Using Grid Diagrams in Knot Theory to Define Gromov–Hausdorff Distance

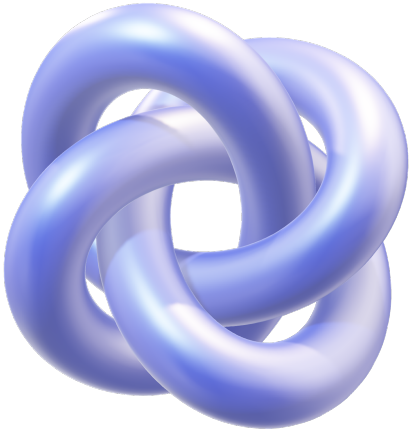
Connecting Knot Theory, Metric Geometry, and the Gordian Distance

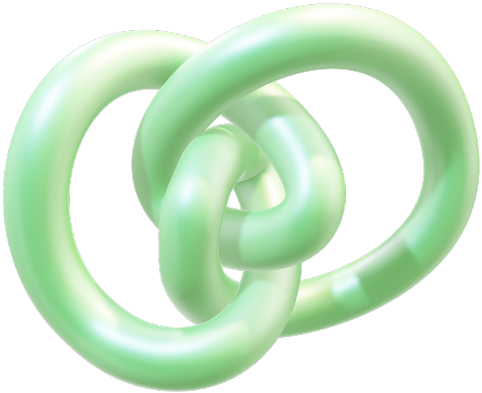
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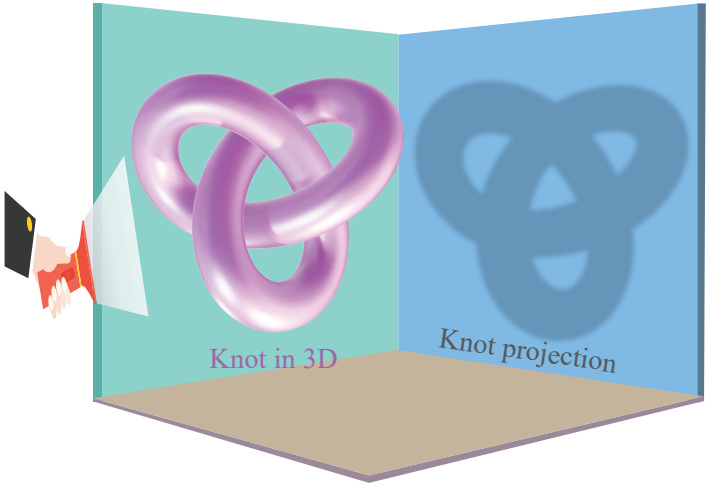
Knot Equivalence

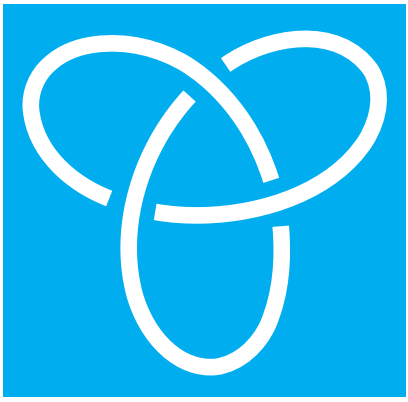
They are the same knot or not?





3D to 2D

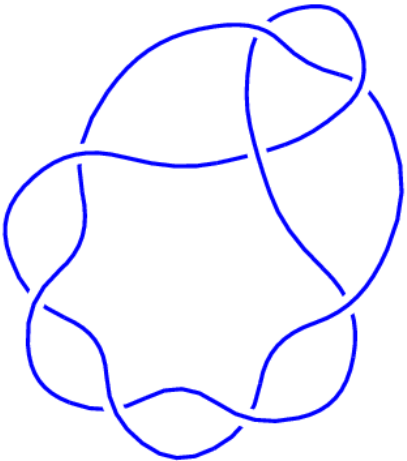


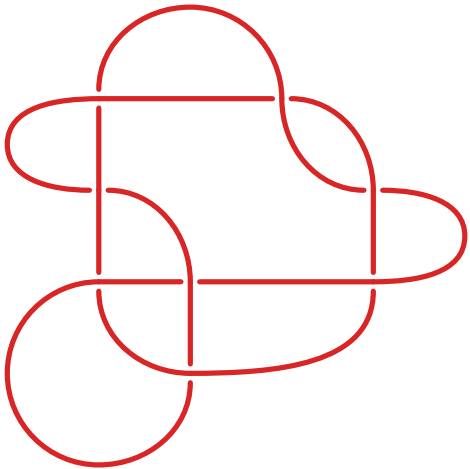


Knot diagram

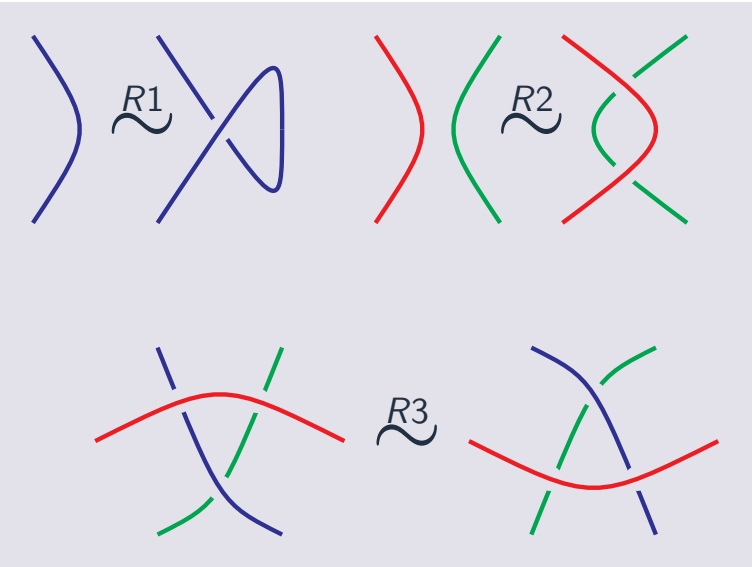
Knot Equivalence

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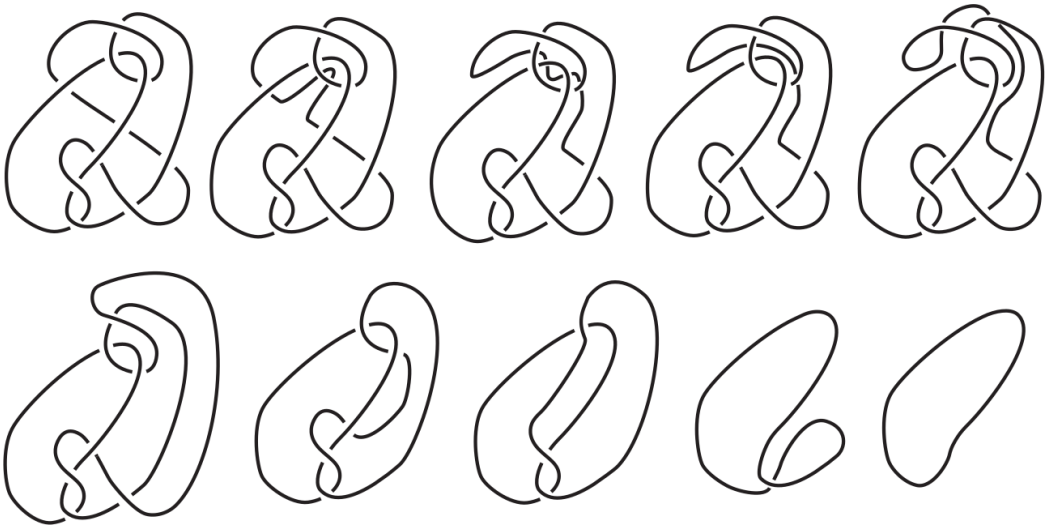




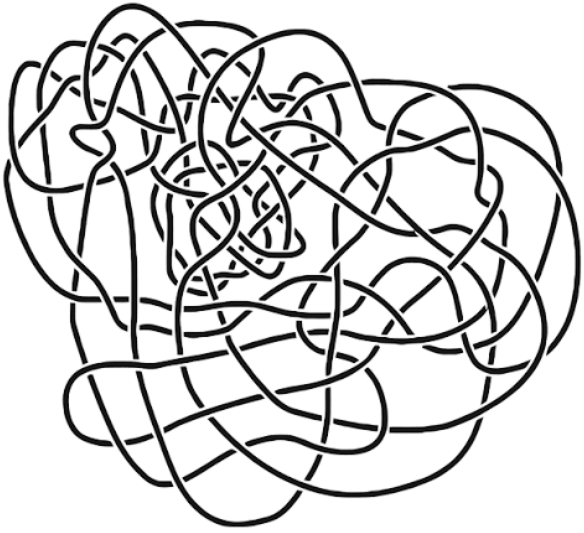
Reidemeister Moves



Knot Equivalence



Knot Equivalence



Knot as metric space

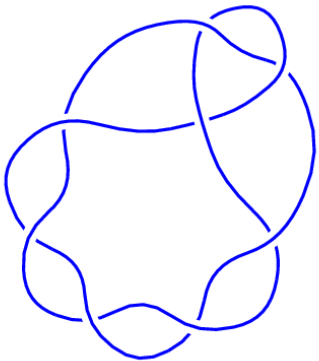


Figure: Metric space X

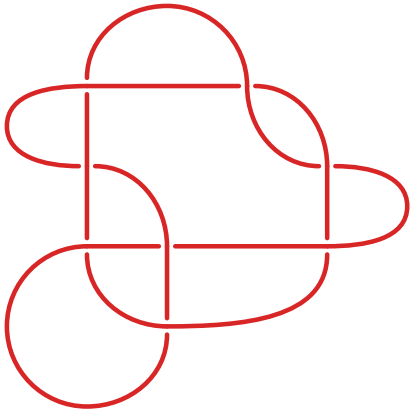


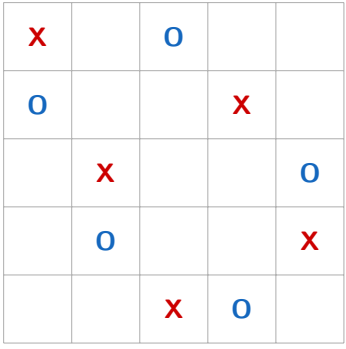
Figure: Metric space Y

Grid diagrams of knots

Definition

A grid diagram of size n is an $n \times n$ grid with exactly one X and one O in every row and every column.

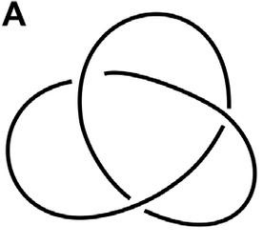
- Connect X–O pairs horizontally and vertically.
- Vertical segments pass over horizontal segments at crossings.
- The diagram encodes a knot or link.



Illustrative grid marking

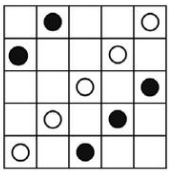
Trefoil knot and it grid diagram

A

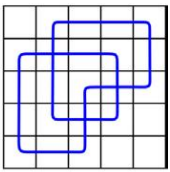


3_1

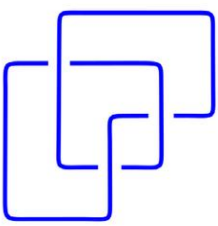
B



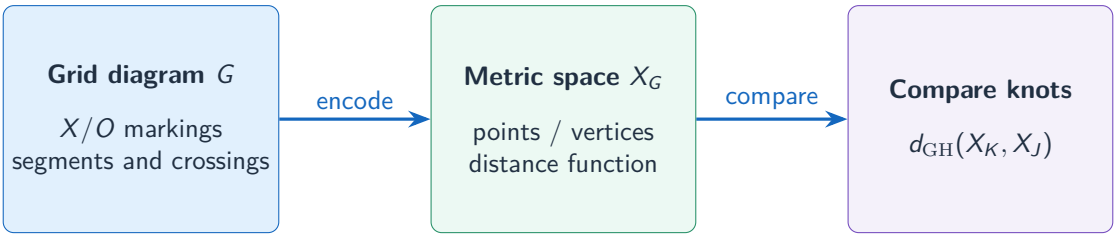
C



D



The central idea: grid diagram $\longrightarrow$ metric space



Goal: define $K \longmapsto X_K$ so the space reflects knot type.

A finite metric space from the grid

1. **Extract combinatorial data**
   X/O markings, segments, endpoints, and crossing information.
2. **Build a weighted graph**
   Choose vertices from grid points, segment endpoints, and crossing-related data.
3. **Define shortest-path distances**
   For vertices u, v, let $d_G(u, v)$ be the length of a shortest path.

$$\boxed{X_G = (V(G), d_G) \quad \text{is finite and compact.}}$$

Why this is attractive

Finite distance matrices make computational experiments possible.

The key design challenge

Possible failure

If every diagram becomes the same unweighted cycle graph at fixed grid size, the resulting metric space may fail to distinguish different knots.

What must be retained?

The construction should encode useful crossing and over/under information, and its behavior under grid moves must be controlled.

A metric is easy to build; a useful knot invariant is harder.

Can GH distance bound Gordian distance?

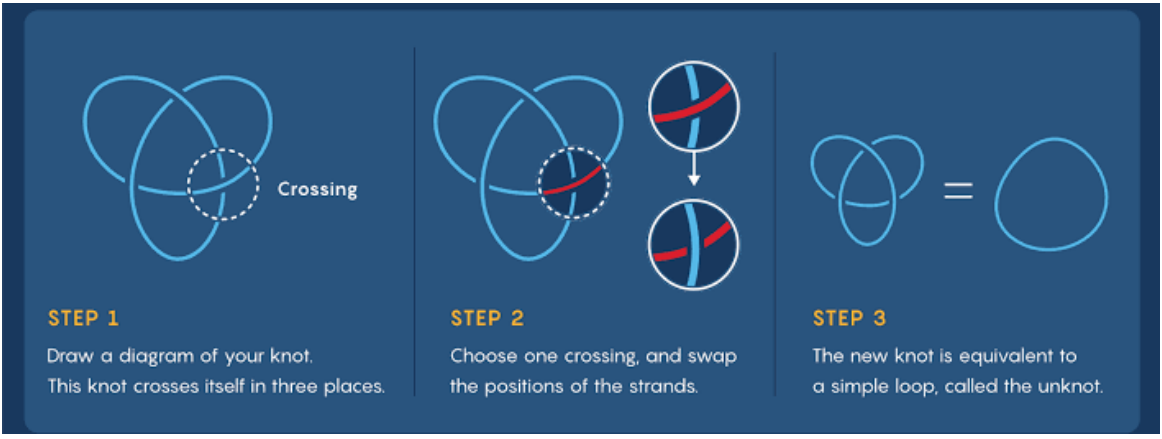
What is Gordian distance?

For knots K and J, the **Gordian distance** $d_G(K, J)$ is the minimum number of crossing changes needed to transform K into J.

$$\boxed{d_G(K, J) = \min\{\text{number of crossing changes taking } K \text{ to } J\}.}$$

- A crossing change switches which strand passes over the other.
- Minimize over all possible sequences of crossing changes.
- If K and J are equivalent, then $d_G(K, J) = 0$.

Example: trefoil and unknot



A single crossing change in a trefoil diagram can produce a diagram of the unknot. Thus

$$\boxed{d_G(T, U) = 1.}$$

Unknotting number

The **unknotting number** $u(K)$ is the minimum number of crossing changes required to transform K into the unknot U.

$$\boxed{u(K) = d_G(K, U).}$$

- $u(U) = 0$.
- The trefoil has $u(T) = 1$.
- A sequence gives an upper bound; proving minimality requires a lower bound.

Why is it difficult?

Even if a short sequence is known, it may be difficult to prove that no shorter sequence exists.

Properties of Gordian distance

For knot types K, J, L:

1. **Non-negativity:** $d_G(K, J) \geq 0$.
2. **Identity:** $d_G(K, J) = 0$ exactly when K, J are equivalent.
3. **Symmetry:** $d_G(K, J) = d_G(J, K)$.
4. **Triangle inequality:**
$$d_G(K, L) \leq d_G(K, J) + d_G(J, L).$$

Thus Gordian distance is a metric on knot types.

Why is Gordian distance hard to compute?

Upper bound

Exhibit a sequence of m crossing changes from K to J. Then $d_G(K, J) \leq m$.

Lower bound

To prove $d_G(K, J) \geq m$, rule out every sequence with fewer than m crossing changes.

Knot invariants and geometric or algebraic techniques can sometimes provide such lower bounds.

Connection to Gromov–Hausdorff distance

Suppose we assign a compact metric space X_K to each knot K, perhaps using a grid diagram.

$$K \longmapsto X_K.$$

Could there be a constant $C > 0$ such that

$$\boxed{d_{\mathrm{GH}}(X_K, X_J) \leq C\, d_G(K, J)?}$$

Then

$$\boxed{d_G(K, J) \geq \frac{1}{C} d_{\mathrm{GH}}(X_K, X_J).}$$

This would give a lower bound for Gordian distance using metric geometry.

A proposed lower bound for unknotting number

Taking $J = U$, the unknot, and using $u(K) = d_G(K, U)$ gives

$$\boxed{u(K) \geq \frac{1}{C} d_{\mathrm{GH}}(X_K, X_U).}$$

Proof strategy

1. Show that one crossing change changes the assigned space by GH distance at most C.
2. Follow a sequence of crossing changes from K to U.
3. Apply the triangle inequality for GH distance.
4. Minimize the number of crossing changes.

9. The main mathematical challenge

The construction must retain knot information

If every knot is assigned the same metric space, then $d_{\mathrm{GH}}(X_K, X_U) = 0$ for every K, and the lower bound is useless.

- Intrinsic path length alone may lose knot type: after length normalization, every simple closed curve is isometric to a circle.
- A grid-derived metric should encode useful crossing and over–under information.
- It must be well-defined under grid moves representing the same knot.

Can GH distance bound Gordian distance?

The Gordian distance $d_G(K, J)$ is the minimum number of crossing changes required to transform knot K into knot J.

$$\boxed{d_{\mathrm{GH}}(X_K, X_J) \leq C\, d_G(K, J)}$$

would imply

$$\boxed{d_G(K, J) \geq \frac{1}{C} d_{\mathrm{GH}}(X_K, X_J).}$$

Proof strategy

Show that one crossing change alters the assigned metric space by GH distance at most C. Apply the GH triangle inequality along a crossing-change sequence, then minimize the number of changes.

This is a proposed theorem to investigate, not an established result.