Chapter 7

Gromov–Hausdorff space.

Schedule. Gromov–Hausdorff space (GH-space), distortions of a correspondence and its closure, calculating GH-distance in terms of closed correspondences, compactness of the set of all closed correspondences for compact metric spaces, continuity of distortion for compact metric spaces, optimal correspondences, existence of closed optimal correspondences for compact metric spaces, GH-space is geodesic, cover number and packing number, there relations, total boundness of families of compact metric spaces in terms of cover and packing numbers, isometric embedding of all compact metric spaces from a totally bounded family to the same compact subset of ℓ_∞, completeness of GH-space, separability of GH-space, mst-spectrum in terms of GH-distances to simplexes, Steiner problem in GH-space.

This section describes some geometrical and topological properties of the space consisting the isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff metric.

We denote by $\mathcal{M}$ the set of all compact metric spaces considered up to isometry (in other words, the set of isometry classes of compact metric spaces). From Propositions 6.6 and 6.7 it follows that the distance d_{GH} is a metric on $\mathcal{M}$. The metric space $(\mathcal{M}, d_{GH})$ is called *the Gromov–Hausdorff space*. Recall that by Δ_1 we denoted a one-point metric space. Note that $\mathcal{M}$ contains all finite metric spaces, in particular, $\Delta_1 \in \mathcal{M}$. The results presented in Examples 6.28–6.32 lead to the following geometric model of $\mathcal{M}$, see Figure 7.1.

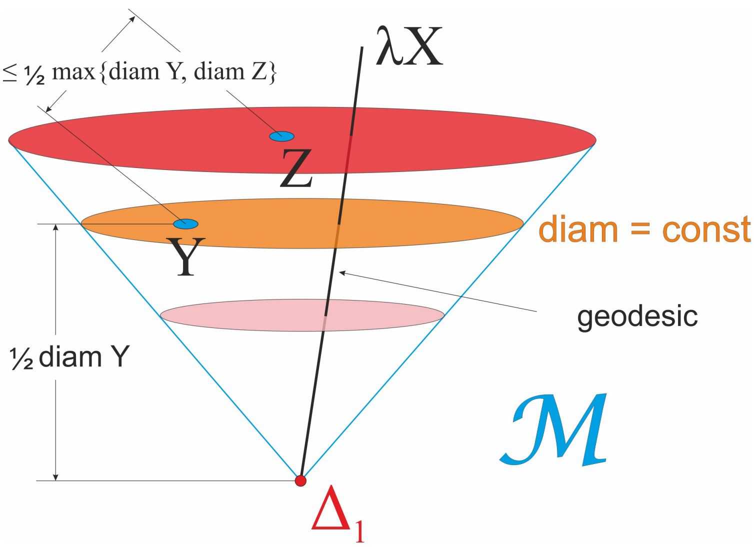

Figure 7.1: Gromov–Hausdorff space: general properties.

Indeed, according to Example 6.32, the operation of multiplying the metric by a number $\lambda > 0$ is a homothety of the space $\mathcal{M}$, centered in the one-point metric space Δ_1, so $\mathcal{M}$ is a cone with the vertex Δ_1. By Example 6.31, the curves $\gamma(t) := t\,X$, $X \in \mathcal{M}$, $X \neq \Delta_1$, are shortest between any of their points, so they are the generators of the cone. Figure 7.1 also illustrates Examples 6.28 and 6.30.

7.1 Existence of optimal correspondences between compact metric spaces

Let X and Y be arbitrary metric spaces.

Agreement 7.1. In what follows, when we work with the metric space $X\times Y$, we always suppose that its distance function is

$$\big|(x,y)(x',y')\big| = \max\big\{|xx'|,\,|yy'|\big\},$$

and just this metric generates the Hausdorff distance on $\mathcal{P}_0(X\times Y)$ and on all its subspaces, e.g., on $\mathcal{R}(X,Y)$.

Problem 7.1. Prove that the product topology on $X\times Y$ coincides with the one generated by the metric from Agreement 7.1.

Proposition 7.2. *For arbitrary metric spaces X and Y, if $\bar\sigma$ is the closure of $\sigma\in\mathcal{P}_0(X\times Y)$, then $\operatorname{dis}\bar\sigma=\operatorname{dis}\sigma$.*

Proof. Since $\sigma\subset\bar\sigma$ then we have $\operatorname{dis}\sigma\le\operatorname{dis}\bar\sigma$, hence in the case $\operatorname{dis}\sigma=\infty$ the result holds.

Now let $\operatorname{dis}\sigma<\infty$. It remains to prove that $\operatorname{dis}\sigma\ge\operatorname{dis}\bar\sigma$.

By definition, for each $\varepsilon>0$ and any $(\bar x,\bar y)$, $(\bar x',\bar y')\in\bar\sigma$ there exist (x,y), $(x',y')\in\sigma$ such that $\max\big\{|\bar xx|,|\bar yy|\big\}<\varepsilon/6$ and $\max\big\{|\bar x'x'|,|\bar y'y'|\big\}<\varepsilon/6$, thus $\big||\bar x\bar x'|-|xx'|\big|<\varepsilon/3$ and $\big||\bar y\bar y'|-|yy'|\big|<\varepsilon/3$. Therefore,

$$\big||\bar x\bar x'|-|\bar y\bar y'|\big|<\big||xx'|-|yy'|\big|+2\varepsilon/3\le\operatorname{dis}\sigma+2\varepsilon/3.$$

Passing to the supremum in definition of $\operatorname{dis}\bar\sigma$, we conclude that $\operatorname{dis}\bar\sigma\le\operatorname{dis}\sigma+2\varepsilon/3$. Since ε is arbitrary, we have $\operatorname{dis}\bar\sigma\le\operatorname{dis}\sigma$. □

Denote by $\mathcal{R}_c(X,Y)$ the subset of $\mathcal{R}(X,Y)$ consisting of all closed correspondences R. Clearly that for any $R\in\mathcal{R}(X,Y)$ its closure $\bar R$ is also a correspondence, i.e., $\bar R\in\mathcal{R}_c(X,Y)$. This fact, together with Proposition 7.2 and Theorem 6.12, immediately implies

Corollary 7.3. *For any metric spaces X and Y we have*

$$d_{GH}(X,Y)=\frac12\inf\big\{\operatorname{dis}R:R\in\mathcal{R}_c(X,Y)\big\}.$$

Now, let X and Y be compact metric spaces. Then $X\times Y$ is compact as well, and $\mathcal{R}_c(X,Y)\subset\mathcal{H}(X\times Y)$. By Corollary 5.37, $\mathcal{H}(X\times Y)$ is compact.

Proposition 7.4. *For X, $Y\in\mathcal{M}$ the set $\mathcal{R}_c(X,Y)$ is closed in $\mathcal{H}(X\times Y)$, thus, $\mathcal{R}_c(X,Y)$ is compact.*

Proof. It suffices to show that for each $\sigma\in\mathcal{H}(X\times Y)\setminus\mathcal{R}_c(X,Y)$ some its neighborhood in $\mathcal{H}(X\times Y)$ does not intersect $\mathcal{R}_c(X,Y)$. Notice that $\sigma\notin\mathcal{R}(X,Y)$ because $\mathcal{H}(X\times Y)$ consists of all closed nonempty subsets of $X\times Y$, and $\mathcal{R}_c(X,Y)$ equals to the set of all closed $R\in\mathcal{R}(X,Y)$. Then either $\pi_X(\sigma)\ne X$, or $\pi_Y(\sigma)\ne Y$, where π_X and π_Y are the canonical projections. To be definite, suppose that the first condition holds, i.e., there exists $x\in X\setminus\pi_X(\sigma)$. Since σ is a closed subset of the compact $X\times Y$, then it is compact itself, and therefore $\pi_X(\sigma)$ is compact in X, thus, $\pi_X(\sigma)$ is closed. The latter implies that there exists an open ball $U_\varepsilon(x)$ such that $U_\varepsilon(x)\cap\pi_X(\sigma)=\emptyset$. Let $U=U_{\varepsilon/2}^{\mathcal{H}(X\times Y)}(\sigma)$, then for each $\sigma'\in U$ we have $d_H(\sigma,\sigma')<\varepsilon/2$, thus for any $(a',b')\in\sigma'$ there exists $(a,b)\in\sigma$ such that $\big|(a,b),(a',b')\big|<\varepsilon/2$. Since $a\in\pi_X(\sigma)$ then $|xa|\ge\varepsilon$. On the other hand, $|aa'|\le\big|(a,b),(a',b')\big|<\varepsilon/2$, therefore, $|xa'|>\varepsilon/2$, i.e., $a'\notin U_{\varepsilon/2}(x)$ and hence $\pi_X(\sigma')\cap U_{\varepsilon/2}(x)=\emptyset$. This implies that σ' is not a correspondence, thus $\sigma'\notin\mathcal{R}_c(X,Y)$ and $U\cap\mathcal{R}_c(X,Y)=\emptyset$. □

Define a function $f\colon(X\times Y)\times(X\times Y)\to\mathbb{R}$ as $f(x,y,x',y')=\big||xx'|-|yy'|\big|$. Clearly that f is continuous. Notice that for each $\sigma\in\mathcal{P}_0(X\times Y)$ we have

$$\operatorname{dis}\sigma=\sup\big\{f(x,y,x',y'):(x,y),\,(x',y')\in\sigma\big\}=\sup f|_{\sigma\times\sigma}.$$

Proposition 7.5. *If X, $Y\in\mathcal{M}$, then the function $\operatorname{dis}\colon\mathcal{H}(X\times Y)\to\mathbb{R}$ is continuous.*

Proof. Since $(X\times Y)\times(X\times Y)$ is compact, then the function f is uniformly continuous, i.e., for any $\varepsilon>0$ there exists $\delta>0$ such that for every (x_1,y_1,x_1',y_1') and (x_2,y_2,x_2',y_2') with $\max\big\{|x_1x_2|,|y_1,y_2|,|x_1'x_2'|,|y_1'y_2'|\big\}<\delta$ it holds

$$\big|f(x_1,y_1,x_1',y_1')-f(x_2,y_2,x_2',y_2')\big|<\varepsilon.$$

Thus, for any $\sigma \in \mathcal{H}(X \times Y)$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for the open δ-neighborhood $U = U_\delta^{X\times Y}(\sigma) \subset X \times Y$ of σ it holds

$$\sup f|_{U\times U} \le \sup f|_{\sigma\times\sigma} + \varepsilon.$$

By V we denote the open ball $U_\delta^{\mathcal{H}(X\times Y)}(\sigma) \subset \mathcal{H}(X \times Y)$ of radius δ centered at σ. Since for any $\sigma' \in V$ we have $\sigma' \subset U$, then it follows that

$$\operatorname{dis}\sigma' = \sup f|_{\sigma'\times\sigma'} \le \sup f|_{U\times U} \le \sup f|_{\sigma\times\sigma} + \varepsilon = \operatorname{dis}\sigma + \varepsilon.$$

Swapping σ and σ', we get $|\operatorname{dis}\sigma - \operatorname{dis}\sigma'| \le \varepsilon$, and hence, the function dis is continuous. □

Definition 7.6. A correspondence $R \in \mathcal{R}(X,Y)$ is called *optimal* if $d_{GH}(X,Y) = \frac{1}{2}\operatorname{dis} R$. By $\mathcal{R}_{\rm opt}(X,Y)$ we denote the set of all optimal correspondences between X and Y.

Theorem 7.7. *For any X, $Y \in \mathcal{M}$ we have $\mathcal{R}_{\rm opt}(X,Y) \cap \mathcal{R}_c(X,Y) \ne \emptyset$.*

Proof. By Proposition 7.5, the function $\operatorname{dis}\colon \mathcal{R}_c(X,Y) \to \mathbb{R}$ is continuous, and by Proposition 7.4 the space $\mathcal{R}_c(X,Y)$ is compact, thus the function dis attains its least value. The half of this value equals $d_{GH}(X,Y)$, and each correspondence R which this least value is attained at is optimal. □

From Theorem 7.7 and Propositions 6.11 and 6.11, we immediately get

Corollary 7.8. *For each nonisometric X, $Y \in \mathcal{M}$ there exist*

1. *a correspondence $R \in \mathcal{R}_c(X,Y)$ such that $d_{GH}(X,Y) = \frac{1}{2}\operatorname{dis} R$;*
2. *an admissible metric $\rho \in \mathcal{D}(X,Y)$ such that $d_{GH}(X,Y) = \rho_H(X,Y)$;*
3. *a metric space Z and isometric embeddings of X and Y into Z such that the Hausdorff distance between their images equals $d_{GH}(X,Y)$.*

7.2 The Gromov–Hausdorff space is geodesic

Let X and Y be arbitrary metric spaces. We assume that the space $X \times Y$ is by default endowed with the metric

$$\big|(x,y)(x',y')\big| = \max\big\{|xx'|, |yy'|\big\}.$$

Also, we introduce on $X \times Y$ the following 1-parametric family of metrics d_t, $t \in (0,1)$:

$$d_t\big((x,y),(x',y')\big) = (1-t)|xx'| + t|yy'|.$$

Problem 7.2. Prove that the topologies generated by all d_t, and by $|\cdot|$ as well, coincide with the product topology of $X \times Y$.

Now we suppose that $d_{GH}(X,Y) < \infty$, and $R \in \mathcal{R}(X,Y)$ is any correspondence with $\operatorname{dis} R < \infty$. By R_t, $t \in [0,1]$, we denote the metric space (R, d_t) if $t \in (0,1)$, and we put $R_0 = X$, $R_1 = Y$.

Proposition 7.9. *The mapping $t \mapsto R_t$ is Lipschitzian with the Lipschitz constant $\frac{1}{2}\operatorname{dis} R$ (w.r.t. the Gromov–Hausdorff distance).*

Proof. We have to show that $2d_{GH}(R_s,R_t) \le |s-t|\operatorname{dis} R$ for arbitrary $s,t \in [0,1]$. Let us start with the case $0 < s,t < 1$. As always, we denote by id the identical mapping. Then

$$2d_{GH}(R_s,R_t) \le \operatorname{dis}\operatorname{id} = \sup\Big\{\Big|d_s\big((x,y),(x',y')\big) - d_t\big((x,y),(x',y')\big)\Big| : (x,y),(x',y') \in R\Big\} =$$

$$= |s-t| \sup\Big\{\big||xx'| - |yy'|\big| : (x,y),(x',y') \in R\Big\} = |s-t|\operatorname{dis} R.$$

Now we consider the case $s = 0$, $0 < t < 1$. As a correspondence we take $S_t \in \mathcal{R}(R_0,R_t)$ of the form

$$S_t = \bigcup_{(x,y)\in R} \Big\{\big(x,(x,y)\big)\Big\}.$$

Notice that such S_t does not depend on t as a set. We have

$$2d_{GH}(R_0,R_t)\le \operatorname{dis} S_t=\sup\Big\{\Big||xx'|-d_t\big((x,y),(x',y')\big)\Big|:\big(x,(x,y)\big),\big(x',(x',y')\big)\in S_t\Big\}=$$
$$=t\sup\Big\{\big||xx'|-|yy'|\big|:\big(x,(x,y)\big),\big(x',(x',y')\big)\in S_t\Big\}=t\operatorname{dis}R=|s-t|\operatorname{dis}R.$$

All remaining cases can be proved similarly. □

Now, let X and Y be compact metric spaces, and $R\in\mathcal{R}_c(X,Y)$. Then R is compact as well, and by Problem 7.2, all metric spaces R_t are compact too. Thus, by Proposition 7.9, the mapping $t\mapsto R_t$ is a Lipschitz curve in $\mathcal{M}$, and it joins X and Y.

Theorem 7.10. *Given $X,Y\in\mathcal{M}$ and $R\in\mathcal{R}_{\rm opt}(X,Y)\cap\mathcal{R}_c(X,Y)$, the curve $\gamma\colon[0,1]\to\mathcal{M}$, $\gamma\colon t\mapsto R_t$, is a shortest geodesic with the speed $d_{GH}(X,Y)$. In particular, the length of γ equals to $d_{GH}(X,Y)$, thus, the space $\mathcal{M}$ is geodesic.*

Proof. Choose arbitrary $0\le s\le t\le 1$, then, by Proposition 7.9, we have

$$d_{GH}(R_s,R_t)\le\frac{t-s}{2}\operatorname{dis}R=(t-s)\,d_{GH}(X,Y),\qquad(7.1)$$

thus, taking into account the triangle inequality, we get

$$d_{GH}(X,Y)\le d_{GH}(X,R_s)+d_{GH}(R_s,R_t)+d_{GH}(R_t,Y)\le d_{GH}(X,Y),$$

therefore, $d_{GH}(X,R_s)+d_{GH}(R_s,R_t)+d_{GH}(R_t,Y)=d_{GH}(X,Y)$. If for some s and t the inequality in Formula (7.1) is strict, then the previous equality is not satisfied, thus $d_{GH}(R_s,R_t)=(t-s)\,d_{GH}(X,Y)$. It remains to note that

$$|\gamma|=\sup_{0=t_0<\cdots<t_n=1}\sum_{i=1}^n d_{GH}(R_{t_{i-1}},R_{t_i})=d_{GH}(X,Y),$$

hence γ is shortest. □

7.3 Cover number and packing number

Let X be an arbitrary metric space and $\varepsilon>0$. The numerical characteristics of the pair (X,ε) defined below will be used by us in the study of totally bounded families of compact metric spaces, in particular, in terms of these numbers, the Gromov criterion for the precompactness of a family of compact metric spaces will be formulated.

Definition 7.11. *The cover number*

$$\operatorname{cov}(X,\varepsilon)=\inf\Big\{n\in\mathbb{N}:\exists\,x_1,\dots,x_n\in X,\quad X=\bigcup_{i=1}^n U_\varepsilon(x_i)\Big\}$$

(as usually, we put $\inf\emptyset=\infty$). In other words, the cover number is the minimum number of open balls of radius ε that cover the space X.

The packing number

$$\operatorname{pack}(X,\varepsilon)=\sup\big\{n\in\mathbb{N}:\exists\,x_1,\dots,x_n\in X\,\forall\,i\ne j\quad U_{\varepsilon/2}(x_i)\cap U_{\varepsilon/2}(x_j)=\emptyset\big\}.$$

In other words, the packing number is the maximum number of open pairwise disjoint balls of radius $\varepsilon/2$ in the space X.

Problem 7.3. Prove that

(1) a metric space X is bounded if only for some $\varepsilon>0$ it holds $\operatorname{cov}(X,\varepsilon)<\infty$ (similarly, $\operatorname{pack}(X,\varepsilon)<\infty$);

(2) a metric space X is finite if and only if there exists n such that $\operatorname{cov}(X,\varepsilon)\le n$ for all $\varepsilon>0$ (similarly, for $\operatorname{pack}(X,\varepsilon)$);

(3) the functions $f(\varepsilon)=\operatorname{cov}(X,\varepsilon)$ and $g(\varepsilon)=\operatorname{pack}(X,\varepsilon)$ are monotonically decreasing.

Proposition 7.12. *For any metric space X and any number $\varepsilon > 0$ we have*

$$\operatorname{cov}(X,\varepsilon) \le \operatorname{pack}(X,\varepsilon) \le \operatorname{cov}(X,\varepsilon/4).$$

Proof. First we prove the first inequality. If $\operatorname{pack}(X,\varepsilon) = \infty$, then the inequality is automatically satisfied. Now let $\operatorname{pack}(X,\varepsilon) < \infty$ and $x_1,\dots,x_n$, $n = \operatorname{pack}(X,\varepsilon)$, be a largest set of points in X for which the balls $U_{\varepsilon/2}(x_i)$ are disjoint. Since this family is maximal, for any $x \in X$ there exists x_k such that $U_{\varepsilon/2}(x) \cap U_{\varepsilon/2}(x_k) \ne \emptyset$, i.e., $|xx_k| < \varepsilon$. Then the family $\{U_\varepsilon(x_i)\}_{i=1}^n$ covers X, so $\operatorname{cov}(X,\varepsilon) \le n = \operatorname{pack}(X,\varepsilon)$.

Let us prove the second inequality. Again, if $\operatorname{cov}(X,\varepsilon/4) = \infty$, then the inequality holds. Now let $\operatorname{cov}(X,\varepsilon/4) < \infty$ and $x_1,\dots,x_m$, $m = \operatorname{cov}(X,\varepsilon/4)$, be the smallest set of points in X for which the balls $U_{\varepsilon/4}(x_i)$ cover X. Suppose that $\operatorname{pack}(X,\varepsilon) > \operatorname{cov}(X,\varepsilon/4)$, then there exist $x_1',\dots,x_n'$, $n > \operatorname{cov}(X,\varepsilon/4)$, such that the balls $U_{\varepsilon/2}(x_i')$ are pairwise disjoint. On the other hand, for some $i \ne j$ there exists k such that $x_i', x_j' \in U_{\varepsilon/4}(x_k)$, therefore, $x_k \in U_{\varepsilon/2}(x_i') \cap U_{\varepsilon/2}(x_j')$, so this intersection is not empty, a contradiction. □

Corollary 7.13. *Let X be an arbitrary metric space, then*

(1) *if* $\operatorname{pack}(X,\varepsilon) < \infty$, *then* $\operatorname{cov}(X,\varepsilon) < \infty$;

(2) *if* $\operatorname{cov}(X,\varepsilon) < \infty$, *then* $\operatorname{pack}(X,4\varepsilon) < \infty$.

Thus, $\operatorname{cov}(X,\varepsilon) < \infty$ *for all* $\varepsilon > 0$, *if and only if* $\operatorname{pack}(X,\varepsilon) < \infty$ *for all* $\varepsilon > 0$.

Proposition 7.14. *Let X be an arbitrary metric space. Then the following statements are equivalent*:

(1) $\operatorname{cov}(X,\varepsilon) < \infty$ *for any* $\varepsilon > 0$;

(2) $\operatorname{pack}(X,\varepsilon) < \infty$ *for any* $\varepsilon > 0$;

(3) *the space X is totally bounded.*

Proof. (1) $\Leftrightarrow$ (2) This follows from Corollary 7.13.

(1) $\Leftrightarrow$ (3) The condition $\operatorname{cov}(X,\varepsilon) < \infty$ is equivalent to the existence of a finite cover $\{U_\varepsilon(x_i)\}_{i=1}^n$, which is equivalent to the existence of a finite ε-net $\{x_i\}_{i=1}^n$. Thus, the condition of Item (1) is equivalent to the total boundedness of the space X. □

Proposition 7.15. *Let X, Y be metric spaces, $\delta > 0$, and $d_{GH}(X,Y) < \delta$, then*

(1) $\operatorname{cov}(X,\varepsilon) \ge \operatorname{cov}(Y,\varepsilon+2\delta)$,

(2) $\operatorname{pack}(X,\varepsilon) \ge \operatorname{pack}(Y,2\varepsilon+4\delta)$.

Proof. (1) The case $\operatorname{cov}(X,\varepsilon) = \infty$ is obvious. Now let $m := \operatorname{cov}(X,\varepsilon) < \infty$ and $\big\{U_\varepsilon(x_i)\big\}_{i=1}^m$ be a cover of X. By Theorem 6.12, there exists $R \in \mathcal{R}(X,Y)$ such that $\operatorname{dis} R < 2\delta$. For each i, we choose an arbitrary $y_i \in R(x_i)$ and show that the set $\{y_i\}_{i=1}^m$ is an $(\varepsilon+2\delta)$-net, thus $\operatorname{cov}(Y,\varepsilon+2\delta) \le m = \operatorname{cov}(X,\varepsilon)$. So, we take arbitrary $y \in Y$ and choose any $x \in R^{-1}(y)$. Then for some j we have $|xx_j| < \varepsilon$. Since $\operatorname{dis} R < 2\delta$, then $|yy_j| < \varepsilon + 2\delta$, as required.

(2) Since the case $\operatorname{pack}(Y,2\varepsilon+4\delta) = \infty$ is trivial, we assume that $n := \operatorname{pack}(Y,2\varepsilon+4\delta) < \infty$ and let $\big\{U_{\varepsilon+2\delta}(y_i)\big\}_{i=1}^n$ be a disjoint family of open balls in Y. Then for any $i \ne j$ we have $|y_iy_j| \ge \varepsilon + 2\delta$. For each i, we choose an arbitrary $x_i \in R^{-1}(y_i)$. Since $\operatorname{dis} R < 2\delta$, we have $|x_ix_j| > \varepsilon$, therefore the family $\big\{U_{\varepsilon/2}(x_i)\big\}_{i=1}^n$ is disjoint and, thus, $\operatorname{pack}(X,\varepsilon) \ge n = \operatorname{pack}(Y,2\varepsilon+4\delta)$. □

7.4 Totally bounded families of compact metric spaces

We will be interested in when a particular family of compact metric spaces is totally bounded. We begin with the following auxiliary statement, which will be needed below. For $n \in \mathbb{N}$ we denote by $\mathcal{M}_n \subset \mathcal{M}$ ($\mathcal{M}_{[n]} \subset \mathcal{M}$) the set of all metric spaces with at most (respectively, exact) n points. For $D \ge 0$, by $\mathcal{M}(D) \subset \mathcal{M}$ we denote the set of all compact metric spaces whose diameters do not exceed D. We also put $\mathcal{M}_n(D) = \mathcal{M}_n \cap \mathcal{M}(D)$ and $\mathcal{M}_{[n]}(D) = \mathcal{M}_{[n]} \cap \mathcal{M}(D)$. It is clear that $\mathcal{M}_n = \cup_{k\le n}\mathcal{M}_{[k]}$ and $\mathcal{M}_n(D) = \cup_{k\le n}\mathcal{M}_{[k]}(D)$.

Proposition 7.16. *The space $\mathcal{M}_{[n]}(D) \subset \mathcal{M}$ is totally bounded.*

Proof. Given $X \in \mathcal{M}_{[n]}(D)$, we consider all possible bijections $\nu\colon X \to \{1,\dots,n\}$, and for every such ν we construct the distance matrix $f(X,\nu) = \rho = (\rho_{ij})$, where $\rho_{ij} = \bigl|\nu^{-1}(i)\nu^{-1}(j)\bigr|$. Let T be the set of all such matrices. It is clear that the mapping $g\colon T \to \mathcal{M}_{[n]}(D)$ such that $g\colon f(X,\nu) \mapsto X$ is surjective.

We define on T the distance function generated by the ℓ_∞-norm, so T will be considered as a subset of $\mathbb{R}^{n^2}_\infty$. Since for every i,j we have $|\rho_{ij}| \le D$, the set T is bounded and, therefore, totally bounded as a subset of $\mathbb{R}^{n^2}_\infty$.

If $X, X' \in \mathcal{M}_{[n]}(D)$, $\rho = f(X,\nu)$ and $\rho' = f(X',\nu')$, then $R = (\nu')^{-1} \circ \nu$ is a bijective correspondence between X and X', and $|\rho\rho'|_\infty = \operatorname{dis} R \ge 2d_{GH}(X,X')$. Thus, the surjection g is Lipschitzian, therefore, $\mathcal{M}_{[n]}(D) = g(T)$ is also totally bounded. □

Corollary 7.17. *The space $\mathcal{M}_n(D) \subset \mathcal{M}$ is totally bounded.*

Problem 7.4. Prove that the set $\mathcal{M}_n(D)$ is compact, while $\mathcal{M}_{[n]}(D)$ for $n > 1$ is not.

Theorem 7.18. *Let $\mathcal{C}$ be a nonempty subset of $\mathcal{M}$. Then the following statements are equivalent.*

(1) *There is a number $D \ge 0$ and a function $N\colon (0,\infty) \to \mathbb{N}$ such that for all $X \in \mathcal{C}$ we have $\operatorname{diam} X \le D$ and $\operatorname{pack}(X,\varepsilon) \le N(\varepsilon)$.*

(2) *There is a number $D \ge 0$ and a function $N\colon (0,\infty) \to \mathbb{N}$ such that for all $X \in \mathcal{C}$ we have $\operatorname{diam} X \le D$ and $\operatorname{cov}(X,\varepsilon) \le N(\varepsilon)$.*

(3) *The space $\mathcal{C}$ with the metric d_{GH} is totally bounded.*

Proof. (3) ⇒ (1). Fix an arbitrary $\varepsilon > 0$. We have to find the corresponding D and $N(\varepsilon)$. Since $\mathcal{C}$ is totally bounded, for any $\delta > 0$ there exists a finite δ-net $\mathcal{C}' \subset \mathcal{C}$. Choose δ such that $4\delta < \varepsilon$. Since all the spaces lying in $\mathcal{C}'$ are totally bounded, by Proposition 7.14, their packing numbers are finite. In addition, their diameters are finite. Put $D' = \max_{X' \in \mathcal{C}'} \operatorname{diam} X'$ and $N'(\varepsilon) = \max_{X' \in \mathcal{C}'} \operatorname{pack}(X',\varepsilon)$. For an arbitrary $X \in \mathcal{C}$ there exists $X' \in \mathcal{C}'$ such that $d_{GH}(X,X') < \delta$. It is easy to see that $\operatorname{diam} X \le \operatorname{diam} X' + 2\delta \le D' + 2\delta$, so that we can put $D = D' + 2\delta$. In addition, by Proposition 7.15, it holds $\operatorname{pack}(X,\varepsilon) \le \operatorname{pack}(X',\varepsilon/2 - 2\delta) \le N'(\varepsilon/2 - 2\delta)$, so we can put $N(\varepsilon) = N'(\varepsilon/2 - 2\delta)$.

(1) ⇔ (2). This immediately follows from Proposition 7.12.

(2) ⇒ (3). Fix some $\varepsilon > 0$, and for each $X \in \mathcal{C}$ consider a finite cover of the space X by at most $n = N(\varepsilon)$ open balls of radius ε. By F^ε_X we denote the set of centers of these balls, then $d_{GH}(X, F^\varepsilon_X) \le \varepsilon$. In addition, $F^\varepsilon_X \in \mathcal{M}_n(D)$, therefore, by Corollary 7.17, the family $\mathcal{F}^\varepsilon = \{F^\varepsilon_X\}_{X \in \mathcal{C}} \subset \mathcal{M}_n(D)$ is totally bounded. Since for any $X \in \mathcal{C}$ and any $\varepsilon' > \varepsilon$ we have $X \in U^{\mathcal{M}}_{\varepsilon'}(F^\varepsilon_X)$, then $\mathcal{C} \subset U^{\mathcal{M}}_{\varepsilon'}(\mathcal{F}^\varepsilon)$. Since ε and ε' are arbitrary, we conclude that $\mathcal{C}$ is also totally bounded (verify that). □

The following theorem allows us to realize all metric spaces from a totally bounded subset of $\mathcal{M}$ as subsets of some compact subset of ℓ_∞.

Theorem 7.19 (Gromov). *For each totally bounded family $\mathcal{C} \subset \mathcal{M}$ there exists a compact $K \subset \ell_\infty$ such that every $X \in \mathcal{C}$ is isometrically embedded into K.*

Proof. The compact K is constructed as follows. By Theorem 7.18, there exist $D \ge 0$ and $N\colon (0,\infty) \to \mathbb{N}$ such that for all $X \in \mathcal{C}$ we have $\operatorname{diam} X \le D$ and $\operatorname{cov}(X,\varepsilon) \le N(\varepsilon)$. Choose an arbitrary decreasing sequence of positive numbers $E = \{\varepsilon_1, \varepsilon_2, \dots\}$ such that $\sum_{i=1}^\infty \varepsilon_i < \infty$. This sequence and the function $N(\varepsilon)$ generate a sequence of natural numbers $N_i = N(\varepsilon_i)$. These two sequences, together with the number D, define the set $F_{D,E} \subset \ell_\infty$ as follows.

Construction 7.1. Put $A = \cup_{j=1}^\infty \bigl(\{1,\dots,N_1\} \times \dots \times \{1,\dots,N_j\}\bigr)$. It is clear that A is a countable set. Let $\ell_\infty(A) = \{f\colon A \to \mathbb{R} : \sup|f| < \infty\}$, then $\ell_\infty(A)$ is isometric to ℓ_∞. For brevity, instead of $f\bigl((n_1,\dots,n_j)\bigr)$ we will write $f(n_1,\dots,n_j)$.

We now define the set $F_{D,E}$, composing it from all $f\colon A \to \mathbb{R}$ that satisfy the following conditions:

(1) $0 \le f(n_1) \le D$ for all $1 \le n_1 \le N_1$;

(2) $\bigl|f(n_1,\dots,n_j,n_{j+1}) - f(n_1,\dots,n_j)\bigr| \le \varepsilon_j$ for all elements $(n_1,\dots,n_j,n_{j+1}) \in A$.

Lemma 7.20. *The set $F_{D,E}$ defined above is a compact subset of $\ell_\infty(A)$.*

Proof. First, note that for each function $f \in F_{D,E}$ it holds $\sup_{a\in A}\bigl|f(a)\bigr| \le D+\sum_{i=1}^{\infty}\varepsilon_i < \infty$, so $f \in \ell_\infty(A)$. Further, since all the inequalities defining $F_{D,E}$ are non-strict, the set $F_{D,E}$ is closed in $\ell_\infty(A)$. Since $\ell_\infty(A)$ is complete, $F_{D,E}$ is also complete. In addition, the diameter of $F_{D,E}$ is finite (it bounded by the number $D+2\sum_{i=1}^{\infty}\varepsilon_i$).

Put $A_{[k]} = \bigl\{(n_1,\ldots,n_k)\in A\bigr\}$ and $A_k = \cup_{j=1}^{k} A_j$. We denote by $\pi_k\colon \ell_\infty(A)\to\ell_\infty(A_k)$ the canonical projection that maps each function $f\colon A\to\mathbb{R}$ to its restriction on $A_k\subset A$, and let $F_k=\pi_k(F_{D,E})$. Note that F_k is a closed and bounded subset of the finite-dimensional vector space $\ell_\infty(A_k)$, therefore F_k is compact.

Define the mapping $\nu\colon F_k\to\ell_\infty(A)$ by extending each function $f_k\in F_k$ to the entire set A as follows:

$$f_k(n_1,\ldots,n_k,n_{k+1},\ldots)=f_k(n_1,\ldots,n_k).$$

It is clear that ν is isometric, therefore $F'_k=\nu(F_k)$ is also a compact set.

Put $e_k=\varepsilon_k+\varepsilon_{k+1}+\ldots$, then $e_k\to\emptyset$ as $k\to\infty$. By Condition (2), we have $F_{D,E}\subset U_{e_k}^{\ell_\infty(A)}(F'_k)$ for all $k\ge 2$, which implies the total boundedness of $F_{D,E}$ (verify that). □

We now take the set $F_{D,2E}$ as K and show that each space $X\in\mathcal{C}$ can be isometrically embedded into this K. We consider points of the form x_a, $a\in A$, and again, for brevity, instead of $x_{(n_1,\ldots,n_j)}$ we write $x_{n_1\cdots n_j}$.

Take an arbitrary $X\in\mathcal{C}$. Since $\operatorname{cov}(X,\varepsilon_1)\le N(\varepsilon_1)=N_1$, then X contains an ε_1-net $\cup_{n_1=1}^{N_1}\bigl\{x_{n_1}\bigr\}$, i.e., the family $\bigl\{U_{\varepsilon_1}(x_{n_1})\bigr\}_{n_1=1}^{N_1}$ forms a cover of X. Note that some points x_{n_1} may coincide.

Further, since $\operatorname{cov}(X,\varepsilon_2)\le N(\varepsilon_2)=N_2$, then X contains an ε_2-net $\cup_{n_2=1}^{N_2}\bigl\{x'_{n_2}\bigr\}$, i.e., the family $\bigl\{U_{\varepsilon_2}(x'_{n_2})\bigr\}_{n_2=1}^{N_2}$ forms a cover of X. Fix n_1 and choose only those balls $U_{\varepsilon_2}(x'_{n_2})$ that satisfy $|x_{n_1}x'_{n_2}|<\varepsilon_1+\varepsilon_2$. In this way we have got at most N_2 balls. Enumerate them and add some copies of them to obtain exactly N_2 balls which we denote by $U_{\varepsilon_2}(x_{n_1n_2})$, $n_2=1,\ldots,N_2$. So, we have got a cover $\bigl\{U_{\varepsilon_2}(x_{n_1n_2})\bigr\}_{n_2=1}^{N_2}$ of the ball $U_{\varepsilon_1}(x_{n_1})$. By construction, it holds $|x_{n_1}x_{n_1n_2}|<\varepsilon_1+\varepsilon_2<2\varepsilon_1$.

Continuing this process, at the j-th step we get the family of balls $\bigl\{U_{\varepsilon_j}(x_a)\bigr\}_{a\in A_{[j]}}$ with $|x_{n_1\cdots n_j}x_{n_1\cdots n_jn_{j+1}}|<2\varepsilon_j$.

It is easy to see that the set $\{x_a\}_{a\in A}$ of centers of these balls is a countable everywhere dense subset of X (some x_a may coincide with each other). By Theorem 2.26, the space X can be isometrically embedded into $\ell_\infty(A)$ by associating with each point x the function $f_x\colon A\to\mathbb{R}$ defined as follows: $f_x(a)=|xx_a|$.

Lemma 7.21. *For every $x\in X$ we have $f_x\in F_{D,2E}$.*

Proof. It is clear that $0\le f_x\le D$, so that Item (1) from the definition of the set $F_{D,2E}$ is satisfied. Further, for each $(n_1,\ldots,n_j,n_{j+1})$, the point $x_{n_1\cdots n_jn_{j+1}}$ lies in $U_{2\varepsilon_j}(x_{n_1\cdots n_j})$, so for every $x\in X$ we have

$$\bigl|f_x(n_1,\ldots,n_j,n_{j+1})-f_x(n_1,\ldots,n_j)\bigr|=\bigl||x_{n_1\cdots n_jn_{j+1}}x|-|x_{n_1\cdots n_j}x|\bigr|\le|x_{n_1\cdots n_jn_{j+1}}x_{n_1\cdots n_j}|<2\varepsilon_j,$$

therefore, Item (2) from the definition of the set $F_{D,2E}$ is also fulfilled. □

Thus, the mapping $x\mapsto f_x$ isometrically embeds X into K. □

7.5 Some other properties of Gromov–Hausdorff space

In this section we apply the previous results to prove a few more properties of the Gromov–Hausdorff space $\mathcal{M}$.

7.5.1 Completeness of Gromov–Hausdorff space

Theorem 7.19 implies the following result.

Theorem 7.22. *The space $\mathcal{M}$ is complete.*

Proof. Consider an arbitrary fundamental sequence $\{X_i\}_{i=1}^{\infty}\subset\mathcal{M}$. Then $\{X_i\}_{i=1}^{\infty}$ is a totally bounded subset of $\mathcal{M}$. By Theorem 7.19, there exists a compact set $K\subset\ell_\infty$ into which all X_i can be isometrically embedded. Denote by Y_i the image of X_i. By Theorem 5.38, the space $\mathcal{H}(K)$ of all closed bounded subsets of K is also compact, therefore the sequence $Y_i\in\mathcal{H}(K)$ contains a convergent subsequence Y_{n_i}. Let Y be the limit of this subsequence. Then Y is a nonempty compact metric space and

$$d_{GH}(X_{n_i},Y)=d_{GH}(Y_{n_i},Y)\le d_H(Y_{n_i},Y)\to 0\ \text{ as } i\to\infty,$$

therefore, $X_{n_i}\xrightarrow{\text{GH}}Y$ and, since the sequence X_i is fundamental, we have $X_i\xrightarrow{\text{GH}}Y$. □

7.5.2 Separability of Gromov–Hausdorff space

Theorem 7.23. *The space $\mathcal{M}$ is separable.*

Proof. By Corollary 7.17, each space $\mathcal{M}_n(D)$ is totally bounded and, therefore, separable. Since $\mathcal{M}_n = \cup_{k=1}^{\infty}\mathcal{M}_n(k)$, then all $\mathcal{M}_n$, as well as their union $\cup_{n=1}^{\infty}\mathcal{M}_n$, are separable. This last union is the set of all finite metric spaces, which, as noted in Example 6.27, is an everywhere dense subset of $\mathcal{M}$, so that $\mathcal{M}$ is separable. □

Recall that a complete separable metric space is called *Polish*. Thus, the following result holds.

Corollary 7.24. *The space $\mathcal{M}$ is Polish.*

By Problem 1.31, for a metric space, the separability is equivalent to having a countable base.

Corollary 7.25. *The space $\mathcal{M}$ has a countable base.*

7.6 Calculating mst-spectrum by means of Gromov–Hausdorff distances

Recall that by Δ_n we denoted n-point metric space such that all its nonzero distances equal 1. Also, given $\lambda > 0$ and any metric space X, if we multiply by λ all the distances in X, then the resulting metric space we denote by λX.

In the present section we show that the mst-spectrum of an arbitrary n-point metric space X can be represented as a linear function on the Gromov–Hausdorff distances from this space to the $\lambda\Delta_2, \ldots, \lambda\Delta_n$ for $\lambda \ge 2\operatorname{diam} X$.

Theorem 7.26. *Let X be a finite metric space, $\sigma(X) = (\sigma_1, \ldots, \sigma_{n-1})$, $\lambda \ge 2\operatorname{diam} X$. Then*

$$\sigma_k = \lambda - 2d_{GH}(\lambda\Delta_{k+1}, X).$$

Proof. Choose any $1 \le k \le n-1$ and arbitrary irreducible correspondence $R \in \mathcal{R}^0(\lambda\Delta_{k+1}, X)$. By Proposition 6.22, there exists partitions $R_{\lambda\Delta_{k+1}} = \{Z_i\}_{i=1}^p$ and $R_X = \{X_i\}_{i=1}^p$ of $\lambda\,\Delta_{k+1}$ and X, respectively, such that $R = \cup_{i=1}^p Z_i \times X_i$, and $\min\{\#Z_i, \#X_i\} = 1$ for all i. By Proposition 6.23, it holds $\operatorname{dis} R \ge \max\{\operatorname{diam} R_{\lambda\Delta_{k+1}}, \operatorname{diam} R_X\}$. Thus, if for some i we have $\#Z_i > 1$, then $\operatorname{dis} R \ge \lambda \ge 2\operatorname{diam} X$. Since $k+1 \le n$, there exists R such that $\#Z_i = 1$ for all i. For such R, again by Proposition 6.23, we have $\operatorname{dis} R \le \operatorname{diam} X$. Therefore, $\inf_{R\in\mathcal{R}^0(\lambda\Delta_{k+1},X)} \operatorname{dis} R$ is achieved on a correspondences of the latter type. The set of these correspondences we denote by $\mathcal{R}$.

Now, if $R \in \mathcal{R}$, then $p = k+1$ and $R_X \in \mathcal{D}_{k+1}(X)$. By Proposition 6.23, we have

$$\operatorname{dis} R = \sup\bigl\{\operatorname{diam} R_X,\, |X_iX_j|' - \lambda,\, \lambda - |X_iX_j| : 1 \le i < j \le k+1\bigr\} = \\ = \sup\bigl\{\lambda - |X_iX_j| : 1 \le i < j \le k+1\bigr\} = \lambda - \alpha(R_X),$$

where the second equality holds because

$$\max\bigl\{|X_iX_j|' - \lambda, \operatorname{diam} R_X\bigr\} \le \operatorname{diam} X \le \lambda - \operatorname{diam} X \le \lambda - |X_iX_j|$$

for any $1 \le i < j \le k+1$. Corollary 6.21, together with above considerations, gives us

$$2d_{GH}(\lambda\Delta_{k+1}, X) = \min_{R\in\mathcal{R}} \operatorname{dis} R = \min_{R\in\mathcal{R}}\bigl(\lambda - \alpha(R_X)\bigr) = \lambda - \max_{D\in\mathcal{D}_{k+1}(X)} \alpha(D),$$

where the last equality holds because each D generates some $R \in \mathcal{R}$.

It remains to use Theorem 4.13 which states that

$$\sigma_k = \max\bigl\{\alpha(D) : D \in \mathcal{D}_{k+1}(X)\bigr\},$$

thus, $2d_{GH}(\lambda\Delta_{k+1}, X) = \lambda - \sigma_k$. □

Corollary 7.27. *Let X be a finite metric space and $\lambda \ge 2\operatorname{diam} X$, then*

$$\operatorname{mst} X = \lambda(\#X - 1) - 2\sum_{k=1}^{\#X-1} d_{GH}(\lambda\Delta_{k+1}, X).$$

7.7 Steiner problem in Gromov–Hausdorff space

In this section we prove the following

Theorem 7.28. *Let $M \subset \mathcal{M}$ be a finite set such that each $X \in M$ is a finite metric space. Then* $\mathrm{SMT}(M, \mathcal{M}) \neq \emptyset$*, i.e., for such M the Steiner problem has a solution.*

Remark 7.29. For arbitrary finite $M \subset \mathcal{M}$ the problem is still open.

Proof. Let $n = \#M$. In Section 4.5 we introduced model full Steiner trees, and we have shown how to use them for calculating the length of a Steiner minimal tree. Recall the corresponding definitions in our case. A full Steiner tree has the vertices of two types only: the ones of degree 1 which we call boundary, and the ones of degree 3 which we call interior. In model full Steiner trees which we use to treat the problem for such M, the vertex set is $\{1, \ldots, 2n-2\}$, where $\{1, \ldots, n\}$ are reserved for the boundary vertices. We called two model full Steiner tree equivalent if there is an isomorphism between them fixed on the boundary. By $\mathcal{B}_n$ we denoted the set of all model full Steiner trees with n boundary vertices considered up to this equivalence.

Enumerate the points from M in an arbitrary way, i.e., we consider a bijection $\varphi\colon \{1, \ldots, n\} \to M$. Choose an arbitrary $G \in \mathcal{B}_n$, and consider a network Γ of the type G for which $\partial\Gamma = \varphi$. Then all such networks for given G differ from each other only in the "positions" of their interior vertices, thus the set $[G, \varphi]$ of such networks can be identified with $\mathcal{M}^{n-2}$. Then we proved (Corollary 4.24) that

$$\mathrm{smt}(M, \mathcal{M}) = \inf\bigl\{|\Gamma| : \Gamma \in [G, \varphi],\, G \in \mathcal{B}_n\bigr\}.$$

Choose an arbitrary $G \in \mathcal{B}_n$ and any $\Gamma \in [G, \varphi]$. We put $X_i = \Gamma(i)$, then $M = \{X_1, \ldots, X_n\}$. For each $ij \in E(G)$ we choose an arbitrary $R_{ij} \in \mathcal{R}_{\mathrm{opt}}(X_i, X_j)$ in such a way that $R_{ji}^{-1} = R_{ij}$. Let $X_k = \{x_k^i\}_{i=1}^{n_k}$, then for any $k \in \{1, \ldots, n\}$ and any $1 \le i \le n_k$ we construct a network $\Gamma_k^i\colon \{1, \ldots, 2n-2\} \to \sqcup_{j=1}^{2n-2} X_j$ as follows: in each X_j we choose one point $x_j^{r_j} =: \Gamma_k^i(j)$ such that

(1) $x_k^{r_k} = x_k^i$;

(2) for any $pq \in E(G)$ we have $(x_p^{r_p}, x_q^{r_q}) \in R_{pq}$

(verify that it is always possible to do). Consider the set $\mathcal{T} = \{\Gamma_k^i\}$ consisting of all Γ_k^i over all possible k and i.

For any $j \in \{1, \ldots, 2n-2\}$ we put $V_j = \cup_{T\in\mathcal{T}}\{T(j)\}$, i.e., we gather in each X_j all points that are the images of the vertices of the constructed networks. Let us note that all V_j have at most $N := \sum_{k=1}^n n_k$ points, and for each $j \in \{1, \ldots, n\}$ it holds $V_j = X_j$.

Further, for each p and q such that $pq \in E(G)$ we denote by R_{pq}^t the set of all pairs (x_p, x_p) such that for some $T \in \mathcal{T}$ we have $x_p = T(p)$ and $x_q = T(q)$, i.e., we gather all pairs forming the images of the edges of the constructed networks. Thus, we obtained correspondences $R_{pq}^t \in \mathcal{R}(V_p, V_q)$ such that $R_{pq}^t \subset R_{pq}$, hence

$$d_{GH}(V_p, V_q) \le \operatorname{dis} R_{pq}^t \le \operatorname{dis} R_{pq} = d_{GH}(X_p, X_q).$$

Denote by Γ^t the network $\Gamma^t\colon p \mapsto V_p$. Since $\Gamma^t|_{\{1,\ldots,n\}} = \varphi$ and Γ^t has the type G, then $\Gamma^t \in [G, \varphi]$. Denote by $[G, \varphi]^t$ the set of all such Γ^t. Since $|\Gamma^t| \le |\Gamma|$, then

$$\mathrm{smt}(M, \mathcal{M}) = \inf\bigl\{|\Gamma^t| : \Gamma^t \in [G, \varphi]^t,\, G \in \mathcal{B}_n\bigr\}.$$

However, all X_i belong to $\mathcal{M}_N$, therefore, $\mathrm{smt}(M, \mathcal{M}) = \mathrm{smt}(M, \mathcal{M}_N)$. Moreover, if we choose Γ^t such that all V_i, $i \ge n+1$, coincide with V_k for some $k \le n$, then $\mathrm{smt}(M, \mathcal{M}_N) \le \sum_{1\le p,q\le n} d_{GH}(X_p, X_q) =: D'$. Thus, it suffices to consider only Γ^t with $|\Gamma^t| \le D'$, in particular, for such Γ^t and any $pq \in E(G)$ we have $d_{GH}(V_p, V_q) \le D'$.

Let us put $d = \max\{\operatorname{diam} X_k : k = 1, \ldots, n\}$. Since for any $X, Y \in \mathcal{M}$ we have $d_{GH}(X, Y) \ge \frac{1}{2}|\operatorname{diam} X - \operatorname{diam} Y|$, then for each X_p we have $\operatorname{diam} X_p \le d + 2(n-2)D' =: D$ (all these estimates are rather rough, however, we do not need exact ones here). In account, we proved that all X_p belongs to $\mathcal{M}_N(D)$. By Problem 7.4, the space $\mathcal{M}_N(D)$ is compact, that is why the continuous function $\ell\colon \mathcal{M}_N(D)^{n-2} \to \mathbb{R}$, $\ell\colon (X_{n+1}, \ldots, X_{2n-2}) \mapsto |\Gamma^t|$, attains its minimum at some Γ_0^t. It remains to notice that $\mathcal{B}_n$ is finite. $\square$

References to Chapter 7

[1] Burago D., Burago Yu., Ivanov S. *A Course in Metric Geometry*. Graduate Studies in Mathematics, vol.33, A.M.S., Providence, RI, 2001.

Exercises to Chapter 7

Exercise 7.1. Prove that the product topology on $X \times Y$ coincides with the one generated by the metric from Agreement 7.1.

Exercise 7.2. Prove that the topologies generated by all d_t, and by $|\cdot|$ as well, coincide with the product topology of $X \times Y$.

Exercise 7.3. Prove that

(1) a metric space X is bounded if only for some $\varepsilon > 0$ it holds $\operatorname{cov}(X, \varepsilon) < \infty$ (similarly, $\operatorname{pack}(X, \varepsilon) < \infty$);

(2) a metric space X is finite if and only if there exists n such that $\operatorname{cov}(X, \varepsilon) \le n$ for all $\varepsilon > 0$ (similarly, for $\operatorname{pack}(X, \varepsilon)$);

(3) the functions $f(\varepsilon) = \operatorname{cov}(X, \varepsilon)$ and $g(\varepsilon) = \operatorname{pack}(X, \varepsilon)$ are monotonically decreasing.

Exercise 7.4. Prove that the set $\mathcal{M}_n(D)$ is compact, while $\mathcal{M}_{[n]}(D)$ for $n > 1$ is not.