Exercises to Chapter 2

Exercise 2.1. Let X be a pseudometric space and $\sim$ is the natural equivalence relation: $x \sim y$ if and only if $|xy| = 0$. For each $x \in X$ denote by $[x]$ the equivalence class containing x. Prove that for any $x, y \in X$, $x' \in [x]$, and $y' \in [y]$ it is true that $|x'y'| = |xy|$. Thus, on the set $X/\!\sim$ the corresponding distance function is correctly defined: $\big|[x][y]\big| = |xy|$. Show that this distance function is a metric.

Exercise 2.2. Let X be an arbitrary metric space, $x, y \in X$, $r \geq 0$, $s, t > 0$, and $A \subset X$ be nonempty. Verify that

(1) $U_s\big(\{x\}\big) = U_s(x)$ and $B_r\big(\{x\}\big) = B_r(x)$;

(2) the functions $y \mapsto |xy|$, $y \mapsto |yA|$ are continuous;

(3) an open neighborhood $U_s(A)$ is an open subset of X, and a closed neighborhood $B_r(A)$ is a closed subset of X;

(4) $U_t\big(U_s(A)\big) \subset U_{s+t}(A)$ and construct an example demonstrating that the left-hand side can be different from the right-hand side;

(5) $B_t\big(B_s(A)\big) \subset B_{s+t}(A)$ and construct an example demonstrating that the left-hand side can be different from the right-hand side;

(6) $\partial U_s(x)$, $\partial B_s(x)$ are not related by any inclusion; $\partial U_s(x) \subset S_s(x)$ and $\partial B_r(x) \subset S_r(x)$; the both previous inclusions can be strict;

(7) $\operatorname{diam} U_s(x) \leq \operatorname{diam} B_s(x) \leq 2s$;

(8) $\operatorname{diam} U_s(A) \leq \operatorname{diam} B_s(A) \leq \operatorname{diam} A + 2s$.

Exercise 2.3. Let $\mathcal{L}(f) \subset \mathbb{R}$ be the set of all Lipschitz constants for a Lipschitz mapping f. Prove that $\inf \mathcal{L}(f)$ is also a Lipschitz constant.

Exercise 2.4. Show that each Lipschitz map is uniformly continuous, and each uniformly continuous map is continuous.

Exercise 2.5. Show that each isometry is a homeomorphism, in particular, each isometric mapping of one metric space into another one is an embedding, i.e., we recall, it is a homeomorphism with an image.

Exercise 2.6. Verify that the identity map, the composition of isometries, and the inverse mapping to an isometry are also isometries, i.e., the set of all isometries of an arbitrary metric space forms a group.

Exercise 2.7. Let X be an arbitrary metric space, $x, y \in X$ and $A \subset X$ be nonempty. Prove that $|Ax| + |xy| \geq |Ay|$, so that the function $\rho_A(x) = |Ax|$ is 1-Lipschitz and, therefore, uniformly continuous.

Exercise 2.8. Describe the Cayley graphs for the following groups G and generating sets S:

(1) $G = \mathbb{Z}$ and $S = \{1\}$;

(2) $G = \mathbb{Z}_m$ and $S = \{1\}$;

(3) $G = \mathbb{Z}^2$ and $S = \big\{(1,0)\,(0,1)\big\}$;

(4) $G = \mathbb{Z}^2$ and $S = \big\{(1,0)\,(0,1),\,(1,1)\big\}$;

(5) G is a free group with generators a and b.

Exercise 2.9. Let (X, ρ) be a metric (pseudometric) space, and $\sim$ an equivalence relation on X. Define the following *quotient distance function* on X:

$$\rho_\sim(x, y) = \inf\Big\{\sum_{i=0}^{n} \rho(p_i, q_i) : p_0 = x,\ q_n = y,\ n \in \mathbb{N},\ q_i \sim p_{i+1} \text{ for all } i\Big\}.$$

Prove that $\rho_\sim$ is a pseudometric on X.

Exercise 2.10. Let $\xi = (p_0 - q_0 \sim p_1 - q_1 \sim \cdots \sim p_n - q_n)$ be an irreducible admissible sequence. Prove that

(1) for any $i < j$ we have $p_i \not\sim p_j$, $q_i \not\sim q_j$;

(2) $q_i \sim p_j$ if and only if $j = i + 1$;

(3) for any i we have $q_i \neq p_{i+1}$.

Exercise 2.11. Let $\sim$ be the trivial equivalence on a metric space (X, ρ), i.e., $x \sim y$ if and only if $x = y$. Prove that $\rho_\sim = \rho$.

Exercise 2.12. Let $\sim$ be an equivalence on a pseudometric space (X, ρ). Prove that for any $x, y \in X$ it holds $\rho_\sim(x, y) \le \rho(x, y)$. Thus, if we define the function $b\colon X \times X \to \mathbb{R}$ such that $b(x, y) = 0$ for $x \sim y$, and $b(x, y) = \rho(x, y)$ otherwise, then $\rho_\sim \le b$.

Exercise 2.13. Let X be a generalized pseudometric space. We can define two equivalence relations: $x \sim_1 y$ if and only if $|xy| = \infty$, and $x \sim_2 y$ if and only if $|xy| = 0$. Prove that each class of equivalence $\sim_1$ is a pseudometric space (with finite distance), and that the distance between points from different classes equals ∞. Thus, if we denote by X_i the classes of equivalence $\sim_1$, then $X = \sqcup X_i$. Prove that the space $X/\!\sim_2$ equals the disjoint union $\sqcup(X_i/\!\sim_2)$ of metric spaces $X_i/\!\sim_2$.

Exercise 2.14. Let X and Y be metric spaces, $Z \subset X$, and $f\colon Z \to Y$ is an isometric mapping. Let ρ be the metric on the $X \sqcup_f Y$. Prove that the restrictions of ρ to X and Y coincides with the initial metrics of X and Y, respectively.

Exercise 2.15. Let $y_0 \in Y$ and $f(X) = y_0$. Prove that $X \sqcup_f Y$ is isometric to Y.

Exercise 2.16. Let X be a metric space and $G \subset \mathrm{Iso}(X)$ a subgroup of its isometry group. For each two elements $G(x), G(y) \in X/G$ we set $d\big(G(x), G(y)\big) = \inf\big\{|x'y'| : x' \in G(x),\, y' \in G(y)\big\}$. Prove that $d = \rho_\sim$, where the equivalence $\sim$ is generated by the action of G on X.

Exercise 2.17. Let S^1 be the standard unit circle in the Euclidean plane. As a distance between $x, y \in S^1$ we take the length of the shortest arc of S_1 between x and y. By *the standard torus* we mean the direct product $T^2 = S^1 \times S^1$ (with the Euclidean binder). We describe the points on the both S^1 by their polar angles φ_1 and φ_2, defined up to 2π. So, *the shifts* $s_{a,b}\colon (\varphi_1, \varphi_2) \mapsto (\varphi_1 + a, \varphi_2 + b)$ are isometries of T^2. Fix some $(a, b) \in \mathbb{R}^2$ and consider a subgroup $G_{a,b} \subset \mathrm{Iso}(T^2)$ consisting of all shifts $s_{ta,tb}$, $t \in \mathbb{R}$. For different a and b, find the corresponding pseudometric and metric quotient spaces.

Exercise 2.18. Represent the standard torus from Exercise 2.17 as a polyhedron space.

Exercise 2.19. Verify that the constructions given in Section 2.3 do define (pseudo-)metrics, as declared.

Exercise 2.20. Let d be a metric. Find the least possible c such that $d + c$ is a pseudometric. Verify that for such c and any $c' > c$ the function $d + c'$ is a metric.

Exercise 2.21. Show that a subspace of a complete metric space is complete if and only if it is closed.

Exercise 2.22. Let X be an arbitrary subspace of a complete metric space. Then the closure $\overline{X}$ of the set X is a completion of the space X.

Exercise 2.23. Let $f\colon X \to Y$ be a bi-Lipschitz mapping of metric spaces. Prove that X is complete if and only of Y is complete. Construct a homeomorphism of metric spaces that does not preserves completeness.

Exercise 2.24. Show that a metric space is complete if and only if the following condition holds: for any sequence of closed subsets $X_1 \supset X_2 \supset X_3 \supset \cdots$ such that $\operatorname{diam} X_n \to 0$ as $n \to 0$, the intersection $\cap_{i=1}^\infty X_i$ is not empty (in fact, it consists of unique element). Show that the condition $\operatorname{diam} X_n \to 0$ is essential.

Exercise 2.25 (Fixed-point theorem)**.** Let $f\colon X \to X$ be a C-Lipschitz mapping of a complete metric spaces X. Prove that for $C < 1$ there exists and unique a point x_0 such that $f(x_0) = x_0$ (it is called *the fixed point of the mapping* f).

Exercise 2.26. Give an example of a topological space that is

(1) compact, but not sequentially compact;

(2) sequentially compact, but not compact.

Exercise 2.27 (Lebesgue's lemma)**.** Let X be a compact metric space. Prove the following statement: for any open cover $\{U_i\}_{i\in I}$ of X there exists $\rho > 0$ such that for any $x \in X$ one can find U_i with $B_\rho(x) \subset U_i$.

Exercise 2.28. Show that each continuous mapping $f\colon X \to Y$ from a compact metric space to an arbitrary metric space is uniformly continuous.

Exercise 2.29. Prove that the diameter $\operatorname{diam} X$ of a compact metric space X if finite, and that there exist $x, y \in X$ such that $\operatorname{diam} X = |xy|$.

Exercise 2.30. Prove that every compact metric space is separable.

Exercise 2.31 (Baire's theorem)**.** A subset of a topological space is called *nowhere dense* if its closure has empty interior. Prove that a complete metric space cannot be covered by at most countably many nowhere dense subsets. Moreover, the complement of the union of at most countably many nowhere dense subsets is everywhere dense.

Exercise 2.32. Prove that a compact metric space X cannot be isometrically mapped to a subspace $Y \subset X$ such that $Y \neq X$. In other words, each isometric mapping $f\colon X \to X$ for a compact metric space X is surjective.

Exercise 2.33. Let X be a compact metric space and $f\colon X \to X$ be a mapping. Prove that

(1) if f is surjective and nonexpanding, then f is an isometry*;*

(2) if $\bigl|f(x)f(y)\bigr| \geq |xy|$ for all $x, y \in X$, then f is an isometry.