Chapter 8

Calculating GH-distances to simplexes and some applications.

Schedule. GH-distance to simplexes with more points, GH-distance to simplexes with at most the same number of points, generalized Borsuk problem, solution of generalized Borsuk problem in terms of GH-distances, clique covering number and chromatic number of simple graphs, their dualities, calculating these numbers in terms of GH-distances.

By *simplex* we mean a metric space in which all non-zero distances equal to each other. If m is an arbitrary cardinal number, a simplex contain m points, and all its non-zero distances equal 1, then we denote this simplex by Δ_m. Thus, $\lambda\Delta_m$, $\lambda > 0$, is a simplex whose non-zero distances equal λ. Also, for arbitrary metric space X and $\lambda = 0$, the space λX coincides with Δ_1.

8.1 Gromov–Hausdorff distance to simplexes with more points

The next result generalizes Theorem 4.1 from [1].

Theorem 8.1. *Let X be an arbitrary metric space, $m > \#X$ a cardinal number, and $\lambda \geq 0$, then*

$$2d_{GH}(\lambda\Delta_m, X) = \max\{\lambda, \operatorname{diam} X - \lambda\}.$$

Proof. If X is unbounded, then $2d_{GH}(\lambda\Delta_m, X) = \infty$ by Example 6.29, and we get what is required.

Now, let $\operatorname{diam} X < \infty$.

If $\#X = 1$, then $\operatorname{diam} X = 0$, and, by Example 6.28, we have

$$2d_{GH}(\lambda\Delta, X) = \operatorname{diam} \lambda\Delta = \lambda = \max\{\lambda, \operatorname{diam} X - \lambda\}.$$

If $\lambda = 0$, then, by Example 6.28, we have

$$2d_{GH}(\Delta_1, X) = \operatorname{diam} X = \max\{\lambda, \operatorname{diam} X - \lambda\}.$$

Let $\#X > 1$ and $\lambda > 0$. Choose an arbitrary $R \in \mathcal{R}(\lambda\Delta_m, X)$. Since $\#X < m$ and $\lambda > 0$, then there exists $x \in X$ such that $\#R^{-1}(x) \geq 2$, thus, $\operatorname{dis} R \geq \lambda$ and $2d_{GH}(\lambda\Delta_m, X) \geq \lambda$.

Consider an arbitrary sequence $(x_i, y_i) \in X \times X$ such that $|x_i y_i| \to \operatorname{diam} X$. If it contains a subsequence (x_{i_k}, y_{i_k}) such that for each i_k there exists $z_k \in \lambda\Delta$, $(z_k, x_{i_k}) \in R$, $(z_k, y_{i_k}) \in R$, then $\operatorname{dis} R \geq \operatorname{diam} X$ and

$$2d_{GH}(\lambda\Delta_m, X) \geq \max\{\lambda, \operatorname{diam} X\} \geq \max\{\lambda, \operatorname{diam} X - \lambda\}.$$

If such subsequence does not exist, then there exists a subsequence (x_{i_k}, y_{i_k}) such that for any i_k there exist distinct $z_k, w_k \in \lambda\Delta_m$, $(z_k, x_{i_k}) \in R$, $(w_k, y_{i_k}) \in R$, and, therefore,

$$2d_{GH}(\lambda\Delta_m, X) \geq \max\big\{\lambda, |\operatorname{diam} X - \lambda|\big\} \geq \max\{\lambda, \operatorname{diam} X - \lambda\}.$$

Thus, in the both cases we have $2d_{GH}(\lambda\Delta, X) \geq \max\big\{\lambda, \operatorname{diam} X - \lambda\big\}$.

Choose an arbitrary $x_0 \in X$, then, by assumption, $\#X > 1$, and, thus, the set $X \setminus \{x_0\}$ is not empty. Since $\#X < m$, then $\lambda\Delta_m$ contains a subset $\lambda\Delta'$ of the same cardinality with $X \setminus \{x_0\}$. Let $g\colon \lambda\Delta' \to X \setminus \{x_0\}$ be an arbitrary bijection, and $\lambda\Delta'' = \lambda\Delta_m \setminus \lambda\Delta'$, then $\#\lambda\Delta'' > 1$. Consider the following correspondence:

$$R_0 = \Big\{\big(z', g(z')\big) : z' \in \lambda\Delta'\Big\} \cup \big(\lambda\Delta'' \times \{x_0\}\big).$$

Then we can apply Proposition 6.23, thus we have

$$\operatorname{dis} R_0 = \sup\{\lambda, |x_1x_1'| - \lambda, \lambda - |x_2x_2'| : x_1, x_1', x_2, x_2' \in X,\, x_1 \ne x_1',\, x_2 \ne x_2'\} = \max\{\lambda, \operatorname{diam} X - \lambda\},$$

therefore,

$$2d_{GH}(\lambda\Delta, X) = \max\{\lambda, \operatorname{diam} X - \lambda\},$$

what is required. □

8.2 Gromov–Hausdorff distance to simplexes with at most the same number of points

Let X be an arbitrary set different from singleton, $2 \le m \le \#X$ a cardinal number, and $\lambda > 0$. Under notations from Section 6.1, consider an arbitrary $D \in \mathcal{D}_m(X)$, any bijection $g\colon \lambda\Delta_m \to D$, and construct the correspondence $R_D \in \mathcal{R}(\lambda\Delta_m, X)$ in the following way:

$$R_D = \bigcup_{z \in \lambda\Delta_m} \{z\} \times g(z).$$

Clearly that each correspondence R_D is irreducible.

From Proposition 6.23 we get

Proposition 8.2. *Let $X \ne \Delta_1$ be an arbitrary metric space, $2 \le m \le \#X$ a cardinal number, and $\lambda > 0$. Then for any $D \in \mathcal{D}_m(X)$ it holds*

$$\operatorname{dis} R_D = \max\{\operatorname{diam} D,\, \lambda - \alpha(D),\, \beta(D) - \lambda\}.$$

Proof. If X is unbounded, then $\operatorname{dis} R = \infty$ for any $R \in \mathcal{R}(\lambda\Delta_m, X)$. Since $m \ge 2$, for any $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ we have either $\operatorname{diam} D = \infty$, or $\beta(D) = \infty$. Indeed, if $\operatorname{diam} D < \infty$ and $\beta(D) < \infty$ then for any $x, y \in X$ either $x, y \in X_i$, thus $|xy| \le \operatorname{diam} D$, or $x \in X_i$, $y \in X_j$, $i \ne j$, and $|xy| \le |X_iX_j| \le \beta(D)$, therefore X is bounded. Thus, for unbounded X the right-hand side of the considered equation is infinite as well, thus we get what is required.

Now, let $\operatorname{diam} X < \infty$. By Proposition 6.23, we have

$$\operatorname{dis} R_D = \sup\big\{\operatorname{diam} D,\, \lambda - |X_iX_j|,\, |X_iX_j|' - \lambda : i, j \in I,\, i \ne j\big\} = \max\{\operatorname{diam} D,\, \lambda - \alpha(D),\, \beta(D) - \lambda\},$$

that completes the proof. □

Corollary 8.3. *Let $X \ne \Delta_1$ be an arbitrary metric space, $2 \le m \le \#X$ a cardinal number, and $\lambda > 0$. Then for any $D \in \mathcal{D}_m(X)$ it holds*

$$\operatorname{dis} R_D = \max\{\operatorname{diam} D,\, \lambda - \alpha(D),\, \operatorname{diam} X - \lambda\}.$$

Proof. Again, for unbounded X the equation evidently holds.

Consider now the case of bounded X. Notice that $\operatorname{diam} D \le \operatorname{diam} X$ and $\beta(D) \le \operatorname{diam} X$. In addition, if $\operatorname{diam} D < \operatorname{diam} X$, and $(x_i, y_i) \in X \times X$ is a sequence such that $|x_iy_i| \to \operatorname{diam} X$, then, starting from some i, the points x_i and y_i belong to different elements of D, therefore, in this case we have $\beta(D) = \operatorname{diam} X$, and the formula is proved.

Now, let $\operatorname{diam} D = \operatorname{diam} X$, then $\beta(D) - \lambda \le \operatorname{diam} X$ and $\operatorname{diam} X - \lambda \le \operatorname{diam} X$, thus

$$\max\{\operatorname{diam} D,\, \lambda - \alpha(D),\, \beta(D) - \lambda\} = \max\{\operatorname{diam} X,\, \lambda - \alpha(D)\} = \max\{\operatorname{diam} D,\, \lambda - \alpha(D),\, \operatorname{diam} X - \lambda\},$$

that completes the proof. □

Proposition 8.4. *Let $X \ne \Delta_1$ be an arbitrary metric space, and $2 \le m \le \#X$ a cardinal number, and $\lambda > 0$. Then*

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D \in \mathcal{D}_m(X)} \operatorname{dis} R_D.$$

Proof. The case of unbounded X is trivial, so, let X be bounded. By Corollary 6.21,

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{R\in\mathcal{R}^0(\lambda\Delta_m,X)} \operatorname{dis} R,$$

thus it suffices to prove that for any irreducible correspondence $R \in \mathcal{R}^0(\lambda\Delta_m, X)$ there exists $D \in \mathcal{D}_m(X)$ such that $\operatorname{dis} R_D \le \operatorname{dis} R$.

Let us choose an arbitrary $R \in \mathcal{R}^0(\lambda\Delta_m, X)$ such that it cannot be represented in the form R_D, then the partition $D^R_{\lambda\Delta_m}$ is not pointwise, i.e., there exists $x \in X$ such that $\#R^{-1}(x) \ge 2$, therefore, $\operatorname{dis} R \ge \lambda$.

Define a metric on the set $D^R_{\lambda\Delta_m}$ to be equal λ between any its distinct elements, then this metric space is isometric to a simplex $\lambda\Delta'_n$, $n \le m$. The correspondence R generates naturally another correspondence $R' \in \mathcal{R}(\lambda\Delta'_n, X)$, namely, if $D^R_{\lambda\Delta_m} = \{\Delta_j\}_{j\in J}$, and $f_R\colon D^R_{\lambda\Delta_m} \to D^R_X$ is the bijection generated by R, then

$$R' = \bigcup_{j\in J} \{\Delta_j\} \times f_R(\Delta_j).$$

It is easy to see that $\operatorname{dis} R = \max\{\lambda, \operatorname{dis} R'\}$. Moreover, R' is generated by the partition $D' = D^R_X$, i.e., $R' = R_{D'}$, thus, by Corollary 8.3, we have

$$\operatorname{dis} R' = \max\{\operatorname{diam} D',\ \lambda - \alpha(D'),\ \operatorname{diam} X - \lambda\},$$

and hence,

$$\operatorname{dis} R = \max\{\lambda, \operatorname{diam} D',\ \lambda - \alpha(D'),\ \operatorname{diam} X - \lambda\} = \max\{\lambda,\ \operatorname{diam} D',\ \operatorname{diam} X - \lambda\}.$$

Since $n \le m$, the partition D' has a subpartition $D \in \mathcal{D}_m(X)$. Clearly, $\operatorname{diam} D \le \operatorname{diam} D'$, therefore,

$$\operatorname{dis} R_D = \max\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\} \le \max\{\operatorname{diam} D',\ \lambda,\ \operatorname{diam} X - \lambda\} = \operatorname{dis} R,$$

q.e.d. □

Considering separately the case $\lambda = 0$, we get the following

Corollary 8.5. *Let $X \ne \Delta_1$ be an arbitrary metric space, $2 \le m \le \#X$ a cardinal number, and $\lambda \ge 0$. Then*

$$2d_{GH}(\lambda\Delta_m, X) = \inf_{D\in\mathcal{D}_m(X)} \max\{\operatorname{diam} D,\ \lambda - \alpha(D),\ \operatorname{diam} X - \lambda\}.$$

For any metric space X put

$$\varepsilon(X) = \inf\big\{|xy| : x, y \in X,\ x \ne y\big\}.$$

Notice that $\varepsilon(X) \le \operatorname{diam} X$, and for a bounded X the equality holds, if and only if X is a simplex.

Corollary 8.5 immediately implies the following result that is proved in [1].

Theorem 8.6 ([1]). *Let $X \ne \Delta_1$ be a finite metric space, $m = \#X$, and $\lambda \ge 0$, then*

$$2d_{GH}(\lambda\Delta_m, X) = \max\big\{\lambda - \varepsilon(X),\ \operatorname{diam} X - \lambda\big\}.$$

8.3 Generalized Borsuk problem

Classical Borsuk Problem deals with partitions of subsets of Euclidean space into parts having smaller diameters. We generalize the Borsuk problem to arbitrary bounded metric spaces and partitions of arbitrary cardinality. Let X be a bounded metric space, m a cardinal number such that $2 \le m \le \#X$, and $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$. We say that D is a partition into subsets having *strictly smaller diameters*, if there exists $\varepsilon > 0$ such that $\operatorname{diam} X_i \le \operatorname{diam} X - \varepsilon$ for all $i \in I$.

By *Generalized Borsuk problem* we call the following one: Is it possible to partition a bounded metric space X into a given, probably infinite, number of subsets, each of which has a strictly smaller diameter than X?

We give a solution to the Generalized Borsuk problem in terms of the Gromov–Hausdorff distance.

Theorem 8.7. *Let X be an arbitrary bounded metric space and m a cardinal number such that $2 \le m \le \#X$. Choose an arbitrary number $0 < \lambda < \operatorname{diam} X$, then X can be partitioned into m subsets having strictly smaller diameters if and only if $2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X$.*

Proof. For the λ chosen, due Corollary 8.5, we have $2d_{GH}(\lambda\Delta_m, X) \le \operatorname{diam} X$, and the equality holds if and only if for each $D \in \mathcal{D}_m(X)$ we have $\operatorname{diam} D = \operatorname{diam} X$. The latter means that there is no partition of the space X into m parts having strictly smaller diameters. ☐

Corollary 8.8. *Let $d > 0$ be a real number, and $m \le n$ cardinal numbers. By $\mathcal{M}_n$ we denote the set of isometry classes of bounded metric spaces of cardinality at most n, endowed with the Gromov–Hausdorff distance. Choose an arbitrary $0 < \lambda < d$. Then the intersection*

$$S_{d/2}(\Delta_1) \cap S_{d/2}(\lambda\Delta_m)$$

of the spheres, considered as the spheres in $\mathcal{M}_n$, does not contain spaces, whose cardinality is less than m, and consists exactly of all metric spaces from $\mathcal{M}_n$, whose diameters are equal to d and that cannot be partitioned into m subsets of strictly smaller diameters.

Proof. Let X belong to the intersection of the spheres, then $\operatorname{diam} X = d$ in accordance with Example 6.28. If $m > \#X$, then, due to Theorem 8.1, we have

$$2d_{GH}(\lambda\Delta, X) = \max\{\lambda, \operatorname{diam} X - \lambda\} < d,$$

therefore $X \notin S_{d/2}(\lambda\Delta_m)$, that proves the first statement of Corollary.

Now let $m \le \#X$. Since $\operatorname{diam} X = d$ and $2d_{GH}(\lambda\Delta_m, X) = d$, then, due to Theorem 8.7, the space X cannot be partitioned into m subsets of strictly smaller diameters.

Conversely, each X of the diameter d, such that $m \le \#X$ and which cannot be partitioned into m subsets of strictly smaller diameter, lies in the intersection of the spheres by Theorem 8.7. ☐

8.4 Calculating clique covering and chromatic numbers of a graph

Recall that a subgraph of an arbitrary simple graph G is called *a clique*, if any its two vertices are connected by an edge, i.e., the clique is a subgraph which is a complete graph itself. Notice that each single-vertex subgraph is also a clique. For convenience, the vertex set of a clique is also referred as *a clique*.

On the set of all cliques, an ordering with respect to inclusion is naturally defined, and hence, due to the above remarks, a family of maximal cliques is uniquely defined; this family forms *a cover of the graph* G in the following sense: the union of all vertex sets of all maximal cliques coincides with the vertex set $V(G)$ of the graph G.

If one does not restrict himself by maximal cliques, then, generally speaking, one can find other families of cliques covering the graph G. One of the classical problems of the Graph Theory is to calculate the minimal possible number of cliques covering a finite simple graph G. This number is referred as *the clique covering number* and is often denoted by $\theta(G)$. It is easy to see that the value $\theta(G)$ is also equal to the least number of cliques whose vertex sets form a partition of $V(G)$.

Another popular problem is to find the least possible number of colors that is necessary to color the vertices of a simple finite graph G in such a way that adjacent vertices have different colors. This number is denoted by $\gamma(G)$ and is referred as *the chromatic number of the graph* G.

For a simple graph G, by G' we denote its *dual graph*, i.e., the graph with the same vertex set and the complementary set of edges (two vertices of G' are adjacent if and only if they are not adjacent in G).

Problem 8.1. For any simple finite graph G it holds $\theta(G) = \gamma(G')$.

Let $G = (V, E)$ be an arbitrary finite graph. Fix two real numbers $a < b \le 2a$ and define a metric on V as follows: the distance between adjacent vertices equals a, and nonadjacent vertices equals b. Then a subset $V' \subset V$ has diameter a if and only if $G(V') \subset G$ is a clique. This implies that each clique covering number equals to the least cardinality of partitions of the metric space V onto subsets of (strictly) smaller diameter. However, this number was calculated in Theorem 8.7. Thus, we get the following

Corollary 8.9. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two real numbers $a < b \le 2a$ and define a metric on V as follows: the distance between adjacent vertices equals a, and nonadjacent vertices equals b. Let m be the greatest positive integer k such that $2d_{GH}(a\Delta_k, V) = b$ (in the case when there is no such k, we put $m = 0$). Then $\theta(G) = m + 1$.*

Problem 8.2. Consider simple finite graphs $G = (V, E)$ for which the clique covering numbers $\theta(G)$ are known, and get the Gromov–Hausdorff distances between the corresponding metric spaces V and simplexes $\lambda\Delta_m$ with $m \le \theta(G)$. Verify explicitly that for $k > m$ these distances are less than $\operatorname{diam} V$.

Because of the duality between clique and chromatic numbers, we get

Corollary 8.10. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two real numbers $a < b \le 2a$ and define a metric on V as follows: the distance between adjacent vertices equals b, and nonadjacent vertices equals a. Let m be the greatest positive integer k such that $2d_{GH}(a\Delta_k, V) = b$ (in the case when there is no such k, we put $m = 0$). Then $\gamma(G) = m + 1$.*

Problem 8.3. Consider simple finite graphs $G = (V, E)$ for which the chromatic numbers $\gamma(G)$ are known, and get the Gromov–Hausdorff distances between the corresponding metric spaces V and simplexes $\lambda\Delta_m$ with $m \le \gamma(G)$. Verify explicitly that for $k > m$ this distances are less than $\operatorname{diam} V$.

References to Chapter 8

[1] A.O.Ivanov, S.Iliadis, and A.A.Tuzhilin, *Geometry of Compact Metric Space in Terms of Gromov-Hausdorff Distances to Regular Simplexes*. ArXiv e-prints, arXiv:1607.06655, 2016.

Exercises to Chapter 8

Exercise 8.1. For any simple finite graph G it holds $\theta(G) = \gamma(G')$.

Exercise 8.2. Consider simple finite graphs $G = (V, E)$ for which the clique covering numbers $\theta(G)$ are known, and get the Gromov–Hausdorff distances between the corresponding metric spaces V and simplexes $\lambda\Delta_m$ with $m \le \theta(G)$. Verify explicitly that for $k > m$ these distances are less than $\operatorname{diam} V$.

Exercise 8.3. Consider simple finite graphs $G = (V, E)$ for which the chromatic numbers $\gamma(G)$ are known, and get the Gromov–Hausdorff distances between the corresponding metric spaces V and simplexes $\lambda\Delta_m$ with $m \le \gamma(G)$. Verify explicitly that for $k > m$ this distances are less than $\operatorname{diam} V$.