(0, r)\big) = r$.

It is not difficult to show that $2\alpha_1(X) = \operatorname{diam} X$ for a connected space X.

Corollary

The set composed by all $X \in 2^Z_{d_B}$ *with* $\dim X \geq n$ *is open.*

Theorem

If X is without fixed point property, then there exists an $\alpha > 0$ such that $d_B(X, Y) < \alpha$ implies that Y is without fixed point property.

Corollary

The set composed by all $X \in 2^Z_{d_B}$ with fpp is closed.

Problem

Let Z be a separable space. Is it true that the space $2^Z_{d_B}$ is also separable?

Theorem

There is a continuum (Cook continuum) K with the following properties:
1) $\dim K = 1 \implies K \subset \mathbb{R}^3$;
2) any non-degenerate subcontinuum $A \subset K$ *is strongly non-planarable in the following meaning: let* A *be the image* $A = f(C)$ *of a continuous map* $f : C \to K$ *of a continuum* C*, then* C *can not be embedded into the plane.*
3) K *is hereditarily indecomposable i.e. any two subcontinua of* K *do not intersect, or one of them lies within the other.*
4) If A *is a subcontinuum of* K *and* $f : A \to B$ *is a continuous mapping onto a non-degenerate subcontinuum* $B \subset K$*, then* $A = B$ *and* f *is the identity mapping of* A *onto itself* $\implies$ $d_B(A, B) \geq \max\{J(A) \operatorname{diam} A, J(B) \operatorname{diam} B\}$ *for any different non-degenerate subcontinua* $A, B \subset K$ *and for any compatible metric on* K $\implies$
all points of $\exp^c K \setminus K$ *are isolated in* $(\exp^c K, d_B)$ $\implies$
the space $(\exp^c K, d_B)$ *is not separable.*

Below other solutions to the Borsuk problem are cited.

Problem

Does there exist an AR-space Z such that the space $2^Z_{d_B}$ is not connected?

Problem

Does there exist an ANR-space Z such that the space $2^Z_{d_B}$ is not locally connected?

Gromov–Borsuk distance d_{GB}

The Gromov–Borsuk distance $d_{GB}(X, Y)$ is the infimum of the Borsuk distances
$d_B(i(X), j(Y))$ over all isometric embeddings
$i\colon (X, \varrho_X) \to (Z, \varrho_Z)$ and $j\colon (Y, \varrho_Y) \to (Z, \varrho_Z)$

$$d_{GB}(X, Y) = \inf\Big\{ d_B\big(i(X), j(Y)\big) : \\ i\colon X \to Z \text{ and } j\colon Y \to Z \text{ are isometries } \Big\}.$$

Lim S., Mémoli F., and Smith Z. in 2023 and Bogatyy–Tuzhilin in 2025 defined and studied the continuous Gromov-Hausdorff distance $d_{GH}^c(X, Y)$, within the framework of approach II (through continuous mappings of X and Y into each other).

Perhaps approach II can be discussed at the very end of the report.

$$d_{GH}^c \le d_{GB}$$

Homology, homotopy, and shape roperties of metric d_{GB}

There exists a relation between homological (homotopical, shape) properties of two compacta X and Y and their distance $d_{GB}(X, Y)$.

Let n be a non-negative integer. We denote by

$$p_n(X) = \operatorname{rank} \check{H}_n(X, \mathbb{Z}) = \dim \check{H}_n(X, \mathbb{Q})$$

the Betti number. It is assumed that the homology groups are Vietoris–Čech homology. For any ANR-compactum, all standard homologies coincide.

$p_0(X) \geq m \Longleftrightarrow$ there exists a partition $X = X_1 \sqcup \ldots \sqcup X_m$ such that all X_i are non-empty closed (and open).

Theorem (Borsuk, 1954)

If $p_n(X) \geq m$, then there exists an $\varepsilon > 0$ such that $d_{GB}(X, Y) < \varepsilon$ implies $p_n(Y) \geq m$.

Theorem (Borsuk, 1977)

Let $\{X_i\}_{i=0}^{\infty}$, be a sequence of compacta such that

$$\lim_{i\to\infty} d_{GB}(X_i, X_0) = 0.$$

If X_i, for $i = 1, 2, \ldots$, is homotopically dominated by a compactum B and $X_0 \in ANR$, then X_0 is homotopically dominated by B.

The assumption $Y \in ANR$ is essential.

There are two ways to strengthen the formulated theorem:

1) Weakening the requirement $X_0 \in ANR$;

2) Replacing the metric d_{GB} with a smaller metric.

For a compact space X the following properties are equivalent:

1) X is contractible;

2) X is homotopically dominated by 1-point space Δ_1;

3) X is homotopically equivalent to the 1-point space Δ_1;

4) identity map $\mathrm{id}_X : X \to X$ is homotopically trivial;

5) every continuous map $f : X \to Y$ is homotopically trivial.

For a compact space X the following properties are equivalent:

1) $X \in FAR$;

2) X is shape dominated by 1-point space Δ_1;

3) X is shape equivalent to the 1-point space Δ_1;

4) identity map $\mathrm{id}_X : X \to X$ is trivial in shape category;

5) every continuous map $f : X \to Y$, into any $Y \in ANR$, is homotopically trivial;

6) every continuous map $f : X \to Y$ is trivial in shape category;

7) every shape map $\overline{f} : X \to Y$ is trivial in shape category;

8) X is contractible in every its neighborhood of some supspace $Z \supset X$;

9) X is contractible in every its neighborhood of every ANR-supspace $Z \supset X$;

10) for some embedding $X \subset Q$ in the Hilbert cube $Q = [0,1]^{\mathbb{N}}$ the factorspace $Q/X \cong Q$ is homeomorphic to Hilbert cube Q;

11) for every embedding $X \subset s$ in the pseudointerior of Hilbert cube $s = (0,1)^{\mathbb{N}} \subset Q$ the factorspace $Q/X \cong Q$ is homeomorphic to Hilbert cube Q;

12) for every embedding $X \subset s$ in the pseudointerior of Hilbert cube s the complement $Q \setminus X \cong Q \setminus \Delta_1$ is homeomorphic to the complement of a (any) point;

13) for some embedding $X \subset Q$ it is true $Q/X \times Q \cong Q$;

14) for some (every) embedding $X \subset Q$ for some (every) AR-compactum Y and for some (every) FAR-subcompactum $Y_0 \subset Y$ it is true $Q \times Y \times [0,1]/X \times Y_0 \times 0 \cong Q$;

15) If $\dim X \leq n$, then for some (every) tame embedding $X \subset S^{2n+1}$ the factorspace $S^{2n+1}/X \cong S^{2n+1}$ is homeomorphic to the sphere S^{2n+1} (is a manifold).

The homotopical metrics d_h and d_h'

6-7. Karol Borsuk also introduced homotopy metrics d_h and d_h' on the space of all ANR-subsets of Z. Defining the metrics d_h and d_h' requires extensive preliminary work, so we shall not discuss them here. The four aforementioned metrics are related by the inequalities

$$d_H(X,Y) \leq d_B(X,Y) \leq d_h(X,Y)$$

and

$$d_H(X,Y) \leq d_h'(X,Y) \leq d_h(X,Y).$$

The metrics d_h and d_h' induce the same topology on the space of all ANR-subsets of Z.

Theorem

If Z is a finite-dimensional compactum then the space $2^Z_{d_h}$ is separable.

Theorem

If Z is a finite-dimensional compactum then the space $2^Z_{d_h}$ is complete.

Theorem

Suppose Z is a finite-dimensional compactum. For any $X \in 2^Z_{d_h}$ the set $[X]$ of all $Y \in 2^Z_{d_h}$ having the same homotopy type as X is open.

The fundamental metric d_f

The Hausdorff distance corresponds to an arbitrary d-shift. The continuous Borsuk distance corresponds to a continuous d-shift. Here we will consider the distance that corresponds to an intermediate class of d-shifts.

8. In 1968, Karol Borsuk created his famous **Shape Theory**, which takes into account the global properties of compacts and overcomes their possible local complexity and the resulting insufficiency of continuous mappings to reveal the similarity of the global properties of two compacta. Based on this new concept of morphism, Karol Borsuk introduced a new distance d_f on the space $\exp Z$ in 1977.

Let X, Y be two compacta lying in a metric space Z and let M be an AR-space containing Z. By $\varrho_{F,M}(X, Y)$ we denote the infimum of the set of all positive numbers d satisfying the following condition:

There exist two fundamental sequences

$$\bar{f} = \{f_k, X, Y\}_{M,M} = \{f_k : M \to M\},$$

$$\bar{g} = \{g_k, X, Y\}_{M,M} = \{g_k : M \to M\},$$

such that

There is a neighborhood (V, U) of the pair (X, Y) in (M, M) such that for almost all k: $|xf_k(x)| < d$, $|yg_k(y)| < d$ for every $x \in V$, $y \in U$.

Theorem

$\varrho_{F,M}$ is a metric.

Theorem

If $X, Y \in \exp Z$ and if $X \cup Y$ is metrically contained in two AR-spaces M, M', then $\varrho_{F,M}(X,Y) = \varrho_{F,M'}(X,Y)$.

Theorem

If $X, Y \in \exp Z$ then

$$d_H(X,Y) \leq d_f(X,Y) \leq d_B(X,Y).$$

Example

Let $M = Z = X = [0,1]$ and $Y = \{0,1\}$ then

$$d_H(X,Y) = \frac{1}{2} < 1 = d_f(X,Y).$$

Theorem

If X, Y are compact ANR-spaces then

$$d_f(X, Y) = d_B(X, Y).$$

Theorem

Let $\{X_i\}_{i=0}^{\infty}$ be compacta lying in a space $Z \in AR$ and let $\lim_{i\to\infty} d_f(X_i, X_0) = 0$. If $p_n(X_i) \leq m$ for every $i = 1, 2, \ldots$ then $p_n(X_0) \leq m$.

Theorem (L. Boxer, R.B. Sher, 1978)

If $\{X_n\}_{n=1}^{\infty}$ is a sequence of compacta such that $d_f(X_n, X) \to 0$ and $X \in FANR$ then $Sh(X) \leq Sh(X_n)$ for almost all n.
This is solution of Borsuk's problem.

Theorem (Borsuk, 1977)

Let $\{X_i\}_{i=0}^{\infty}$ be compacta lying in ANR-space M and let $\lim_{i\to\infty} d_f(X_i, X_0) = 0$. If B is a compactum such that $B \geq_q X_i$ for every $i \in \mathbb{N}$ then $B \geq_q X_0$, where $\geq_q$ is the quasi-domination relation.

The shape metric d_{sh}

9. Here we consider the shape distance d_{sh} on $\exp Z$ introduced by the V.F. Laguna, M.A. Moron, Nguyen To Nhu and J.M R. Sanjurjo in 1993.

Here, we assume that $Z \subset M = \ell_2 = \left\{(x_i)_{i=1}^{\infty} : x_i \in \mathbb{R}, \ \sum_{i=1}^{\infty} x_i^2 < \infty\right\}$. We would like **to replace** the subset Z of the Hilbert space $M = \ell_2$ with an arbitrary (compact) metric space.

We denote by $i_{A,B} : A \to B$ the inclusion map and by $S(f)$ the shape morphism induced by f.

We define what we call the shape metric on $\exp Z$ as follows:

Let $X, Y \in \exp Z$. We consider the non-empty set of positive numbers

$S(X,Y) = \{d > 0 : X \subset O(Y,d), \ Y \subset O(X,d)$ and there exist shape morphisms $\bar{f} : X \to Y$ and $\bar{g} : Y \to X$ with $S(i_{Y,O(X,d)})\bar{f} = S(i_{X,O(X,d)})$ and $S(i_{X,O(Y,d)})\bar{g} = S(i_{Y,O(Y,d)})\}$.

We define $d_{sh}(X,Y) = \inf S(X,Y)$.

Theorem

d_{sh} is a metric on $\exp X$.

Theorem

Let Z be a subset of ℓ_2. Then

$$d_H \leq d_{sh} \leq d_f \leq d_B \quad \text{in} \quad \exp Z.$$

Theorem

Let P_n be a regular n-polygon. Then

$$d_H(P_n, S^1) \to 0 \qquad \text{and} \qquad d_{sh}(P_n, S^1) \not\to 0.$$

The next proposition is very useful in the case when we may use ambient spaces, such as Euclidean space $\mathbb{R}^n$ or the Hilbert cube Q instead of ℓ_2.

Proposition

Let C be a closed convex set of ℓ_2 and $Z \subset C$. Then the metric defined on $\exp Z$ *exactly in the same way, but using C instead of ℓ_2 as ambient space, agrees with the shape metric.*

Properties of the metric d_{sh}

Proposition

Let Z be a metric space. Then the family $T(Z)$ of all compacta of trivial shape in Z is closed in 2^Z_{sh}.

Proposition

For all compacta X and Y of trivial shape we have

$$d_H(X, Y) = d_{sh}(X, Y).$$

Proposition

A compactum X lying in the Hilbert cube Q is of trivial shape if and only if X is the limit of a sequence of arcs in $2^Q_{d_{sh}}$.

Theorem

Let X be a compact subset of the Hilbert cube Q. Then the following conditions are equivalent.
(1) X is movable.
(2) For every $\varepsilon > 0$ there exists a closed neighbourhood U of X in Q such that $d_{sh}(X, U) < \varepsilon$.
(3) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $Y \subset O(X, \delta)$ is a compactum containing X then $d_{sh}(X, Y) < \varepsilon$.
(4) There exists a sequence (P_n) of polyhedra in Q such that $\lim_{n\to\infty} d_{sh}(X, P_n) = 0$.

It can be proved in an analogous way that a compactum $X \subset \mathbb{R}^n$ is movable if and only if X is the d_{sh}-limit of a sequence of polyhedra in $\mathbb{R}^n$.

Corollary

The family of all movable compacta in Q is closed in $2^Q_{d_{sh}}$ and is separable.

Corollary

The family of all zero-dimensional compacta in Q is separable in the shape metric.

Corollary

$2^{\mathbb{R}^2}_{d_{sh}}$ *is separable.*

Theorem

There exists an uncountable family $\{S_\alpha\}$ of compacta in the plane such that $d_f(S_\alpha, S_{\alpha'}) \geq 1/8$ for every $\alpha \neq \alpha'$.
$2^{\mathbb{R}^2}_{d_f}$ *is not separable.*

Theorem

$2^{\mathbb{R}^3}_{d_{sh}}$ *is not separable.*

Theorem (V.F. Laguna, M.A. Moron, Nguyen To Nhu, J.M R. Sanjurjo, 1993)

Let $\{X_i\}_{i=0}^{\infty}$ *be compacta and let* $\lim_{i\to\infty} d_{sh}(X_i, X_0) = 0$. *If* $p_n(X_i) \leq m$ *for every* $i = 1, 2, \ldots$ *then* $p_n(X_0) \leq m$.
The case $d_f(X_n, X) \to 0$ *was done by Karol Borsuk in 1977 (??).*

Theorem

Let X *be a subset of* ℓ_2. *Let* $\{A_n\}$ *be a sequence of compacta lying in* X *such that* $d_{sh}(A_n, A) \to 0$. *If* $A \in FANR$ *then* $Sh(A) \leq Sh(A_n)$ *for almost all* n.
The case $d_f(X_n, X) \to 0$ *was done by L. Boxer, R.B. Sher (??).*

Theorem

Let $\{A_n\}$ *be a sequence of compacta in* ℓ_2 *such that* $A_n \to A$ *in the shape metric. If* B *is a compactum such that* $B \geq_q A_n$ *for every* $n \in \mathbb{N}$ *then* $B \geq_q A$, *where* $\geq_q$ *is the quasi-domination relation.*
The case $d_f(X_n, X) \to 0$ *was done by Karol Borsuk in 1977 (??).*

d_S

The concept of an S-morphism allows us to define a distance d_S such that

$$d_H \leq d_S \leq d_{sh}.$$

Consider approximate sequences.

For spaces with additional structures (properties), one can consider subclasses of (closed) subsets and subclasses of continuous mappings between them. Note that this is precisely what happens when defining the distances d_h and $d_h^{'}$ — they are defined between ANR-spaces.

There are many different distances between subspaces of a normed vector space.

Whitney function (map) $\mu\colon \exp X \to [0,\infty)$

The definition of a Whitney map is essentially an axiomatic formulation of what one might mean by a general measure of the relative geometric sizes of sets.

Definition

Let X be a compactum, and let $\mathscr{H} \subset 2^X$. A Whitney map for $\mathscr{H}$ is a continuous function $\mu : \mathscr{H} \to [0,\infty)$ that satisfies the following two conditions:
(1) for any $A, B \in \mathscr{H}$ such that $A \subset B$ and $A \neq B$, $\mu(A) < \mu(B)$;
(2) $\mu(A) = 0$ if and only if $A \in \mathscr{H} \cap \exp_1 X$.

For each $n = 1, 2, \ldots$, let $\exp_n X = \{A \subset X : 1 \leq |A| \leq n\}$, which we consider as a subspace of $\exp X$. The space $\exp_n X$ is called the n-fold hypersymmetric product of X; the 1-fold hypersymmetric product $\exp_1 X \cong X$ of X is also called the space of singletons.

For example, assuming that $[0, 1]$ has its usual (absolute value) metric, the diameter map is a Whitney map for $\exp^c[0, 1]$.

Here $\exp^c X \subset \exp X$ denotes the space of all non-empty compact connected subsets of X.

On the other hand, let X be an arc in the unit circle S^1 such that X has arc length greater than π, and assume that X has the Euclidean metric; then the diameter map is not a Whitney map for $\exp^c X$. In fact, a diameter map is rarely a Whitney map for $\exp^c X$, and a diameter map is never a Whitney map for $\exp X$ when X contains more than two points. Nevertheless, diameter maps are useful in constructing Whitney maps.

Theorem

If X is a compactum, then there is a Whitney map for the hyperspace $\exp X$.

Proof. It is evident from the definition that if μ is a Whitney map for $\exp X$ and $\mathscr{H} \subset \exp X$ then the restricted map $\mu|_{\mathscr{H}}$ is a Whitney map for $\mathscr{H}$. Therefore, it suffices to show that there is a Whitney map for $\exp X$.

I. The construction is due to Hassler Whitney.

Fix $A \in \exp X$. For any given integer $n \geq 2$, define $\lambda_n \colon \exp_n A \to [0, \infty)$ as follows: For each $K \in \exp_n A$, index points of K using all the integers $1, 2, \ldots, n$ (thus requiring the indexing not to be one-to-one when $|K| < n$); then, if $K \in \exp_n A$ and $K = \{a_1, a_2, \ldots, a_n\}$, let

$$\lambda_n(K) = min\big\{|a_i a_j| \colon i \neq j\big\}.$$

Next, for each $n \geq 2$, let $\mu_n(A) = \sup\big\{\lambda_n(K) \colon K \in \exp_n A\big\}$. Finally, let

$$\mu_W(A) = \sum_{n=2}^{\infty} 2^{1-n} \cdot \mu_n(A).$$

μ_W is a Whitney map for $\exp X$.

The Whitney map μ_W was, historically, the first Whitney map. It has a very natural and useful property that Whitney maps in general do not have.

The Whitney map μ_W has the following property:

(W) For any $A, B \in \exp X$ such that A and B are isometric (as subsets of X), $\mu_W(A) = \mu_W(B)$.

II. The construction is due to Alejandro Illanes.

Fix $A \in \exp X$. For any given integer $n \geq 1$, let $\mu_n wn(A) = \inf\{\varepsilon > 0 : \text{ there exist } x_1, \ldots, x_n \in X \text{ such that } A \subset \cup_{i=1}^{n} O(x_i, \varepsilon)\}$. Then let

$$\mu_I(A) = \sum_{i=1}^{\infty} 2^{-n} \cdot \mu_n(A) \text{ for each } A \in \exp X.$$

μ_I is a Whitney map for $\exp X$.

The Whitney map μ_I satisfies the Whitney condition (W).

III. Let $T = \{t_1, t_2, \ldots, t_n, \ldots\}$ be a countable, dense subset of X. For each $n = 1, 2, \ldots$, define $f_n \colon X \to [0, 1]$ as follows:

$$f_n(x) = \frac{1}{1 + |xt_n|} \text{ for each } x \in X.$$

Next, for each $n = 1, 2, \ldots$, define $\mu_n \colon \exp X \to [0, 1]$ as follows:

$$\mu_n(A) = \text{diameter } f_n(A) \text{ for each } A \in \exp X.$$

Now, define $\mu \colon \exp X \to [0, 1]$ as follows:

$$\mu(A) = \sum_{i=1}^{\infty} 2^{-n} \cdot \mu_n(A) \text{ for each } A \in \exp X.$$

μ is a Whitney map for $\exp X$.

The Whitney map μ depends on the given dense sequence $T = \{t_i\}_{i=1}^{\infty}$.

IV. The construction is due to J. Krasinkiewicz.

Let $\mathscr{B} = \{B_1, B_2, \ldots\}$ be a countable base for the topology for X. We assume that for each $i = 1, 2, \ldots$, $B_i \neq \emptyset$ and $\overline{B_i} \neq X$. Then, for each pair (B_i, B_j) such that $\overline{B_i} \subset B_j$, there is a continuous function $f_i^j : X \to [0, 1]$ such that $f_i^j(B_i) = 0$ and $f_i^j(X \setminus B_j) = 1$. Reindex all the functions f_i^j so that they form a simple sequence $f_1, f_2, \ldots, f_n, \ldots$. For each $n = 1, 2, \ldots$, define $\mu_n : \exp X \to [0, 1]$ by

$$\mu_n(A) = \text{diameter } f_n(A) \text{ for each } A \in \exp X.$$

Now, define $\mu_K : \exp X \to [0, 1]$ by

$$\mu_K(A) = \sum_{i=1}^{\infty} 2^{-n} \cdot \mu_n(A) \text{ for each } A \in \exp X.$$

μ_K is a Whitney map for $\exp X$.

The Whitney map μ_K depends on the given base $\mathscr{B}$, Tietze-Urysohn functions f_i^j, and the reindexing of functions f_i^j.



Properties

Theorem

If X is a compactumn that contains more than two points, then the diameter map with respect to any metric for X is not a Whitney map for $\exp X$.

Theorem

If X is a nondegenerate, arcwise connected continuum such that the diameter map with respect to some metric for X is a Whitney map for $\exp^c X$, *then X is an arc.*

Theorem

Let X be a metric compactum, and let μ be any Whitney map for $\exp X$. *Then a sequence, $\{A_i\}_{i=1}^{\infty}$, converges in* $\exp X$ *to A if and only if $\lim_{i\to\infty}\mu(A_i) = \mu(A)$ and for any $\varepsilon > 0$, $A_i \subset O(A, \varepsilon)$ for all i sufficiently large.*

Applications

Any Whitney function μ on a metric compactum X with property (W) and any (pseudo)metric ϱ on the set $[0, \infty)$ define a (pseudo)metric on $\exp X$.

$$d_{\mu,\varrho}(A, B) = \varrho\big(\mu(A), \mu(B)\big) \text{ for ecery } A, B \in expX.$$

If X is a finite set, then

$$|A| = \text{ Cardinality of } A \text{ for every } A \in \exp X$$

is a Whitney function.

If X is a set with a measure, then

$$\mu(A) = \text{ Measure of } A \text{ for every } A \in \exp X$$

is a Whitney function.

Dile distance

$$D(A, B) = \frac{\mu(A \cap B)}{\mu(A) + \mu(B)}.$$

Jalavd distance

$$J(A,B) = \frac{\mu(A \cap B)}{\mu(A \cup B)}.$$

? distance

$$H(A,B) = \max \dots .$$

Let us recall that the symmetric difference of two subsets $A, B \subset X$ is defined as

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

$$d(A,B) = \frac{\mu(A \triangle B)}{\mu(A \cup B)}.$$

Extra application

Let X be a continuum. W.J. Charatonik observed that the Whitney map μ_{III} defined in the above theorem has the following additional property:

(H) For every $A, B \in \exp X$ with $A \subset B$ and for every $C \in \exp X$,

$$\mu(B \cup C) - \mu(A \cup C) \leq \mu(B) - \mu(A).$$

W.J. Charatonik has shown that Whitney maps with property (H) induce a nice metric on the hyperspace $\exp X$.

Theorem

Let X be a continuum (compactum), μ be a Whitney map for $\exp X$ that satisfies (H). For $A, B \in \exp X$, define

$$d_\mu(A, B) = \mu(A \cup B) - \min\{\mu(A), \mu(B)\}.$$

Then d_μ is a metric on $\exp X$ and d_μ is equivalent to the Hausdorff metric d_H on $\exp X$.

Since the work by W.J. Charatonik itself and all its citations use the term “continuum” (not compactum), we will discuss the proof. In W.J. Charatonik’s original work, the proof is not presented as obvious. The speaker does not understand why W.J. Charatonik assumes the connectedness of the compactum X.

Proof. The equivalence $d_\mu(A, B) = 0 \iff A = B$ follows from condition (1) of the definition of a Whitney map.

The symmetry of d_μ is obvious from the definition.

Let us verify the triangle inequality. Let $A, B, C \in \exp X$. Without loss of generality, we can assume that $\mu(A) \leq \mu(B)$. We need to show that

$$\mu(A \cup C) - \min\{\mu(A), \mu(C)\} + \mu(B \cup C) - \min\{\mu(B), \mu(C)\} \geq \mu(A \cup B) - \mu(A).$$

It is enough to show that

$$\mu(A \cup C) - \mu(A) + \mu(B \cup C) - \mu(C) - \mu(A \cup B) + \mu(A) \geq 0. \quad (*)$$

Inequality (H) for $\overline{A} = C$, $\overline{B} = A \cup C$, and $\overline{C} = B$ takes the form

$$\mu(A \cup B \cup C) - \mu(B \cup C) \leq \mu(A \cup C) - \mu(C).$$

By replacing $\mu(A \cup B \cup C)$ with the no greater number $\mu(A \cup B)$, we obtain the required inequality $(*)$. □

W.J. Charatonik posed the following question:

Given any continuum X and any Whitney map $\mu\colon \exp X \to [0, \mu(X)]$, does there exist an embedding h from $[0, \mu(X)]$ into $[0, \infty)$ such that $h \circ \mu$ is a Whitney map satisfying property (H)?

When determining distances on $\exp Z$, we are talking about a d-shift, i.e., we compare the distance $|xf(x)|$ between a point and its image under the considered mapping. This automatically means that we are comparing two objects located in the same universe. However, in fact, we have already replaced the purely pointwise conditions $X \subset O(Y, d)$ and $Y \subset O(X, d)$ with the existence of mappings $f : X \to Y$ and $g : Y \to X$, which are compared with the identity mappings $i_{X,Z} : X \to Z$ and $i_{Y,Z} : Y \to Z$.

The corresponding quantities depend not on the entire space Z, but on the subset $X \cup Y \subset Z$ (see Theorem 7.2), i.e., on the metric on $X \cup Y$, i.e., on the pseudo-metric on $X \sqcup Y$.

The classical Hausdorff theorem on extending a metric from a closed subset here turns into the problem of extending metrics on two compact sets to a pseudo-metric on their disjoint union. If we cannot solve this problem (or want to avoid considering **all** extending pseudo-metrics), then based on two mappings $f : X \to Y$ and $g : Y \to X$, it is necessary to construct (specify, define) some numerical characteristics analogous to the shift value in the presence of a common pseudo-metric on the disjoint union. In fact, this will mean that we will then solve the problem using the ambient space in internal terms.

In the definition of the Hausdorff distance d_H (continuous Borsuk distance d_B), we are already considering usual (continuous) mappings of given spaces into each other, and we need to somehow determine the magnitude of the shift without using the identity mapping. The simplest way to do this is by using the triangle inequality

$$|xx^{'}| - 2d \leq |xx^{'}| - |xf(x)| - |f(x^{'})x^{'}| \leq |f(x)f(x^{'})|$$
$$\leq |f(x)x| + |xx^{'}| + |x^{'}f(x^{'})| \leq |xx^{'}| + 2d.$$

Moreover, it can be assumed that for an optimal embedding-isometry, the $2d$ estimate should not improve — there must be a Chebyshev alternation — the magnitudes of the maximum increase and maximum decrease must be equal. The proposed heuristic consideration shows that the following quantity should be of significant importance.

By distortion of the mapping $f : X \to Y$ we mean the magnitude

$$\operatorname{dis} f = \sup\Big\{\Big||f(x)f(x')| - |xx'\Big| : x, x' \in X\Big\}.$$

By the codistortion of a pair of mappings $f : X \to Y$ and $g : Y \to X$ is meant the value

$$\operatorname{codis} f = \sup\Big\{\Big||f(x)y| - |xg(y)|\Big| : x \in X,\ y \in Y\Big\}.$$

It is natural to consider also the quantities defined by one (or three) points.

$$\operatorname{dis}(f, g) = \sup\Big\{\Big|xg(f(x))\Big| : x \in X\Big\}.$$

Now we will not assume the existence of an encompassing universe Z.

1. Change the class of isometric embeddings i and j.
We set

$$d'_?(X,Y) = \inf\bigl\{ ch(i) + d_?\bigl(i(X), j(Y)\bigr) + ch(g) : \\ i\colon X \to Z,\ j\colon Y \to Z\bigr\}.$$

Obviously, for any distance $d_?$, we have

$$d'_? \le d_?.$$

a. The class of isometric embeddings can be narrowed. For normed vector spaces, linear isometric embeddings are usually considered, but in this case, the Hausdorff distance between subspaces (the images of the normed spaces X and Y in the normed space Z) can also be reasonably replaced with something that takes into account the linear structure in Z.

b. The class of isometric embeddings can be extended. It is clear that in this case, the relationship between the Gromov–Hausdorff distance and the Hausdorff distance must include the characteristics of the pair of mappings i and j.

2. Change the Hausdorff distance $d_H(i(X), j(Y))$ to another distance between the subsets $i(X)$ and $j(Y)$.

a. For normed spaces, we have already been forced to indicate in 1.a the need to switch to "linear" variants of the Hausdorff distance.

b. Włodzimierz J. Charatonik, in 1985, constructed a new metric $d_{Ch}(i(X), j(Y))$ on the exponent $\exp Z$ based on a Whitney function satisfying a certain condition. Unfortunately, all Whitney functions known to the author that satisfy Charatonik's condition are defined by an additional object that cannot be chosen in a natural way. Perhaps a separate report will be devoted to Charatonik's distance.

c. The metric of Karol Borsuk is also known on the exponent $\exp Z$. The main content of the report is devoted to the Borsuk metric d_B and the Gromov–Borsuk metric $d_{GB}(X, Y)$ generated by it.

II. The second "functional – internal" approach is based on considering pairs of mappings $f : (X, \varrho_X) \to (Y, \varrho_Y)$, $g : (Y, \varrho_Y) \to (X, \varrho_X)$ and assessing the distortion of distances under the mappings under consideration.

1. The resulting analogues of the Gromov–Hausdorff distance are determined by the class of mappings under consideration.

a. For normed spaces, one usually limits oneself to linear isomorphisms.

b. In the case of multivalued mappings, the corresponding distances are determined, for example, by the nature of the continuity of these mappings. Perhaps such analogues of the Gromov–Hausdorff distance will be the focus of the reports by Alexey A. Tuzhilin and his colleagues.

c. For single-valued mappings $f : (X, \varrho_X) \to (Y, \varrho_Y)$, $g : (Y, \varrho_Y) \to (X, \varrho_X)$, the following distances are best studied.

α. For arbitrary mappings, the Gromov–Hausdorff distance and the modified Gromov–Hausdorff distance;

β. For continuous mappings, the continuous Gromov–Hausdorff distance and the modified continuous Gromov–Hausdorff distance;

γ. Bijective, surjective, and injective mappings.

2. Various numerical characteristics of the pair of mappings f and g are possible.

a. For normed spaces, the classical Banach–Mazur distance is defined in this way using the norms of the operators f and g. It is intended to touch upon the topological properties of the Banach–Mazur compact (Minkowski compact) $B(n)$ of all n-dimensional norm vector spaces.

b. For the classical Gromov–Hausdorff distance, the distortions $\operatorname{dis} f$, $\operatorname{dis} g$ of mappings and the codistortion $\operatorname{codis}(f, g)$ of a pair of mappings acting in opposite directions are taken:

$$d_{GH}(X, Y) = \inf\{\max\{\operatorname{dis} f, \operatorname{dis} g, \operatorname{codis}(f, g)\} : f\colon X \to Y,\ g\colon Y \to X\}.$$

Here

$$\operatorname{dis} f = \sup\{\big||f(x)f(x')|_Y - |xx'|_X\big| : x, x' \in X\ \},$$

$$\operatorname{codis}(f, g) = \sup\{\big||f(x)y|_Y - |xg(y)|_X\big| : x \in X,\ y \in Y\ \}.$$

c. The definition of the modified Gromov–Hausdorff distance introduced by Mémoli is simpler:

$$d^m_{GH}(X, Y) = \inf\{\max\{\operatorname{dis} f, \operatorname{dis} g : f\colon X \to Y,\ g\colon Y \to X\}.$$

It may be possible to demonstrate the similarity (parallelism) of certain definitions of the Gromov–Hausdorff distance with certain constructions in K. Borsuk's shape theory (particularly the use of ε-mappings and multivalued mappings).

It is perhaps not surprising that the pioneer in defining the distance between metric spaces, D. Edwards, was a specialist in the shape theory.

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