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Gromov's Conjecture

M.Gromov in "Metric structures for Riemannian and non-Riemannian spaces" discussed possibility of considering the (Gromov–)Hausdorff distance between any spaces, not necessarily compact. He wrote: "One can also make a moduli space of isometry classes of non-compact spaces X lying within a finite Hausdorff distance from a given X_0, e.g. $X_0 = \mathbb{R}^n$. Such moduli spaces are also complete and contractible".

It was not very difficult to verify completeness (Bogaty–AT). However, contractibility needs additional consideration becuase it is a topological notion.

We denote by $\mathcal{GH}$ the family of all metric spaces considered upto isometry. Notice that $\mathcal{GH}$ is a proper class in terms of von Neumann–Bernays–Gödel (NGB) set theory.

Recall that all objects of NBG are called *classes*. There are two types of classes:

- a *set* $\mathcal{A}$: there exists a class $\mathcal{C}$ such that $\mathcal{A} \in \mathcal{C}$;
- a *proper class* $\mathcal{A}$: for any class $\mathcal{C}$ it holds $\mathcal{A} \notin \mathcal{C}$.

For all classes $\mathcal{A}$, $\mathcal{B}$, it is defined $\mathcal{A} \times \mathcal{A}$, $f\colon \mathcal{A} \to \mathcal{B}$, etc., in particular, we can speak about *distance*, (*extended*) *pseudometric*, and (*extended*) *metric* on a class.

However, *we cannot define topology* τ *on a proper class* $\mathcal{A}$ in the standard way: otherwise, $\mathcal{A} \in \tau$, hence $\mathcal{A}$ is a set.

To overcome, let $\mathcal{A}_n \subset \mathcal{A}$ consists of all elements (sets) whose cardinality is at most n. We get a "filtration" of $\mathcal{A}$ w.r.t. various n:

$$\cdots \mathcal{A}_m \subset \mathcal{A}_n \cdots, \quad m < n.$$

Suppose that each $\mathcal{A}_n$ is a set. We call such $\mathcal{A}$ *set-filtered*.

For a set-filtered $\mathcal{A}$, we define a *topology* τ on $\mathcal{A}$ as a mapping $\tau\colon n \mapsto \tau_n$, where τ_n is a topology on $\mathcal{A}_n$ such that τ_m is induced from t_n for all $m < n$ (in this case we say that the topologies τ_n are *consistent*). We call a set-filtered class $\mathcal{A}$ with a topology τ a *topological class*. Evidently, each topological space is a topological class.

If ρ is an extended pseudometric on a set-filtered class $\mathcal{A}$, then ρ induces the corresponding topology τ_n on $\mathcal{A}_n$ for each n, and the topologies τ_n are obviously consistent. We call such τ a *pseudometric topology*.

Let X be a set, $\mathcal{A}$ a class, and $f\colon X \to \mathcal{A}$ a mapping.

Proposition. *The image $f(X)$ belongs to some $\mathcal{A}_n$.*

As a corollary, for any topological space X and any topological class $\mathcal{A}$, we can define the notion of continuous mapping:

$f\colon X \to \mathcal{A}$ is *continuous* if the restriction $f\colon X \to \mathcal{A}_n$ is continuous for some (and, therefore, for each) $\mathcal{A}_n \supset f(X)$.

Moreover, we can define continuous mappings between topological classes: for any topological classes $\mathcal{A}$ and $\mathcal{B}$, a mapping $f\colon \mathcal{A} \to \mathcal{B}$ is *continuous* if the restrictions $f\colon \mathcal{A}_n \to \mathcal{B}$ are continuous for all n. In particular, for topological classes, we can speak about continuous curves, linear connectivity, intrinsic distance function, homotopy, contractibility, etc.

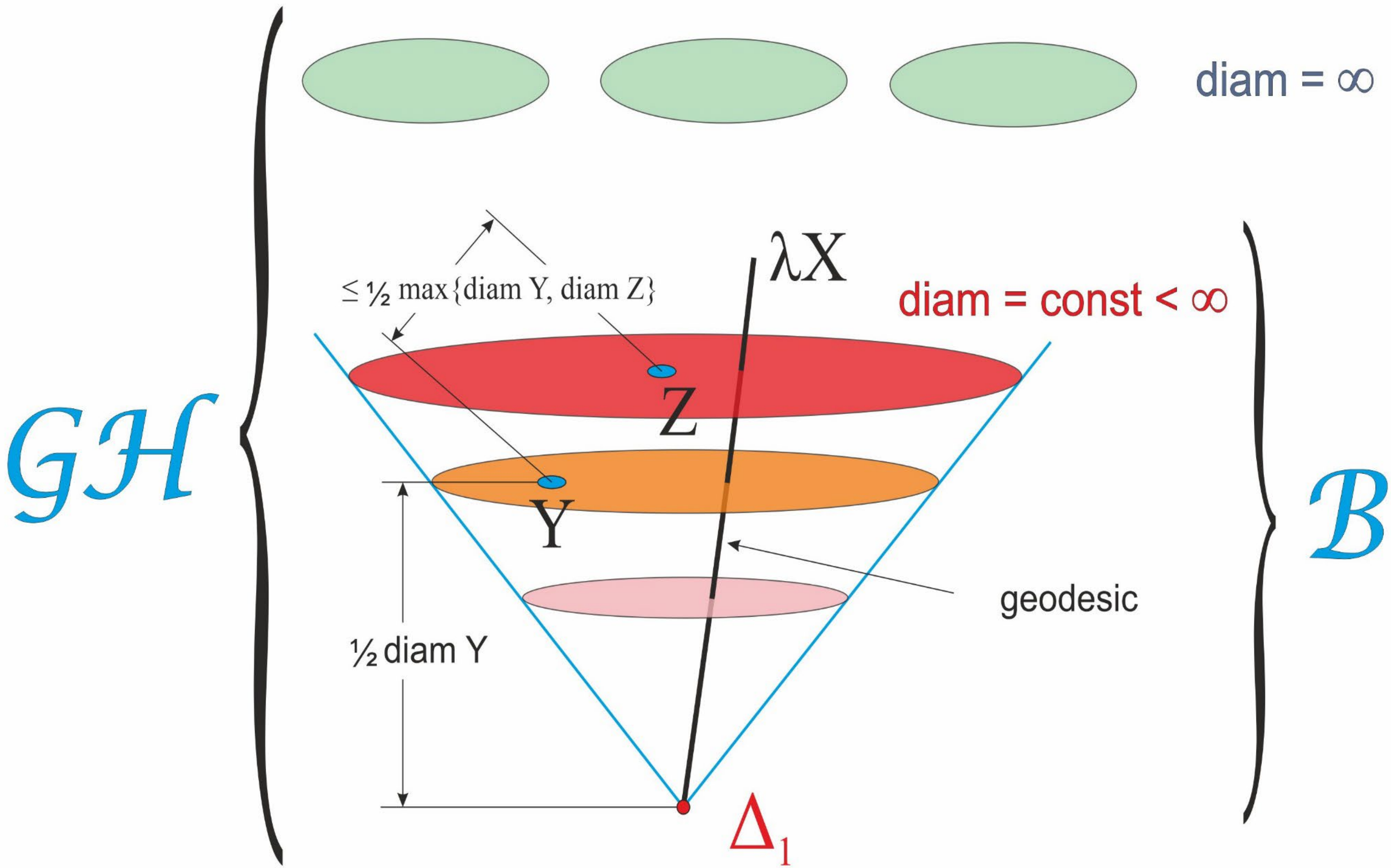
Proposition. *The class* $\mathcal{GH}$ *of all metric spaces considered upto isometry, together with all its subclasses, is set-filtered.*

There are a few important subclasses in $\mathcal{GH}$:

- the class $\mathcal{GH}_0$ consising of representatives of the equivalence ($X \sim Y$ iff $d_{GH}(X,Y) = 0$);
- the classes $\mathcal{B} \subset \mathcal{GH}$ and $\mathcal{B}_0 \subset \mathcal{GH}_0$ consisting of all bounded metric spaces;
- the classes $\mathcal{M} \subset \mathcal{B}$ and $\mathcal{M}_0 \subset \mathcal{B}_0$ consisting of all compact metric spaces (indeed, $\mathcal{M} = \mathcal{M}_0$, and the class is called the *Gromov–Hausdorff space*).

Theorem (S.Bogaty, A.Ivanov, N.Nikolaeva, AT). *Gromov-Hausdorff distance is intrinsic and complete on* $\mathcal{GH}$, $\mathcal{GH}_0$, $\mathcal{B}$, $\mathcal{B}_0$, *and* $\mathcal{M}$. *On* $\mathcal{M}$ *it is geodesic, i.e., each* $X, Y \in \mathcal{M}$ *are joined by a shortest curve* γ *with* $|\gamma| = d_{GH}(X,Y)$.

Problem. Is Gromov-Hausdorff distance on any of $\mathcal{GH}$, $\mathcal{GH}_0$, $\mathcal{B}$, $\mathcal{B}_0$ geodesic? (Later Anton Vikhrov will show some of his results.).



diam = ∞
λX
≤ ½ max{diam Y, diam Z}
diam = const < ∞
GH
Z
Y
B
geodesic
½ diam Y
Δ1

Clouds and their Homothety

Consider the following equivalence $\sim$ on $\mathcal{GH}_0$:

$X \sim Y$ iff $d_{GH}(X,Y) < \infty$.

Its equivalence classes we call *clouds*. Thus, the clouds are maximal subclasses such that the restriction of the Gromov–Hausdorff metric on them is a (finite) metric.

For $X \in \mathcal{GH}_0$, denote by $[X]$ the cloud containing X.

Proposition (B.Nesterov). *All clouds are proper classes.*

Example. The class $\mathcal{B}_0$ is a cloud containing Δ_1, i.e., $\mathcal{B}_0 = [\Delta_1]$.

Consider the mapping

$\Psi\colon (0,\infty) \times \mathcal{GH}_0 \to \mathcal{GH}_0$, $\Psi\colon (\lambda, X) \mapsto \lambda X$, $\Psi_\lambda\colon X \mapsto \lambda X$.

Problem. Investigate the properties of the mapping Ψ.

The class $\mathcal{B}_0$ is invariant under each Ψ_λ, and Ψ_λ is a homothety with the center at Δ_1.

Question. Is it true that each cloud $\mathcal{C}$ is invariant under all mappings Ψ_λ, i.e., for any $X \in \mathcal{C}$ and any $\lambda > 0$, it holds $d_{GH}(X, \lambda X) < \infty$?

Example (S.Bogaty, AT)**.** Let $p > 2$ be a prime number, and $X = \{p, p^2, p^3, \dots\} \subset \mathbb{R}$. Then $d_{GH}(X, 2X) = \infty$, however, $d_{GH}(X, pX) < \infty$.

Theorem. *For each cloud $\mathcal{C}$, any $X, Y \in \mathcal{C}$, and any $\lambda > 0$, if $\lambda X \in \mathcal{C}$, then $\lambda Y \in \mathcal{C}$. The set of all $\lambda > 0$ such that $\lambda X \in \mathcal{C}$ for some (and, thus, for all) $X \in \mathcal{C}$ forms a subgroup of the multiplicative group $\mathbb{R}_+$ of positive real numbers, which we call the* stabilizer of the cloud *$\mathcal{C}$.*

Problem. Is it true that clouds with different stabilizers are not isometric? What can we say about clouds with equal stabilizers? Boris Nesterov will give us some details.

Gromov's Conjecture on Contractibility of Clouds

By means of our definition of the topology for set-filtered classes, it is possible to give meaning to the notion of contractibility in this case. Nevertheless, we do not know how to prove the contractibility of clouds different from $\mathcal{B}_0$. Moreover, for $\mathcal{B}_0$, we can take as the corresponding contraction the homothety Ψ defined above. For $[\mathbb{R}^n]$, which is "similar" with $\mathcal{B}_0$ as Gromov mentioned, Ψ is not continuous (Ivan Mikhailov will give us some details). And what to do with clouds whose stabilizers differ from $\mathbb{R}_+$?.

Some problems

- Geodesics in GH-class.
- Calculate the distance between clouds.
- Is it true that each unbounded metric space can be isometrically embedded into GH-class? Which spaces can be isometrically embedded into GH-space (of compact metric spaces)?
- Continue investigation of GH-distances to spaces with one non-zero distance.
- Is it true that the Hausdorff mapping that takes a metric spaces to its hyperspace (the space of all closed bounded subsets, endowed with Hausdorff metric) is isometric?
- When the Steiner problem of existance of the shortest tree has solutiuons in GH-class?
- Investigate isometries of GH-class.
- Application of GH-distance to graph theory and some problems like Borsuk conjecture.
- ETC.