Exercises to Chapter 1

Exercise 1.1. Show that the family of all closed subsets of a topological space X contains $\emptyset$ and X, and that the intersection of any collection of closed subsets, as well as the union of any finite collection of closed subsets are some closed sets.

Exercise 1.2. Let X be a topological space and $Y \subset X$. Consider the family $\tau_Y := \{U \cap Y : U \in \tau_X\}$. Prove that τ_Y is a topology on Y.

Exercise 1.3. For a metric space (X, ρ) define $\tau_\rho \subset 2^X$ as the collection consisting of the empty set and all possible unions of open balls. Prove that the family τ_ρ is a topology.

Exercise 1.4. Prove that a family $\beta \subset 2^X$ is a base of some topology τ on X if and only if β is a cover of X, and for any intersecting $B_1, B_2 \in \beta$ their intersection $B_1 \cap B_2$ is the union of some elements from β. Moreover, each family satisfying these properties, generates a unique topology.

Exercise 1.5. Let some topologies τ_1 and τ_2 with bases β_1 and β_2 be given on a set X. Then $\tau_1 = \tau_2$ if and only iff for any $x \in X$ the following condition is fulfilled: for any $B_2 \in \beta_2$, $x \in B_2$, there is $B_1 \in \beta_1$ for which $x \in B_1 \subset B_2$, and vice versa, for any $B_1 \in \beta_1$, $x \in B_1$, there exists $B_2 \in \beta_2$ for which $x \in B_2 \subset B_1$. In particular, for a topological space X, a collection β of open sets satisfying the condition of Exercise 1.4 is a base of the topology τ_X if and only if for each open set $U \in \tau_X$ and any point $x \in U$ there exists some $B \in \beta$ such that $x \in B \subset U$.

Exercise 1.6. Let X be an infinite set of cardinality n, and m be an infinite cardinal number, with $m \le n$. Consider a family $\mathcal{F}$ of all $F \subset X$ such that $\#F < m$, and let $\beta_m = \{X \setminus F : F \in \mathcal{F}\}$. Prove that the family β_m is a base of some topology.

Exercise 1.7. Show that the collection of all possible half-intervals of the form $[a, b) \subset \mathbb{R}$ form a base of some topology that contains the standard topology.

Exercise 1.8. Prove that a family $\sigma \subset 2^X$ is a subbase of some topology on X if and only if σ is a cover of X. Moreover, each cover of X generates a unique topology.

Exercise 1.9. Show that the standard topology of the Euclidean space $\mathbb{R}^n$ coincides with the topology of the Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$ of real lines endowed with the standard topology.

Exercise 1.10. Let X be an arbitrary topological space. For each finite collection $U_1, \ldots, U_n \in \tau_X$ we put

$$\langle U_1, \ldots, U_n \rangle = \{Y \subset X : Y \subset \cup_{i=1}^n U_i, \text{ and } Y \cap U_i \ne \emptyset \text{ for all } i = 1, \ldots, n\}.$$

Show that the families

$$\sigma = \big\{\langle U \rangle : U \in \tau_X\big\} \cup \big\{\langle X, U \rangle : U \in \tau_X\big\} \quad \text{and} \quad \beta = \big\{\langle U_1, \ldots, U_n \rangle : U_1, \ldots, U_n \in \tau_X\big\}$$

form respectively a subbase and the corresponding base of some topology on 2^X.

Exercise 1.11. Prove that each construction from the section “Standard constructions of topologies” provides a topology.

Exercise 1.12. Prove that the definitions 1.18, 1.19, and 1.20 are equivalent.

Exercise 1.13. Let $f\colon X \to Y$ be a mapping of topological spaces and σ be a subbase of the topology on the space Y. Prove that f is continuous if and only if f-preimage of each element from the subbase σ is open in X.

Exercise 1.14. Show that the identity mapping and the composition of continuous mappings are continuous.

Exercise 1.15. Let X be a topological space, and $Z \subset X$ be its subspace. Show that the inclusion mapping $i\colon Z \to X$, $i(z) = z$ for each point $z \in Z$, is continuous.

Exercise 1.16. Let X, Y be topological spaces, $W \subset Y$ be a subspace of Y, and $f\colon X \to W$ be a continuous mapping. Let $g\colon X \to Y$ be a mapping coinciding with f: for each $x \in X$ it holds $f(x) = g(x)$. Prove that the mapping g is continuous.

Exercise 1.17. Let $f\colon X \to Y$ be a continuous mapping of topological spaces, $Z \subset X$, $W \subset Y$, $f(Z) \subset W$. Then the restriction $f|_{Z,W}\colon Z \to W$ is also continuous as the mapping of the topological spaces Z and W with the topologies induced on them from X and Y, respectively.

Exercise 1.18. Let $\{X_i\}_{i\in I}$ be a cover of a topological space X by open subsets X_i, and $f\colon X \to Y$ a mapping to a topological space Y. Show that f is continuous if and only if all the restrictions $f|_{X_i}$ are continuous. In particular, this holds when $X = \sqcup_{i\in I} X_i$ is the disjoint union of some topological spaces. Will this result remain true if we replace $\{X_i\}$ with a cover of X by arbitrary sets?

Exercise 1.19. Let $X = \sqcup_{i\in I} X_i$ be the disjoint union of some topological spaces and $f\colon X \to Y$ be a map into the topological space Y. Show that f is continuous if and only if all its restrictions $f|_{X_i}$ are continuous.

Exercise 1.20. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$. We define the canonical projection $\pi_i\colon X \to X_i$ by setting $\pi_i(x) = x_i$. Prove that the product topology on X is the smallest of those topologies in which all the projections π_i are continuous.

Exercise 1.21. Let $\{Y_i\}_{i\in I}$ be a family of topological spaces, $Y = \prod_{i\in I} Y_i$, and $f_i\colon X \to Y_i$ are mappings from some topological space X. We construct the mapping $F := \prod_{i\in I} f_i\colon X \to Y$ by associating with each point $x \in X$ the element $y \in Y$ defined as follows: $y_i = f_i(x)$. Prove that the mapping F is continuous if and only if all f_i are continuous.

Exercise 1.22. Let $A \subset \mathbb{R}^n$ be an arbitrary subset, $(x^1, \dots, x^n)$ the Cartesian coordinates on $\mathbb{R}^n$, $f\colon A \to \mathbb{R}^m$ a continuous mapping, $(y^1, \dots, y^m)$ the Cartesian coordinates on $\mathbb{R}^m$, and $y^i = y^i(x^1, \dots, x^n)$ the coordinate functions of the mapping f. Prove that the mapping f is continuous if and only if all the coordinate functions $y^i = y^i(x^1, \dots, x^n)$ are continuous.

Exercise 1.23. Describe all continuous functions on a topological space with Zariski topology.

Exercise 1.24. Let $f\colon X \to Y$ be a homeomorphism, and $Z \subset X$, $W = f(Z)$. Prove that the restriction $f|_{Z,W}\colon Z \to W$ is also a homeomorphism. Show that the letters b, c, f, g, i, h, o are pairwise non-homeomorphic.

Exercise 1.25. Show that every embedding is continuous. Give an example of a continuous injective mapping of topological spaces that is not an embedding.

Exercise 1.26. Let ω be a character not contained in $\mathbb{N}$. We define a topology on the set $\bar{N} = \mathbb{N} \cup \{\omega\}$, taking as a base all points from $\mathbb{N}$, as well as all sets of the form $\{n \geq N\} \cup \{\omega\}$, $N \in \mathbb{N}$. Show that a sequence $x\colon \mathbb{N} \to X$ converges if and only if the mapping x can be extended to a continuous mapping on $\bar{\mathbb{N}}$.

Exercise 1.27. Show that a continuous mapping $f\colon X \to Y$ of topological spaces takes convergent sequences to convergent ones. Show that if X is a metric space, then every mapping $g\colon X \to Y$ that takes convergent sequences into convergent ones is continuous. Give an example of a topological space X and a mapping $h\colon X \to Y$ into a topological space Y, which takes convergent sequences into convergent ones, but is not continuous nonetheless.

Exercise 1.28. Let $x_1, x_2, \dots$ be a sequence of points in a metric space X. Suppose that for some point $x \in X$ each neighborhood of x intersects the set $\{x_i\}_{i=1}^{\infty} \setminus \{x\}$. Prove that the sequence $x_1, x_2, \dots$ contains a convergent subsequence. Extract from this that if a sequence of points in a metric space does not contain any convergent subsequence, then for each $x \in X$ there exists $r > 0$ such that the open ball $U_r(x)$ does not contain points of this sequence other than x.

Exercise 1.29. Prove that the closure of a set $Y \subset X$ is the smallest closed subset of X containing Y, i.e., $\overline{Y}$ is the intersection of all closed sets containing Y.

Exercise 1.30. Let the topology of Zariski of weight m be given on an infinite set X. Then a subset $Y \subset X$ is everywhere dense in X if and only if $\#Y \geq m$.

Exercise 1.31. Show that in metric space, separability is equivalent to having a countable base. Extract from this that every subset of a separable metric space is separable. Show that an open subset of an arbitrary separable topological space is separable. Give an example of a separable topological space containing an non-separable subset (use the Sorgenfrey plane).

Exercise 1.32. Show that the interior $\operatorname{Int} Y$ is the largest open subset of X contained in Y.

Exercise 1.33. Prove that a subset Y of the topological space X is closed if and only if $Y = \overline{Y}$, and is open if and only if $Y = \operatorname{Int} Y$.

Exercise 1.34. Prove that the boundary ∂Y is a closed subset of X, and

$$\partial Y = \overline{Y} \setminus \operatorname{Int} Y = \overline{Y} \cap \overline{X \setminus Y}.$$

Exercise 1.35. Show that in a Hausdorff topological space every point is closed. Give an example of a non-Hausdorff topological space in which all points are closed.

Exercise 1.36. Show that the disjoint union and the Cartesian product of Hausdorff topological spaces are also Hausdorff.

Exercise 1.37. Show that in a Hausdorff topological space the limit of a convergent sequence is uniquely defined. Give an example of topological space in which each sequence converges to each point.

Exercise 1.38. Describe what sequences in a space with Zariski topology are convergent, and what limits each convergent sequence has.

Exercise 1.39. Prove that each segment $[a, b] \subset \mathbb{R}$ is connected.

Exercise 1.40. Prove that the closure of a connected subset of a topological space is connected.

Exercise 1.41. Let $\{A_i\}_{i\in I}$ be a family of connected pairwise intersecting subsets of a topological space X, then the set $\cup_{i\in I} A_i$ is connected.

Exercise 1.42. Show that the image of a connected topological space under a continuous mapping is also connected.

Exercise 1.43. Prove that every continuous function on a connected topological space takes all intermediate values.

Exercise 1.44. Show that each connected component is closed, and that each topological space is uniquely partitioned into its connected components. If such a partition is finite, then connected components are also open. Give an example of a topological space in which some connected components are not open.

Exercise 1.45. Prove that a path-connected topological space is connected. Give an example of a connected space that is not path-connected.

Exercise 1.46. Show that a finite union of compact subsets of a topological space is compact.

Exercise 1.47. Prove the following statements:

(1) the image under a continuous mapping from a compact topological space is compact;

(2) a closed subset of a compact topological space is compact;

(3) a compact subset of a Hausdorff topological space is closed;

(4) a continuous bijective mapping from a compact topological space to a Hausdorff space is a homeomorphism;

(5) give an example of an infinite topological space in which all subsets are compact. Note that in such a space there are compact subsets that are not closed;

(6) give an example of a continuous bijective mapping of topological spaces that is not a homeomorphism.

Exercise 1.48 (Alexander subbase theorem)**.** Let X be a topological space and σ its subbase. Prove that X is compact if and only if each cover of X by elements of the subbase σ has a finite subcover.

Exercise 1.49 (Tychonoff's theorem)**.** Prove that the Cartesian product $\prod_{i\in I} X_i$ of topological spaces X_i, endowed with Tychonoff topology, is compact if and only if all X_i are compact.

Exercise 1.50. Prove that each segment $[a, b] \subset \mathbb{R}$ is compact.

Exercise 1.51. Prove that a subset of a Euclidean space is compact if and only if it is closed and bounded.

Exercise 1.52. Prove that every compact metric space is bounded. Prove that a continuous function on a compact topological space is bounded and takes its largest and smallest values.

Exercise 1.53. Prove that every sequentially compact metric space is bounded. Prove that a continuous function on a sequentially compact topological space is bounded and takes its largest and smallest values.

Notation. The following matrix groups consist of real matrices of size $n \times n$ and are considered as subsets of $\mathbb{R}^{n^2}$ with the induced topology (their rows or columns are written out one after another and form vectors): $\mathrm{O}(n)$ consists of all orthogonal matrices (*orthogonal group*); $\mathrm{SO}(n)$ consists of all orthogonal matrices with determinant 1 (*special orthogonal group*); $\mathrm{GL}(n)$ consists of all non-degenerate matrices (*general linear group*); $\mathrm{SL}(n)$ consists of all matrices with determinant 1 (*special linear group*).

Exercise 1.54. Find out which of the following matrix groups are connected, which are compact:

$$\mathrm{O}(n),\ \mathrm{SO}(n),\ \mathrm{GL}(n),\ \mathrm{SL}(n).$$

Exercise 1.55. Let $X = \{a, b\}$. We define the following topology on X: $\tau = \big\{\emptyset, X, \{a\}\big\}$. Find out what the space $\mathrm{CL}(X)$ is.

Definition 1.33. A topological space is called *a space of class* T_0 if, for any two different points of this space, at least one of them has a neighborhood that does not contain the second point.

Exercise 1.56. Prove that the space $\mathrm{CL}(X)$ is always a space of class T_0.

Definition 1.34. A topological space is called *a space of class* T_1 if, for any two different points of this space, each of them has a neighborhood that does not contain the remaining point.

Exercise 1.57. Prove that if X is a space of class T_1, then $\mathrm{CL}(X)$ is also a space of class T_1. Give an example that demonstrates that the converse is not true.

Exercise 1.58. Prove that the space $\mathcal{P}_0(X)$ belongs to the class T_1 if and only if the space X is discrete.