Geodesics in Gromov—Hausdorff proper class.

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Definition

A correspondence $R \in \mathcal{R}(X, Y)$ is called *optimal* if $2\,\mathrm{d}_{\mathrm{GH}}(X, Y) = \operatorname{dis} R$.

Definition

An *optimal Hausdorff realization* of two metric spaces A and B is defined as a triple $(X, \tilde{A}, \tilde{B})$ realizing the pair A, B such that $\mathrm{d}_H^X(A, B) = \mathrm{d}_{\mathrm{GH}}(A, B)$. If the Gromov—Hausdorff distance does not coincide with the Hausdorff distance between $\tilde{A}$ and $\tilde{B}$, we call such a triple a *Hausdorff realization*.

Theorem (S. Chowdhury, F. Mémoli; A.O. Ivanov, S. Iliadis, A.A. Tuzhilin)

If R is an optimal correspondence between metric spaces A and B, then the curve $\gamma : [0,1] \to \mathcal{GH}$, where $\gamma(t) = (R, d_t)$ and $d_t((a,b),(a',b')) = (1-t)d_A(a,a') + t\,\mathrm{d}^B(b,b')$, is a geodesic connecting the metric spaces A and B.

Such geodesics are called ***linear***. Patrick Ghanaat found an example of two unbounded metric spaces, with no optimal correspondence between them.

Example

The space A is $\triangle_2$. The space C is constructed as follows. Let $C = (\mathbb{N}, \mathrm{d}^C)$, where

$$\mathrm{d}^C(i,j) = \begin{cases} 0, & i = j, \\ 1/4 - 1/2^{\max(i,j)+2}, & i \neq j. \end{cases}$$

For such A and C there is no optimal correspondence. However, there found metric space $\tilde{C} = (C \cup \{p\}, \mathrm{d}^{\tilde{C}})$, where $\mathrm{d}^{\tilde{C}}|_C = \mathrm{d}^C$ and $\mathrm{d}^{\tilde{C}}(p, c) = 1/4$ for $c \in C$. For $\tilde{C}$ we have the following:

1. $C \neq \tilde{C}$ up to isometry
2. $\mathrm{d}_{GH}(C, \tilde{C}) = 0$
3. $\mathcal{R}_{opt}(A, \tilde{C}) \neq \emptyset$

Big example

The following example obtained:

Theorem

There exist metric spaces X and Y with the following properties:

1. $0 < \mathrm{d_{GH}}(X, Y) < \infty$
2. $\mathcal{R}_{opt}(X, Y) = \emptyset$, *i.e., a linear geodesic does not exist*
3. $\nexists \tilde{X}, \tilde{Y}$ *such that* $\mathrm{d_{GH}}(X, \tilde{X}) = \mathrm{d_{GH}}(Y, \tilde{Y}) = 0$ *and* $\tilde{X} \neq X$ *and* $\tilde{Y} \neq Y$

Moreover, it is shown that there is no geodesic $\gamma(t), t \in [0, 1]$ between the spaces X and Y that is piecewise linear on any segment $[\varepsilon, 1 - \varepsilon]$. Furthermore, there is no optimal Hausdorff realization between these spaces. (the infimum in the definition of Gromov–Hausdorff distance is not reached) However, another geodesic was found between them.

Examples of "Good Spaces"

Let $X \in \mathcal{GH}$. Denote by $S(X)$ the set of all bijective mappings of X onto itself. We introduce the following notations:

$$\mathrm{s}(X) = \inf\left\{|xx'| \mid x \neq x';\ x, x' \in X\right\},$$
$$\mathrm{t}(X) = \inf\left\{|xx'| + |x'x''| - |xx''| \mid x \neq x' \neq x'' \neq x; x, x', x'' \in X\right\},$$
$$\mathrm{e}(X) = \inf\left\{\mathrm{dis}(f) \mid f \in S(X), f \neq \mathrm{id}\right\},$$
$$\mathrm{e}'(X) = \inf\left\{\mathrm{dis}(f) \mid f \in S(X) \setminus \mathrm{ISO}(X)\right\}.$$

Definition

A ***general position space*** is a metric space X in which $\mathrm{s}(X)$, $\mathrm{t}(X)$, and $\mathrm{e}(X)$ are positive.

Definition

A ***generalized general position space*** is a metric space X in which $\mathrm{s}(X)$ and $\mathrm{e}'(X)$ are positive.

Main Theorem about this Space

Theorem

If a space M satisfies the conditions $\mathrm{e}'(M) > 0$ and $\mathrm{s}(M) > 0$, then for any $\varepsilon > 0$ such that $\varepsilon < \mathrm{s}(M)/4$ and $\varepsilon < \mathrm{e}'(M)/4$, any metric space X, and any correspondence $R \in \mathcal{R}(M, X)$ satisfying $\mathrm{dis}(R) < 2\varepsilon$, R is an optimal correspondence.

Theorem

General position spaces form a dense subclass in the class of metric spaces, i.e., in an arbitrarily small neighborhood of any metric space, there exists a general position space.

Metric-Preserving Functions

Definition

We call a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *metric-preserving* if and only if, for every metric space (X, d_X), the space $(X, f \circ d_X)$ is a metric space.

Theorem

A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is metric-preserving if and only if $f(x) = 0$ only for $x = 0$, and for any nonnegative a, b, c such that $|b - c| \leq a \leq b + c$, the inequality $f(a) \leq f(b) + f(c)$ holds.

Metric-Preserving Functions

Some properties:

Lemma

For any metric space X and any metric-preserving function f, the following inequality holds:

$$2\,\mathrm{d_{GH}}\big(X, f(X)\big) \le ||\,\mathrm{id} - f\,||_A$$

where $A = \big[\mathrm{s}(X), \mathrm{diam}(X)\big]$ *if* $\mathrm{diam}(X) < \infty$, *and* $A = \big[\mathrm{s}(X), \infty\big)$ *otherwise.*

Lemma

Let $X, Y \in \mathcal{GH}$ *and* $\mathrm{d_{GH}}(X, Y) < \infty$, *and let* f *be a metric-preserving function. Then*

$$\mathrm{d_{GH}}(f(X), f(Y)) \le \liminf_{r \to \mathrm{d_{GH}}(X,Y)+} f(r).$$

Examples

1. $f(x) = x$
2. $f(x) = \begin{cases} x, x \leq 2 \\ \frac{x}{x-1}, x > 2 \end{cases}$
3. $f(x) = \begin{cases} x, x < 1 \\ \frac{1+x+\sin^2(x-1)}{2x}, x > 1 \end{cases}$
4. $l_\varepsilon(x) = \varepsilon \lceil x/\varepsilon \rceil$ (ε-ladder)

A Good Class

Let us take $\varepsilon = 2^{-n}$ for a positive integer n. Applying $I_{2^{-n}}$ to all metric spaces for all $n \in \mathbb{N}$, we obtain a certain subclass of the class $\mathcal{GH}$. We denote this class as D. Here are some properties of this class:

1. For any space X in D, there exists $n \in \mathbb{N}$ such that $\sigma(X) \in \left\{m/2^n \mid m \in \mathbb{N} \cup \{0\}\right\}$.
2. For any two spaces X and Y in this class, if $\mathrm{d_{GH}}(X, Y) < \infty$, then the set $\mathcal{R}_{opt}(X, Y)$ is nonempty.
3. For any metric space X and any $\varepsilon > 0$, it holds that $s(I_\varepsilon(X)) \geq \varepsilon$ and $e'(I_\varepsilon(X)) \geq \varepsilon$, since the minimal nonzero difference between numbers of the form $\varepsilon\lceil x/\varepsilon \rceil$ is ε.
4. The class D is dense everywhere in $\mathcal{GH}$.

0-Modification

Definition

A metric space Z is called a 0-***modification*** of the metric space X if $\mathrm{d}_{\mathrm{GH}}(Z, X) = 0$ and $X \neq Z$.

All metric spaces at zero Gromov–Hausdorff distance from each other are partitioned into equivalence classes such that there exists an optimal Hausdorff realization between any two elements from the same class, but not between elements from different classes. There is only one full metric space in every equivalence class. There is no optimal correspondence between elements from different classes.