Exercises to Chapter 3

Exercise 3.1. Let X be an arbitrary metric space and $\Omega(X)$ the family of all curves in X. Verify that

(1) if $\gamma \in \Omega(X)$ joins the points $x, y \in X$, then $|\gamma| \ge |xy|$;

(2) if $\gamma = \gamma_1 \cdot \gamma_2$ is the gluing of curves $\gamma_1, \gamma_2 \in \Omega(X)$ then $|\gamma| = |\gamma_1| + |\gamma_2|$;

(3) for each $\gamma \in \Omega(X)$, $\gamma\colon [a,b] \to X$, and reparametrization $\varphi\colon [c,d] \to [a,b]$, it holds $|\gamma| = |\gamma \circ \psi|$;

(4) for each $x \in X$, $\varepsilon > 0$, $y \in X \setminus U_\varepsilon(x)$ and the curve $\gamma \in \Omega(X)$ joining x and y, $|\gamma| \ge \varepsilon$ holds;

(5) is it true that for any $\gamma \in \Omega(X)$, $\gamma\colon [a,b] \to X$, the function $f(t) = \bigl|\gamma|_{[a,t]}\bigr|$ is continuous?

(6) is it true that for any sequence $\gamma_n \in \Omega(X)$ converging pointwise to some $\gamma \in \Omega(X)$, we have

$$|\gamma| \le \liminf_{n\to\infty} |\gamma_n|?$$

Exercise 3.2. Will the items (2) and (5) of Exercise 3.1 remain true if we change $\Omega_0(X)$ to $\Omega(X)$?

Exercise 3.3. Show that the piecewise smooth curve in $\mathbb{R}^n$ is Lipschitzian with a Lipschitz constant equal to the maximum modulus of the velocity vector of the curve, therefore each such curve is rectifiable.

Exercise 3.4. Let X be a metric space in which any two points are connected by a rectifiable curve.

(1) Prove that d_{in} is a metric.

(2) Denote by τ the metric topology of X w.r.t. the initial metric on X, by τ_{in} the metric topology w.r.t. d_{in}, by $X_\in$ the set X with metric d_{in} and topology τ_{in}. Show that $\tau \subset \tau_{in}$. In particular, if a mapping $\gamma\colon [a,b] \to X_{in}$ is continuous, then the mapping $\gamma\colon [a,b] \to X$ is continuous as well.

(3) Construct an example when $\tau \ne \tau_{in}$.

(4) Prove that for each rectifiable curve $\gamma\colon [a,b] \to X$ the mapping $\gamma\colon [a,b] \to X_{in}$ is continuous.

(5) Denote by $|\gamma|_{in}$ the length of a curve $\gamma\colon [a,b] \to X_{in}$. Show that for each curve $\gamma\colon [a,b] \to X$ which is also a curve in X_{in}, it holds $|\gamma| = |\gamma|_{in}$. Thus, the sets of rectifiable curves for X and X_{in} coincide, and each non-rectifiable curve in X is either a non-rectifiable one in X_{in}, or the mapping $\gamma\colon [a,b] \to X_{in}$ is discontinuous.

(6) Construct an example of continuous mapping $\gamma\colon [a,b] \to X$ such that the mapping $\gamma\colon [a,b] \to X_{in}$ is not continuous. Notice that the curve $\gamma\colon [a,b] \to X$ can not be rectifiable.

Exercise 3.5. Let X be a metric space in which any two points are connected by a rectifiable curve. Prove that the metric d_{in} is intrinsic.

Exercise 3.6. Let $\rho_1 \le \rho_2$ be generalized pseudometrics on a set X, and Y be a topological space. Prove that each mapping $f\colon Y \to X$, continuous w.r.t. ρ_2, is also continuous w.r.t. ρ_1, in particular, if γ is a curve in (X, ρ_2), then γ is also a curve in (X, ρ_1); moreover, if ρ_1' and ρ_2' denote the corresponding generalized intrinsic pseudometrics, then $\rho_1' \le \rho_2'$.

Exercise 3.7. Let X be an arbitrary set covered by a family $\{X_i\}_{i\in I}$ of generalized pseudometric spaces. Denote the distance function on X_i by ρ_i, and consider the set $\mathcal{D}$ of all generalized pseudometrics d on X such that for any i and $x, y \in X_i$ it holds $d(x,y) \le \rho_i(x,y)$. Extend each ρ_i to the whole X by setting $\rho_i'(x,y) = \infty$ if at least one of x, y does not belong to X_i, and $\rho_i'(x,y) = \rho(x,y)$ otherwise (it is easy to see that each ρ_i' is a generalized pseudometric). Denote by $\mathcal{D}'$ the set of all such ρ_i'. Prove that $\sup \mathcal{D} = \inf \mathcal{D}'$, and if all ρ_i are intrinsic, then $\sup \mathcal{D}$ is intrinsic as well.

Exercise 3.8. Let $\mathcal{D}$ be a collection of generalized pseudometrics defined on the same set X, and X_d for $d \in \mathcal{D}$ denote the generalized pseudometric space (X, d). Put $W = \sqcup_{d\in\mathcal{D}} X_d$ and denote by ρ the generalized pseudometric of W. Define on W an equivalence relation $\sim$ by identifying those points $x_d \in X_d$ and $x_{d'} \in X_{d'}$ which correspond to the same point x of the set X. The equivalence class of these points x_d and $x_{d'}$ we denote by $[x]$. Denote by $\rho_\sim$ the quotient generalized pseudometric on $W/\!\sim$. Define the mapping $\varphi\colon W/\!\sim\, \to X$ as $\varphi\colon [x] \to x$, then φ is bijective, and $\rho_\sim$ can be considered as a generalized pseudometric on X. Prove that $\rho_\sim = \inf \mathcal{D}$.

Exercise 3.9. Let ρ_1 and ρ_2 be intrinsic metrics on a set X. Suppose that these metrics generate the same topology, and that each $x \in X$ has a neighborhood U^x such that the restrictions of ρ_1 and ρ_2 to U^x coincide. Prove that $\rho_1 = \rho_2$. Show that the condition "ρ_1 and ρ_2 are intrinsic" is essential.

Exercise 3.10. Prove that a metric space X is locally compact if and only if for each point $x \in X$ there exists a neighborhood with compact closure.

Exercise 3.11. Show that a metric spaces is boundedly compact if and only if its compact subsets are exactly those subsets that are closed and bounded.

Exercise 3.12. Let X be an arbitrary set and Y an arbitrary metric space. Consider the collection of sets of the form $\prod_{x\in X} V(x) \subset \prod_{x\in X} Y$, where $\big\{V(x)\big\}_{x\in X}$ is the family of nonempty open subsets of Y such that for all $x \in X$, except for their finite number, $V(x) = Y$. Show that the family defined in this way forms a basis of a topology, and the convergence in this topology of points f_n to a point f is equivalent to pointwise convergence of the mappings f_n to the mapping f.

Exercise 3.13. Let X be an arbitrary set and Y an arbitrary metric space. A mapping $f\colon X \to Y$ is called *bounded* if its image $f(X)$ is a bounded subset of Y. The family of all bounded mappings from X to Y we denote by $\mathcal{B}(X, Y)$. We define the following distance function on $\mathcal{B}(X, Y)$: $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$. Prove that the distance function defined above is a metric, and that the convergence in this metric of a sequence $f_n \in \mathcal{B}(X, Y)$ to some $f \in \mathcal{B}(X, Y)$ is equivalent to uniform convergence of the mappings f_n to the mapping f.

Exercise 3.14. Let X be an arbitrary set and Y an arbitrary metric space. Define the following generalized distance function on Y^X: $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$. Prove that the generalized distance function defined above is a generalized metric, and that the convergence in this generalized metric of a sequence $f_n \in Y^X$ to some $f \in Y^X$ is equivalent to uniform convergence of the mappings f_n to the mapping f.

Exercise 3.15. Let γ be a curve in a metric space. Prove that

(1) nondegenerate γ can be reparameterized to an arc-length or, more generally, to a uniform one if and only if γ is rectifiable and non-stop;

(2) degenerate γ can be reparameterized to an arc-length one if and only if its domain is singleton (indeed, such γ is arc-length itself and, thus, it need not a reparametrization);

(3) degenerate γ is always uniform.

Exercise 3.16. Prove that a curve γ in a metric space can be monotonically reparameterized to an arc-length or, more generally, a uniform one if and only if γ is rectifiable.

The reparameterized curve is unique upto the choice of its domain and direction. In arc-length case one can choose any segment of the length $|\gamma|$. In the uniform case the domain can be arbitrary nondegenerate segment for nondegenerate γ, and arbitrary segment for degenerate γ.

Exercise 3.17. Prove that an arc-length curve $\gamma\colon [a, b] \to X$ in a space X with an intrinsic metric is shortest if and only if γ is an isometric embedding.

Exercise 3.18. Let Z be an everywhere dense subset of a metric space X, and $f\colon Z \to Y$ be some C-Lipschitz map into a complete metric space Y. Then there exists a unique continuous mapping $F\colon X \to Y$ extending f. Moreover, the mapping F is also C-Lipschitz.

Exercise 3.19. Show that in a space with an intrinsic (strictly intrinsic) metric, for any two points and any $\varepsilon > 0$ there is an ε-midpoint (a midpoint), respectively.