Gromov–Hausdorff distance between clouds of special type

Calculating distance to cloud of bounded metric spaces

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Main question

- Are there any **isometric** clouds?

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- Are there clouds 0 distance from each other?

Cloud of bounded metric spaces

Cloud $[\Delta_1]$ — cloud of all **bounded** metric spaces.

Lemma (Ultrametric inequality)

For all bounded metric spaces X, Y
$|X, Y| \leq \frac{1}{2} \max\{\operatorname{diam} X, \operatorname{diam} Y\} = \max\{|X, \Delta_1|, |Y, \Delta_1|\}$

Geodesic:

$$|\lambda X, \mu X| = |\lambda - \mu||X, \Delta_1|$$

Cloud of bounded metric spaces

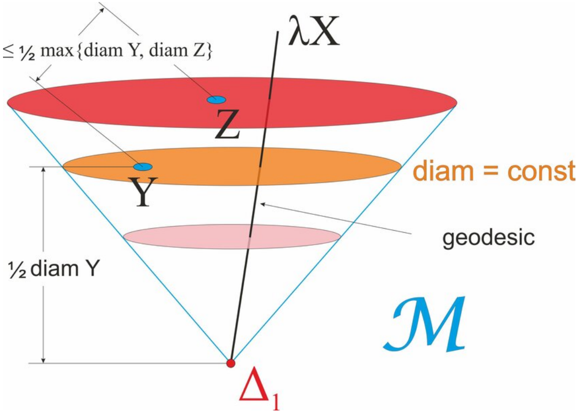

Figure: Cloud of bounded metric spaces has a cone shape

Stabilizer

Stabilizer: all $\lambda > 0$ such that $[X] = [\lambda X]$.

Definition

A stabilizer group is called **trivial** if it is equal to $\{1\}$

Definition

The **center** of a cloud is a metric space which doesn't change under Ψ_λ, $\lambda X = X$.

Proposition (S. Bogataya, S. Bogatyy, V. Redkozubov, A. Tuzhilin)

Each cloud with a non-trivial stabilizer group has a ***unique*** *center.*

Gromov–Hausdorff distance between clouds

Metric spaces are **sets** by definition. If we want to define Gromov–Hausdorff distance between clouds, we need to determine if they are sets too.

Theorem

If a cloud contains a metric space of cardinality $\aleph$, it contains metric spaces of all cardinalities greater than $\aleph$. So, all clouds are ***proper classes****.*

Distance and stabilizer

We examine what happens when two clouds have a nontrivial intersection of stabilizers. From the definition of the Gromov–Hausdorff distance, we can derive

$$\Big|[\lambda X],[\lambda Y]\Big| = \lambda\Big|[X],[Y]\Big|.$$

Suppose that $\lambda \in \mathrm{St}([X]) \cap \mathrm{St}([Y])$. Then,

$$\Big|[X],[Y]\Big| = \Big|[\lambda X],[\lambda Y]\Big| = \lambda\Big|[X],[Y]\Big|.$$

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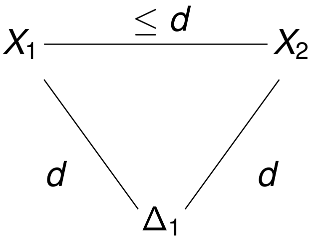
Lemma

If $\mathsf{St}([X]) \cap \mathsf{St}([Y]) \neq \{1\}$ *then* $\big|[X], [Y]\big|$ *equals* 0 *or* ∞

Principal idea

$[\Delta_1]$:



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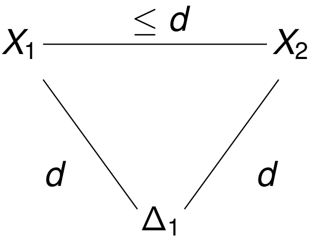
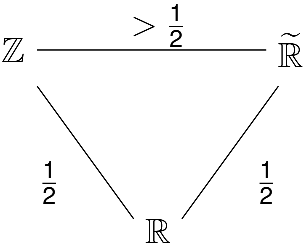


$[\mathbb{R}]$:



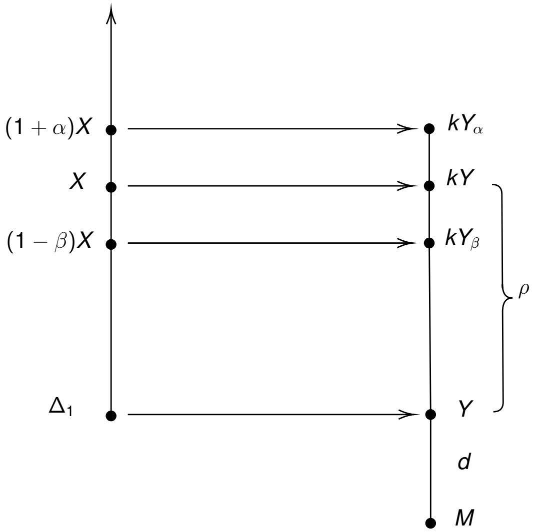
Center image Theorem

The following theorem gives insight about the stricture of correspondences with finite distortion.

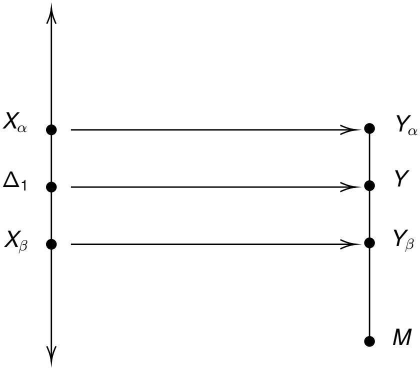
Theorem

Suppose that a cloud $[M]$ has a nontrivial stabilizer and M is its center. R is a correspondence between $[M]$ and $[\Delta_1]$, $\operatorname{dis} R = \epsilon < \infty$. Then $\left|R(\Delta_1), M\right| \leq 2\epsilon$.

Center image Theorem



Center image Theorem



Distance theorem

Earlier, it was shown that the distance between clouds with nontrivial stabilizers and the cloud of bounded metric spaces is either 0 or infinite. The following theorem shows that, under certain restrictions, this distance is infinite.

Theorem

*If a cloud has a nontrivial stabilizer, and contains two spaces which break the ultrametric inequality, then it's distance to $[\Delta_1]$ is **infinite**.*

Distance theorem

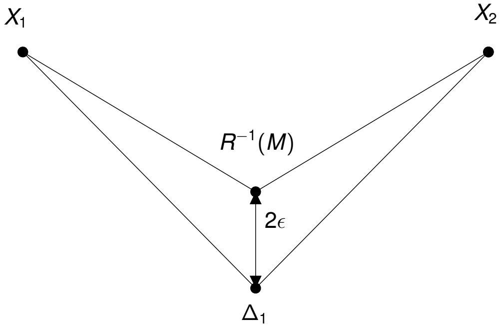
Space M is the center of $[M]$, Y_1 and Y_2 are such metric spaces that

$$|Y_1, M| \leq d,$$
$$|Y_2, M| \leq d,$$
$$|Y_1, Y_2| > d.$$

In the cloud $[\mathbb{R}]$, these inequalities are true for metric spaces $\mathbb{R}$, $\mathbb{Z}$ and the space composed of the real line and a point $(0, 1)$ which are considered a subset of $\mathbb{R}^2$ with the L_1 metric.

Distance Theorem

$$X_1 = R^{-1}(Y_1), X_2 = R^{-1}(Y_2)$$



Further problems

- Generalizing to $\left|[\mathbb{R}^n], [\Delta_1]\right|$
- Generalizing the notion of breaking the ultrametric inequality