Chapter 6

Gromov–Hausdorff Distance.

Schedule. Realization of a pair of metric space, Gromov–Hausdorff distance, admissible metric on disjoin union of metric spaces, calculation of Gromov–Hausdorff distance in terms of admissible metrics, triangle inequality for Gromov–Hausdorff distance, positive definiteness of Gromov–Hausdorff distance for isometry classes of compact spaces, counterexample for boundedly compact spaces, Gromov–Hausdorff distance for separable spaces in terms of their isometric images in ℓ_∞, relations, distortion of a relation between metric spaces, correspondences, Gromov–Hausdorff distance in terms of correspondences, ε-isometries and Gromov–Hausdorff distance, irreducible correspondences between sets, existence of irreducible correspondences, irreducible correspondences as bijections of partitions of sets, Gromov–Hausdorff distance in terms of irreducible correspondences, examples: Gromov–Hausdorff distance between 2- or 3-point metric spaces, simple general properties of Gromov–Hausdorff distance, Gromov–Hausdorff convergence, inheritance of metric and topological properties while Gromov–Hausdorff convergence.

In this section, we will study the Gromov–Hausdorff distance, see [1] and [2] for more details. All metric spaces are supposed to be nonempty.

Construction 6.1. Let X and Y be metric spaces. A triple (X',Y',Z) consisting of a metric space Z and its two subsets X' and Y', which are isometric respectively to X and Y, will be called *a realization of the pair* (X,Y). We put

$$d_{GH}(X,Y)=\inf\bigl\{r\in\mathbb{R}:\text{there exists a realization }(X',Y',Z)\text{ of }(X,Y)\text{ such that }d_H(X',Y')\le r\bigr\}.$$

Remark 6.1. The value $d_{GH}(X,Y)$ is evidently non-negative, symmetric, and $d_{GH}(X,X)=0$ for any metric space X. Thus, d_{GH} is a distance function on each set of metric spaces.

Definition 6.2. The value $d_{GH}(X,Y)$ from Construction 6.1 is called *the Gromov–Hausdorff distance* between the metric spaces X and Y.

Problem 6.1. Prove that for any metric spaces X and Y there exists a realization of (X,Y).

Remark 6.3. In some monographs the Gromov–Hausdorff distance $d_{GH}(X,Y)$ is defined as the infimum of the values $d_H(X',Y')$ over all realizations (X',Y',Z) of the pair (X,Y). However, we give such, at first glance, a more technically complicated definition of d_{GH}, to avoid the Cantor paradox, because the family of all realizations is no longer a set. Introducing r and talking about *the existence of a realization*, we thereby get rid of the need to consider all the realizations.

Notation. In what follows, it will sometimes be necessary for us to explicitly indicate the space X in which a particular metric is considered, as well as the corresponding Hausdorff metric. Thus, the distance between points $x,x'\in X$ will sometimes be denoted by $|xx'|_X$, and the corresponding Hausdorff distance between nonempty subsets A and B of the space X by $d_H^X(A,B)$. In addition, if ρ is some metric on X, then the Hausdorff distance generated by this metric will sometimes be denoted by ρ_H.

It turns out that, to define the Gromov–Hausdorff distance, it suffices to consider only metric spaces of the form $(X\sqcup Y,\rho)$, where ρ extends the original metrics of X and Y. Such ρ will be called *an admissible metric for X and Y*, and the set of all admissible metrics for given X and Y will be denoted by $\mathcal{D}(X,Y)$.

Problem 6.2. Prove that for any metric spaces X and Y there exists at least one admissible metric, i.e., the set $\mathcal{D}(X,Y)$ is not empty.

Theorem 6.4. *For any metric spaces X and Y, we have*

$$d_{GH}(X,Y)=\inf\bigl\{\rho_H(X,Y):\rho\in\mathcal{D}(X,Y)\bigr\}. \tag{6.1}$$

Proof. Denote by $d'_{GH}(X,Y)$ the right hand side of the equation (6.1). Then $d_{GH}(X,Y) \le d'_{GH}(X,Y)$, because for every $\rho \in \mathcal{D}(X,Y)$ and $Z = (X \sqcup Y, \rho)$, the triple (X,Y,Z) is a realization of the pair (X,Y) with the same Hausdorff distance $\rho_H(X,Y)$. We now prove the inverse inequality.

By Construction 6.1, for any $\varepsilon > 0$ there exists a realization (X',Y',Z) of the pair (X,Y) such that

$$d_H(X',Y') \le d_{GH}(X,Y) + \varepsilon.$$

If X' and Y' do not intersect each other, then we restrict the metric from Z to $X' \sqcup Y'$, and, after identifying X' and Y' with X and Y, respectively, we obtain an admissible metric ρ on $X \sqcup Y$ for which $\rho_H(X,Y) \le d_{GH}(X,Y) + \varepsilon$. If $X' \cap Y' \ne \emptyset$, we replace Z by $Z \times \mathbb{R}$ with the metric $\big|(z,t)(z',s)\big| = |zz'| + |ts|$, also replace X' and Y' by the sets $X'' = X' \times \{0\}$ and $Y'' = Y' \times \{\varepsilon\}$, respectively, then $(X'',Y'',Z \times \mathbb{R})$ is a realization of the (X,Y) such that $X'' \cap Y'' = \emptyset$, $d_H(X'',Y'') \le d_{GH}(X,Y) + 2\varepsilon$, and aging we can restrict the metric of $Z \times \mathbb{R}$ to $X'' \sqcup Y''$, identify X with X'', Y'' with Y, and obtain an admissible metric ρ with $\rho_H(X,Y) \le d_{GH}(X,Y) + 2\varepsilon$. Since ε is arbitrary, we get $d'_{GH}(X,Y) \le d_{GH}(X,Y)$. □

Remark 6.5. If X and Y are subsets of some metric space, then $d_{GH}(X,Y) \le d_H(X,Y)$. In particular, if $d_H(X,Y) = 0$, then $d_{GH}(X,Y) = 0$, so the Gromov–Hausdorff distance, like the Hausdorff distance, is not positively defined: for example, the Gromov–Hausdorff distance between the segment $[0,1]$ and the interval $(0,1)$ vanishes. However, if we restrict ourselves to compact metric spaces, then $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric (see below for the proof).

Proposition 6.6. *The function d_{GH} satisfies the triangle inequality.*

Proof. Choose arbitrary metric spaces X, Y, and Z. We have to show that $d_{GH}(X,Z) \le d_{GH}(X,Y) + d_{GH}(Y,Z)$.

Choose any admissible metrics $\mu \in \mathcal{D}(X,Y)$ and $\nu \in \mathcal{D}(Y,Z)$. Recall that in Example 2.15 we have defined the gluing $U \sqcup_f V$ of metric spaces U and V over a mapping $f\colon W \to V$, where $W \subset U$. Also, Problem 2.15 states that if f is isometric, then the restrictions of the metric of $U \sqcup_f V$ to U and V coincides with the original ones. We apply this result to $U = X \sqcup Y$ and $V = Y \sqcup Z$ with the metrics μ and ν, respectively, and to $W = Y$ and $f = \mathrm{id}\colon Y \to Y \subset Y \sqcup Z$.

Consider the metric space $A = (X \sqcup Y) \sqcup_f (Y \sqcup Z)$ with the corresponding quotient metric $\rho := \rho_\sim$. By Problem 2.15, the restrictions of the metric ρ from A to X and Z coincide with the original ones, thus (X,Z,A) is a realization of (X,Z). Also, by definition of ρ, for any $x \in X$ and $z \in Z$ we have $\rho(x,z) \le \mu(x,y) + \nu(y,z)$. Let us put $a = \mu_H(X,Y)$, $b = \nu_H(Y,Z)$, and choose some $\delta > 0$. Then for any $x \in X$ there exists $y \in Y$ such that $\mu(x,y) < a + \delta/2$; also, for any $y \in Y$ there exists $z \in Z$ such that $\nu(y,z) < b + \delta/2$, thus for any $x \in X$ there exists $z \in Z$ and, similarly, for any $z \in Z$ there exists $x \in X$ such that $\rho(x,z) \le \mu(x,y) + \nu(y,z) < a + b + \delta$, thus $\rho_H(X,Z) \le a + b + \delta$. Since δ is arbitrary, we get $\rho_H(X,Z) \le \mu_H(X,Y) + \nu_H(Y,Z)$. Thus,

$$\begin{aligned} d_{GH}(X,Z) = \inf_{d \in \mathcal{D}(X,Z)} d_H(X,Z) \le \inf_{\rho} \rho_H(X,Z) &\le \inf_{\mu \in \mathcal{D}(X,Y),\ \nu \in \mathcal{D}(Y,Z)} \big(\mu_H(X,Y) + \nu_H(Y,Z)\big) = \\ &= \inf_{\mu \in \mathcal{D}(X,Y)} \mu_H(X,Y) + \inf_{\nu \in \mathcal{D}(Y,Z)} \nu_H(Y,Z) = d_{GH}(X,Y) + d_{GH}(Y,Z), \end{aligned}$$

where the second equality holds because the values we minimize over μ and ν are independent. □

Thus, we have shown that on every set of metric spaces, the function d_{GH} is a generalized pseudometric. If the diameters of all spaces in the family are bounded by the same number, then d_{GH} is a (finite) pseudometric. As we already noted, d_{GH} is not a metric in general. However, if we restrict ourselves to compact metric spaces considered upto isometry, then d_{GH} will already be a metric.

Proposition 6.7. *For compact metric spaces X and Y, it holds $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric.*

Proof. If X is isometric to Y then we can take $Z = X' = Y' = X$, thus (Z,X',Y') is a realization of (X,Y), and $d_H(X',Y') = 0$. Now we prove the converse statement.

Consider a sequence of admissible metrics $d^k \in \mathcal{D}(X,Y)$ such that $d^k_H(X,Y) < 1/k$, $k = 1, 2, \ldots$, then for each $x \in X \subset X \sqcup Y$ there exists $y \in Y \subset X \sqcup Y$ such that $d^k(x,y) < 1/k$. We choose any such y and put $I_k(x) = y$. Thus, we have constructed a mapping $I_k\colon X \to Y$ (possibly discontinuous). Similarly, we define a mapping $J_k\colon Y \to X$.

From the triangle inequality it follows that for any $x, x' \in X$ and $y, y' \in Y$ we have

$$\begin{gathered} d^k\big(I_k(x), I_k(x')\big) < \frac{2}{k} + d^k(x,x'), \quad d^k\big(J_k(y), J_k(y')\big) < \frac{2}{k} + d^k(y,y'), \\ d^k\big(x, J_k \circ I_k(x)\big) < \frac{2}{k}, \quad d^k\big(y, I_k \circ J_k(y)\big) < \frac{2}{k}. \end{gathered} \tag{6.2}$$

Similarly to what was done in the proof of the Arzela–Ascoli theorem, we construct the "limit" mappings $I\colon X \to Y$ and $J\colon Y \to X$. Namely, we choose in X a countable everywhere dense subset $S = \{x_1, x_2, \ldots\}$; using the Cantor diagonal process, we construct a subsequence $\{I_{k_1}, I_{k_2}, \ldots\}$ such that for every i the sequence $I_{k_p}(x_i)$ converges to some $I(x_i) \in Y$ (here we use the compactness of Y). The mapping I is 1-Lipschitz because for every $x_i, x_j \in S$ we have

$$\big|I(x_i)I(x_j)\big| = d^k\big(I(x_i), I(x_j)\big) \le \frac{2}{k} + d^k(x_i, x_j) = \frac{2}{k} + |x_i x_j|,$$

and since k is arbitrary, we have $\big|I(x_i)I(x_j)\big| \le |x_i x_j|$. Since Y is complete, by Lemma 3.48 we can extend I onto the whole X by continuity to a 1-Lipschitz mapping (we denote this mapping by the same symbol I). We proceed similarly with the sequence J_k. Passing to the limit in the last two inequalities (6.2) as $k \to \infty$, we conclude that for any $x \in X$ and $y \in Y$ we have

$$\big|x\,(J \circ I)(x)\big| = 0, \quad \big|y\,(I \circ J)(y)\big| = 0. \tag{6.3}$$

The relations (6.3) indicates that I and J are mutually inverse bijections, and the 1-Lipschitz condition for the both I and J guarantees that them preserve the distance, i.e., I and J are isometric. □

The following exercise gives one possible generalization of Proposition 6.7.

Problem 6.3. Prove that if a metric space X is compact, a metric space Y is complete, and $d_{GH}(X, Y) = 0$, then X is isometric to Y.

Solution. By Theorem 2.24, it suffices to prove that Y is a totally bounded space. Choose an arbitrary $\varepsilon > 0$. Since $d_{GH}(X, Y) = 0$ then there exists an admissible metric $d \in \mathcal{D}(X, Y)$ such that $d_H(X, Y) < \varepsilon/3$. Since X is compact, there exists a finite $(\varepsilon/3)$-net $S \subset X$. Since $d_H(X, Y) < \varepsilon/3$, for each $s \in S$ there exists $y_s \in Y$ for which $d(s, y_s) < \varepsilon/3$, and for each $y \in Y$ there exists $x \in X$ such that $|xy| < \varepsilon/3$. Since S is an $(\varepsilon/3)$-net in X, there exists $s \in S$ for which $|xs| < \varepsilon/3$. Thus

$$|yy_s| \le |yx| + |xs| + d(s, y_s) < \varepsilon,$$

so the set $\{y_s\}_{s \in S}$ is a finite ε-net in Y.

Remark 6.8. Even if both metric spaces X and Y are boundedly compact, Proposition 6.7 may not hold. To describe the corresponding example, we denote by X and Y the subsets of the Euclidean plane $\mathbb{R}^2$ constructed as follows. Each of these spaces is obtained from the abscissa by adding vertical segments between $(m, 0)$ and (m, ℓ_m), $m \in \mathbb{Z}$. In the case of X we put $\ell_m = |\sin m|$, and in the case of Y let $\ell_m = \big|\sin(m + 1/2)\big|$. For the distances on X and Y we take the corresponding internal metrics. Notice that the spaces X and Y can be considered as graphs whose points $(m, 0)$ correspond to the vertices of degree 3, and the points (m, ℓ_m) to the vertices of degree 1.

It is easy to see that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{Z}$ it holds

$$\Big|\big|\sin(m + n)\big| - \big|\sin(m + 1/2)\big|\Big| < \varepsilon.$$

Therefore, the subset X can be shifted along the abscissa by the vector $(n, 0)$ in such a way that the resulting subset X', which is isometric to X, satisfies $d_H(X', Y) < \varepsilon$. Thus, $d_{GH}(X, Y) = 0$.

On the other hand, it is easy to see that the sets $\big\{|\sin m|\big\}_{m \in \mathbb{Z}}$ and $\Big\{\big|\sin(m+1/2)\big|\Big\}_{m \in \mathbb{Z}}$ do not intersect. Suppose that X and Y are isometric, and $f\colon X \to Y$ is an isometry. Since each isometry is a homeomorphism, then f takes the vertices of degree 1 and 3 to the vertices of the same degree, thus each segment in X between $(m, 0)$ and $\big(m, |\sin m|\big)$ is mapped by f onto a segment of Y between some $(n, 0)$ and $\big(n, |\sin(n + 1/2)|\big)$. However, since f is an isometry, it has to preserve the lengths of these segments, a contradiction.

As follows from Theorem 6.4, the Gromov–Hausdorff distance between X and Y measures the least "discrepancy" for all possible "alignments" of these spaces inside metric spaces constructed on $X \sqcup Y$. A natural question arises: can all these alignments, perhaps for some special classes of metric spaces, be realized inside the same ambient metric space? The following result answers the question for the class of separable spaces.

Consider the metric space ℓ_∞ of all bounded sequences introduced in Example 2.10. Recall that, by Theorem 2.26, each separable metric space is isometrically embedded into ℓ_∞.

Proposition 6.9. *Let X and Y be separable metric spaces. Then*

$$d_{GH}(X, Y) = \inf d_H^{\ell_\infty}\big(\varphi(X), \psi(Y)\big),$$

where the infimum is taken over all isometric embeddings $\varphi\colon X \to \ell_\infty$ and $\psi\colon Y \to \ell_\infty$.

Proof. The space $X \sqcup Y$ with an admissible metric $d \in \mathcal{D}(X,Y)$ is also separable, therefore, by Theorem 2.26, it can be isometrically embedded into ℓ_∞. □

For specific calculations of the Gromov–Hausdorff distance, other equivalent definitions of this distance are useful.

Recall that *a relation* between sets X and Y is a subset of the Cartesian product $X \times Y$. Recall that $\mathcal{P}_0(X \times Y)$ denotes the set of all nonempty subsets of $X \times Y$, i.e., the set of all nonempty relations between X and Y. Similarly to the case of mappings, for each $\sigma \in \mathcal{P}_0(X \times Y)$ and for every $x \in X$ and $y \in Y$, there are defined *the image* $\sigma(x) := \big\{y \in Y : (x,y) \in \sigma\big\}$ and *the preimage* $\sigma^{-1}(y) = \big\{x \in X : (x,y) \in \sigma\big\}$. Also, for $A \subset X$ and $B \subset Y$ their *image* and *preimage* are defined as the union of the images and preimages of their elements, respectively.

Let $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ be the canonical projections $\pi_X(x,y) = x$ and $\pi_Y(x,y) = y$. We denote in the same way the restrictions of these mappings to each relation $\sigma \subset X \times Y$. A relation R between X and Y is called *a correspondence* if the restrictions of the canonical projections π_X and π_Y to R are surjective. In other words, for every $x \in X$ there exists $y \in Y$, and for every $y \in Y$ there exists $x \in X$, such that $(x,y) \in R$. Thus, the correspondence can be considered as a surjective multivalued mapping. The set of all correspondences between X and Y is denoted by $\mathcal{R}(X,Y)$.

If X and Y are metric spaces, then for each relation $\sigma \in \mathcal{P}_0(X \times Y)$ we define its *distortion* $\operatorname{dis}\sigma$ as follows

$$\operatorname{dis}\sigma = \sup\Big\{\big||xx'| - |yy'|\big| : (x,y),\ (x',y') \in \sigma\Big\}.$$

Problem 6.4. Prove that for any $\sigma_1, \sigma_2 \in \mathcal{P}_0(X \times Y)$ such that $\sigma_1 \subset \sigma_2$, we have $\operatorname{dis}\sigma_1 \le \operatorname{dis}\sigma_2$.

Problem 6.5. Prove that for $R \in \mathcal{R}(X,Y)$ it holds $\operatorname{dis}R = 0$ if and only if R is an isometry.

The next two constructions establish a link between correspondences from $\mathcal{R}(X,Y)$ and admissible metrics on $X \sqcup Y$.

To start with, we consider an arbitrary admissible metric and use it to construct a specific correspondence.

Let $\rho \in \mathcal{D}(X,Y)$ be an arbitrary admissible metric for metric spaces X and Y, and suppose that $\rho_H(X,Y) < \infty$. Choose arbitrary $r \ge \rho_H(X,Y)$ for which the set $R^\rho_r = \{(x,y) : \rho(x,y) \le r\}$ is a correspondence between X and Y. Notice that we always can take arbitrary $r > \rho_H(X,Y)$. Sometimes, for example, for compact X and Y, we can take $r = \rho_H(X,Y)$.

Proposition 6.10. *Under above notations, it holds* $\operatorname{dis}R^\rho_r \le 2r$.

Proof. For any (x,y), $(x',y') \in R^\rho_r$ we have

$$\big||xx'| - |yy'|\big| = \big|\rho(x,x') - \rho(y,y')\big| \le \rho(x,y) + \rho(x',y') \le 2r.$$

It remains to pass to supremum in definition of $\operatorname{dis}R^\rho_r$. □

Now we start from a correspondence and construct a specific admissible metric.

Consider arbitrary correspondence $R \in \mathcal{R}(X,Y)$. Suppose that $\operatorname{dis}R < \infty$. Extend the metrics of X and Y upto a symmetric function ρ^R defined on pairs of points from $X \sqcup Y$: for $x \in X$ and $y \in Y$ put

$$\rho^R(x,y) = \rho^R(y,x) = \inf\big\{|xx'| + |yy'| + \frac{1}{2}\operatorname{dis}R : (x',y') \in R\big\}.$$

Proposition 6.11. *Under above notations, let* $\operatorname{dis}R > 0$*, then* ρ^R *is an admissible metric, and* $\rho^R_H(X,Y) = \frac{1}{2}\operatorname{dis}R$.

Proof. To simplify notation, we put $\rho := \rho^R$.

Since $\operatorname{dis}R > 0$ then ρ is positively defined. Now we verify the triangle inequality. It suffices to consider the case $x_1, x_2 \in X$, $y \in Y$ and to prove the inequalities for the triangle x_1x_2y. Due to symmetry reasons, we prove only two inequalities $\rho(x_1,y) + \rho(x_2,y) \ge |x_1x_2|$ and $\rho(x_2,y) + |x_1x_2| \ge \rho(x_1,y)$.

Let us start from the first one. Choose arbitrary (x_1',y_1'), $(x_2',y_2') \in R$, then

$$|x_1x_1'| + |y_1'y| + |x_2x_2'| + |y_2'y| + \operatorname{dis}R \ge |x_1x_2| - |x_1'x_2'| + |y_1'y_2'| + \operatorname{dis}R \ge |x_1x_2|,$$

where the last inequality holds because $|y_1'y_2'| - |x_1'x_2'| \ge -\operatorname{dis}R$ by definition. Thus,

$$\rho(x_1,y) + \rho(x_2,y) = \inf\big\{|x_1x_1'| + |y_1'y| + |x_2x_2'| + |y_2'y| + \operatorname{dis}R : (x_1',y_1'),\ (x_2',y_2') \in R\big\} \ge |x_1x_2|.$$

Now we prove the second inequality. We have

$$\rho(x_2,y)+|x_1x_2| = \inf\bigl\{|x_2x_2'|+|y_2'y|+\frac12\operatorname{dis}R+|x_1x_2| : (x_2',y_2')\in R\bigr\} \ge$$
$$\ge \inf\bigl\{|x_1x_2'|+|y_2'y|+\frac12\operatorname{dis}R : (x_2',y_2')\in R\bigr\} = \rho(x_1,y).$$

It remains to prove that $\rho_H(X,Y)=\frac12\operatorname{dis}R$. Since $R\in\mathcal{R}(X,Y)$ then for each $x\in X$ there exists $y\in Y$, and for each $y\in Y$ there exists $x\in X$, such that $(x,y)\in R$ and, thus, $\rho(x,y)=\frac12\operatorname{dis}R$, therefore $\rho_H(X,Y)\le\frac12\operatorname{dis}R$. Besides that, for any $x\in X$ and $y\in Y$ it holds $\rho(x,y)\ge\frac12\operatorname{dis}R$, thus $\rho_H(X,Y)\ge\frac12\operatorname{dis}R$, and the proof is complete. □

Theorem 6.12. *For any metric spaces X and Y we have*

$$d_{GH}(X,Y)=\frac12\inf\bigl\{\operatorname{dis}R : R\in\mathcal{R}(X,Y)\bigr\}.$$

Proof. Denote by $I(X,Y)$ the right-hand side of the equality from the statement of the theorem. We prove first that $d_{GH}(X,Y)\ge I(X,Y)$. If $d_{GH}(X,Y)=\infty$, then the inequality holds.

Now, suppose that $d_{GH}(X,Y)<\infty$. Choose an arbitrary $r>d_{GH}(X,Y)$, then, by Theorem 6.4, there exists $\rho\in\mathcal{D}(X,Y)$ for which $\rho_H(X,Y)<r$. Consider the correspondence R_r^ρ constructed above. Then, by Proposition 6.10, we have $\operatorname{dis}R_r^\rho\le 2r$, thus $I(X,Y)\le r$. Since r is arbitrary, we get $I(X,Y)\le d_{GH}(X,Y)$.

We now prove that $d_{GH}(X,Y)\le I(X,Y)$. If $I(X,Y)=\infty$, then the inequality holds.

Now, suppose that $I(X,Y)<\infty$. If there exists $R\in\mathcal{R}(X,Y)$ such that $\operatorname{dis}R=0$, then, by Problem 6.5, we have $d_{GH}(X,Y)=0$, and the equality holds. Now suppose that for all $R\in\mathcal{R}(X,Y)$ we have $\operatorname{dis}R>0$. Choose an arbitrary $R\in\mathcal{R}(X,Y)$ such that $\operatorname{dis}R<\infty$ and consider $\rho^R\in\mathcal{D}(X,Y)$ constructed above. By Proposition 6.11, we have $\rho_H^R(X,Y)=\frac12\operatorname{dis}R$, thus

$$d_{GH}(X,Y)=\inf_{\rho\in\mathcal{D}(X,Y)}\rho_H(X,Y)\le\inf\bigl\{\rho_H^R(X,Y) : R\in\mathcal{R}(X,Y),\ \operatorname{dis}R<\infty\bigr\}=I(X,Y).$$

□

Recall that for a relation σ between X and Y, and a relation θ between Y and Z, *the composition* $\theta\circ\sigma$ is defined by the following condition: $(x,z)\in\theta\circ\sigma$ if and only if there exists $y\in Y$ such that $(x,y)\in\sigma$ and $(y,z)\in\theta$.

Problem 6.6. Let X, Y, and Z be metric spaces, $R_1\in\mathcal{R}(X,Y)$, $R_2\in\mathcal{R}(Y,Z)$. Prove that

(1) $R_2\circ R_1\in\mathcal{R}(X,Z)$;

(2) $\operatorname{dis}(R_2\circ R_1)\le\operatorname{dis}R_1+\operatorname{dis}R_2$;

(3) derive from the previous items the triangle inequality for the Gromov–Hausdorff distance.

Solution. (1) Let x be an arbitrary point in X. Since R_1 is a correspondence, there exists $y\in Y$ such that $(x,y)\in R_1$. Since R_2 is a correspondence, there exists $z\in Y$ such that $(y,z)\in R_2$. Therefore, $(x,z)\in R_2\circ R_1$. Similarly, for any $z\in Z$ there exists $x\in X$ such that $(x,z)\in R_2\circ R_1$. Thus, $R_2\circ R_1$ is a correspondence.

(2) For any (x,z), $(x',z')\in R_2\circ R_1$ there exist y, $y'\in Y$ such that (x,y), $(x',y')\in R_1$ and (y,z), $(y',z')\in R_2$, hence

$$\bigl||zz'|-|xx'|\bigr|=\bigl||zz'|-|yy'|+|yy'|-|xx'|\bigr|\le\bigl||zz'|-|yy'|\bigr|+\bigl||yy'|-|xx'|\bigr|,$$

and, passing to the suprema, we obtain what is required.

(3) Denote by $R(X,Y,Z)$ the subset of $R(X,Z)$ consisting of all correspondences represented in the form $R_2\circ R_1$, where $R_1\in R(X,Y)$ and $R_2\in R(Y,Z)$. Then, by Theorem 6.12 and the previous item, we have

$$d_{GH}(X,Z)=\frac12\inf_{R\in\mathcal{R}(X,Z)}\operatorname{dis}R\le\frac12\inf_{R\in\mathcal{R}(X,Y,Z)}\operatorname{dis}R=\frac12\inf_{\substack{R_1\in\mathcal{R}(X,Y)\\ R_2\in\mathcal{R}(Y,Z)}}\operatorname{dis}(R_2\circ R_1)\le$$
$$\le\frac12\inf_{\substack{R_1\in\mathcal{R}(X,Y)\\ R_2\in\mathcal{R}(Y,Z)}}(\operatorname{dis}R_1+\operatorname{dis}R_2)\le\frac12\inf_{R_1\in\mathcal{R}(X,Y)}\operatorname{dis}R_1+\frac12\inf_{R_2\in\mathcal{R}(Y,Z)}\operatorname{dis}R_2=d_{GH}(X,Y)+d_{GH}(Y,Z),$$

as required.

We show one more approach to the study of the Gromov–Hausdorff distance.

Definition 6.13. A mapping $f\colon X\to Y$ of metric spaces is called *an ε-isometry* if $\operatorname{dis} f<\varepsilon$ and $f(X)$ is an ε-net in Y.

Theorem 6.14. *Let X and Y be arbitrary metric spaces and $\varepsilon>0$. Then*

(1) *if $d_{GH}(X,Y)<\varepsilon$ then there exists a (2ε)-isometry $f\colon X\to Y$;*

(2) *if there exists an ε-isometry $f\colon X\to Y$ then it holds $d_{GH}(X,Y)<2\varepsilon$.*

Proof. (1) By Theorem 6.12, there exists a relation $R\in\mathcal{R}(X,Y)$ such that $\operatorname{dis} R<2\varepsilon$. For each $x\in X$ we choose an arbitrary $y\in R(x)$ and put $f(x)=y$. Thus, we determined a mapping $f\colon X\to Y$, and since $f\subset R$ (here we identify f with its graph), by Problem 6.4 it holds $\operatorname{dis} f\le\operatorname{dis} R<2\varepsilon$. Now we choose an arbitrary $y'\in Y$, an arbitrary $x\in R^{-1}(y')$, and let $y=f(x)$. Since $\operatorname{diam} R(x)\le\operatorname{dis} R<2\varepsilon$, it follows that $|yy'|<2\varepsilon$, therefore $f(X)$ is a (2ε)-net.

(2) Consider the relation $R=\big\{(x,y):|f(x)y|<\varepsilon\big\}$. Since $f(X)$ is an ε-net, R is a correspondence. To estimate the distortion of R, we choose arbitrary $(x,y),(x',y')\in R$, then

$$\big||xx'|-|yy'|\big|\le\Big||xx'|-\big|f(x)f(x')\big|\Big|+\Big|\big|f(x)f(x')\big|-|yy'|\Big|\le\varepsilon+\big|f(x)y\big|+\big|y'f(x')\big|<3\varepsilon,$$

therefore, $\operatorname{dis} R\le3\varepsilon<4\varepsilon$ and $d_{GH}(X,Y)\le\frac12\operatorname{dis} R<2\varepsilon$. □

6.1 Irreducible correspondences

For arbitrary nonempty sets X and Y, a correspondence $R\in\mathcal{R}(X,Y)$ is called *irreducible* if it is a minimal element of the set $\mathcal{R}(X,Y)$ w.r.t. the order given by the inclusion relation. The set of all irreducible correspondences between X and Y is denoted by $\mathcal{R}^0(X,Y)$.

The following result is evident.

Proposition 6.15. *A correspondence $R\in\mathcal{R}(X,Y)$ is irreducible if and only if for any $(x,y)\in R$ it holds*

$$\min\big\{\#R(x),\#R^{-1}(y)\big\}=1.$$

Theorem 6.16. *Let X, Y be arbitrary nonempty sets. Then for every $R\in\mathcal{R}(X,Y)$ there exists $R^0\in\mathcal{R}^0(X,Y)$ such that $R^0\subset R$. In particular, $\mathcal{R}^0(X,Y)\ne\emptyset$.*

Remark 6.17. One could use the standard technique based on the Zorn lemma, but then we need to guarantee that each chain $R_1\supset R_2\supset\cdots$ has a lower bound, i.e., there exists a correspondence that belongs to all R_i. However, this, generally speaking, is not true. As an example, consider $X=Y=\mathbb{N}$ and set $R_k=\big\{(i,j):\max(i,j)\ge k\big\}$. It is clear that every R_k belongs to $\mathcal{R}(X,Y)$, and that these R_k form a decreasing chain. However, $\cap R_k=\emptyset$, since for any i and j there is k for which $i<k$ and $j<k$, therefore $(i,j)\notin R_k$.

Proof of Theorem 6.16. For each $x\in X$, choose an arbitrary $y\in R(x)$ and define a mapping $f\colon X\to Y$ by setting $y=f(x)$. Let us note that $f\subset R$. Put $Y_1=f(X)$ and $Y_2=Y\setminus Y_1$.

Now for each $y\in Y_2$ we choose an arbitrary $x\in R^{-1}(y)$ and define a mapping $g\colon Y_1\to X$ by setting $x=g(y)$. In this case we have $g^{-1}\subset R$. Put $X_2=g(Y_2)$ and $X_1=X\setminus X_2$.

Let $Y_3=f(X_2)$. It is clear that $Y_3\subset Y_1$.

Using f and g, we define another relation: $h=f\cup g^{-1}$.

Lemma 6.18. *We have $h\in\mathcal{R}(X,Y)$.*

Proof. By the definition of f, for each $x\in X$ it holds $\big(x,f(x)\big)\in f\subset h$.

Now consider an arbitrary $y\in Y$. If $y\in Y_1$, then, since $Y_1=\operatorname{im} f$, there exists $x\in X$ such that $y=f(x)$, therefore $(x,y)\in f\subset h$. If $y\in Y_2$, then, by the definition of g, we have $\big(g(y),y\big)\in g^{-1}\subset h$. □

Now we define the relation R^0 by removing from h some (x,y) for each $y\in Y_3$ according to the following rule:

(1) if $h^{-1}(y)\cap X_1\ne\emptyset$, then we remove $\big(h^{-1}(y)\cap X_2\big)\times\{y\}$;

(2) if $h^{-1}(y)\cap X_1=\emptyset$, i.e., $h^{-1}(y)\subset X_2$, then we remove all the elements from $h^{-1}(y)\times\{y\}$, except any one.

Lemma 6.19. *We have $R^0 \in \mathcal{R}(X,Y)$.*

Proof. For every $y \in Y \setminus Y_3$ we remove nothing, therefore for such y there always exists $x \in X$ such that $(x,y) \in R^0$.

Now let $y \in Y_3$. If $h^{-1}(y) \cap X_1 \ne \emptyset$, then we do not remove (x,y) with $x \in h^{-1}(y) \cap X_1$, thus $(x,y) \in R^0$ for such x. If $h^{-1}(y) \cap X_1 = \emptyset$, then we do not remove some $x \in h^{-1}(y)$, thus $(x,y) \in R^0$ for such x.

Now let us deal with $x \in X$. If $x \in X_1$, then we remove nothing, therefore $(x,y) \in R^0$ for some $y \in Y$. If $x \in X_2$, then, since $X_2 = \operatorname{im} g$, there exists $y \in Y_2$ for which $x = g(y)$, but we remove nothing for such y, thus $(x,y) \in R^0$. □

Lemma 6.20. *We have $R^0 \in \mathcal{R}^0(X,Y)$.*

Proof. It is sufficient to show that for each pair $(x,y) \in R^0$ either x, or y do not belong to other pairs.

If $y \in Y_2$, then y is included in the only pair $\big(g(y),y\big) \in R^0$. If $y \in Y_1 \setminus Y_3$, then $y = f(x)$ for some $x \in X_1$, however such x is included in the only pair $\big(x, f(x)\big) \in R^0$.

Finally, now let $y \in Y_3$. If $h^{-1}(y) \cap X_1 \ne \emptyset$, then we removed all the pairs of the form (x',y), $x' \in X_2$, so $x \in X_1$, however, each such x is included in exactly one pair, namely, in $\big(x, f(x)\big)$. If $h^{-1}(y) \cap X_1 = \emptyset$, then we removed all the pairs (x',y), $x' \in h^{-1}(y) \subset X_2$, except some one, so y is included in exactly one such pair. □

This lemma completes the proof of the theorem. □

Theorems 6.16 and 6.12, together with Problem 6.4, implies

Corollary 6.21. *For any metric spaces X and Y we have*

$$d_{GH}(X,Y) = \frac{1}{2} \inf\big\{\operatorname{dis} R \mid R \in \mathcal{R}^0(X,Y)\big\}.$$

Now we give another useful description of irreducible.

Proposition 6.22. *For any nonempty sets X, Y, and each $R \in \mathcal{R}^0(X,Y)$, there exist and unique partitions $R_X = \{X_i\}_{i\in I}$ and $R_Y = \{Y_i\}_{i\in I}$ of the sets X and Y, respectively, such that $R = \cup_{i\in I} X_i \times Y_i$. Moreover, $R_X = \cup_{y\in Y}\big\{R^{-1}(y)\big\}$, $R_Y := \cup_{x\in X}\big\{R(x)\big\}$,*

$$\{X_i \times Y_i\}_{i\in I} = \cup_{(x,y)\in R}\{R^{-1}(y) \times R(x)\},$$

and for each i it holds $\min\{\#X_i, \#Y_i\} = 1$.

Conversely, each set $R = \cup_{i\in I} X_i \times Y_i$, where $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are partitions of nonempty sets X and Y, respectively, such that for each i it holds $\min\{\#X_i, \#Y_i\} = 1$, is an irreducible correspondence between X and Y.

Proof. First, let $R \in \mathcal{R}^0(X,Y)$. Put $R_X = \cup_{y\in Y}\big\{R^{-1}(y)\big\}$, $R_Y = \cup_{x\in X}\big\{R(x)\big\}$, and show that R_X and R_Y are partitions. We show it for R_Y (for R_X the proof is the same).

Notice first that R_X and R_Y are covers because R is a correspondence. Now, suppose to the contrary that R_Y is not a partition, i.e., there exist two different elements of R_Y, say $R(x)$ and $R(x')$, such that $R(x) \cap R(x') \ne \emptyset$. Since $R(x) \ne R(x')$, one of them, say $R(x)$, contains two different elements y and y', such that one of them, say y, belongs to $R(x) \cap R(x')$. This means that for the pair (x,y) it holds $\#R(x) \ge 2$ and $R^{-1}(y) \ge 2$, a contradiction to Proposition 6.15.

Now, we put $R_X = \{X_i\}_{i\in I}$. Note that for any $x, x' \in X_i$ we have $R(x) = R(x')$. Indeed, by definition, $X_i = R^{-1}(y)$ for some $y \in Y$, thus $R(x) \cap R(x') \ne \emptyset$ and, therefore, $R(x) = R(x')$ because R_Y is a partition.

Choose arbitrary $i \in I$, $x \in X_i$, and put $Y_i = R(x)$. Then, this definition is correct (does not depend on the choice of x). We show that the mapping $\varphi\colon X_i \mapsto Y_i$ is a bijection between R_X and R_Y.

If φ is not injective, then there exist $x, x' \in X$ lying in different elements of the partition R_X for which $R(x) = R(x')$. Thus $x, x' \in R^{-1}(y) \in R_X$ for $y \in R(x)$, a contradiction.

Finally, φ is surjective, since for any Y_i, $y \in Y_i$, the set $R^{-1}(y)$ is an element of the partition R_X. Choose an arbitrary $x \in R^{-1}(y)$, then $R(x) \in R_Y$ contains y, therefore $\varphi\big(R^{-1}(y)\big) = Y_i$.

Since for any $x, x' \in X_i$ we have $R(x) = R(x') = Y_i$, then $X_i \times Y_i \subset R$. On the other hand, since R_X is a partition of X, for any $x \in X$ there exists $X_i \in R_X$ such that $x \in X_i$, therefore each $(x,y) \in R$ is contained in some $X_i \times Y_i$.

To prove the equality $\min\{\#X_i, \#Y_i\} = 1$, let us note that for any $(x,y) \in R$, $x \in X_i$, and $y \in Y_i$, we have $X_i = R^{-1}(y)$ and $Y_i = R(x)$, thus, $\{X_i \times Y_i\}_{i\in I} = \cup_{(x,y)\in R}\{R^{-1}(y) \times R(x)\}$. It suffices to apply Proposition 6.15.

The uniqueness of the partitions is the standard fact from the set theory.

The converse is trivial. □

Let X be an arbitrary set different from singleton, and m a cardinal number, $2 \le m \le \#X$. By $\mathcal{D}_m(X)$ we denote the family of all possible partitions of the set X into m nonempty subsets.

Now let X be a metric space. Then for each $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ we put

$$\operatorname{diam} D = \sup_{i\in I} \operatorname{diam} X_i.$$

Further, for any nonempty $A, B \subset X$, we have already defined $|AB|$ as $\inf\bigl\{|ab| : (a,b) \in A \times B\bigr\}$. We also need $|AB|' := \sup\bigl\{|ab| : (a,b) \in A \times B\bigr\}$. Further, for each $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ we put

$$\alpha(D) = \inf\bigl\{|X_iX_j| : i \ne j\bigr\} \quad \text{and} \quad \beta(D) = \sup\bigl\{|X_iX_j|' : i \ne j\bigr\}.$$

Also notice that $|X_iX_i| = 0$, $|X_iX_i|' = \operatorname{diam} X_i$ and hence $\operatorname{diam} D = \sup_{i\in I} |X_iX_i|'$.

The next result follows easily from the definition of distortion, as well as from Proposition 6.22.

Proposition 6.23. *Let X and Y be arbitrary metric spaces, $D_X = \{X_i\}_{i\in I}$, $D_Y = \{Y_i\}_{i\in I}$, $\#I \ge 2$, be some partitions of the spaces X and Y, respectively, and $R = \cup_{i\in I} X_i \times Y_i \in \mathcal{R}(X,Y)$. Then*

$$\begin{aligned}\operatorname{dis} R = \sup\bigl\{|X_iX_j|' - |Y_iY_j|,\, |Y_iY_j|' - |X_iX_j| : i,j \in I\bigr\} = \\ = \sup\bigl\{\operatorname{diam} D_X,\, \operatorname{diam} D_Y,\, |X_iX_j|' - |Y_iY_j|,\, |Y_iY_j|' - |X_iX_j| : i,j \in I,\, i \ne j\bigr\} \le \\ \le \max\bigl\{\operatorname{diam} D_X, \operatorname{diam} D_Y, \beta(D_X) - \alpha(D_Y), \beta(D_Y) - \alpha(D_X)\bigr\}.\end{aligned}$$

In particular, if $R \in \mathcal{R}^0(X,Y)$, then in the previous formula we can take R_X and R_Y defined in Proposition 6.22 for D_X and D_Y, respectively.

It will also be convenient for us to represent a relation $\sigma \in \mathcal{P}_0(X \times Y)$ as a bipartite graph. Then the degree deg of each vertex is defined: $\deg_\sigma(x) = \#\sigma(x)$ and $\deg_\sigma(y) = \#\sigma^{-1}(y)$.

Problem 6.7. Let $R \in \mathcal{R}^0(X,Y)$, $x \in X$, $\deg_R(x) > 1$. Prove that for each $x' \in X$, $x' \ne x$, it holds $R(x) \cap R(x') = \emptyset$.

Problem 6.7 immediately implies

Corollary 6.24. *Let $\#X \ge 2$ and $\#Y \ge 2$, then for any $R \in \mathcal{R}^0(X,Y)$ there is no $x \in X$ such that $\{x\} \times Y \subset R$.*

Example 6.25. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, then, by Corollary 6.24, the set $\mathcal{R}^0(X,Y)$ consists only of bijections, therefore

$$2d_{GH}(X,Y) = \bigl||x_1x_2| - |y_1y_2|\bigr|.$$

Thus, the set of isometric classes of two-point metric spaces endowed with the Gromov–Hausdorff distance is a metric space isometric to the open ray $x > 0$ of the real line $\mathbb{R}$ with coordinate x.

Example 6.26. Consider two three-point metric spaces $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Put $\rho_{ij} = |x_ix_j| =: \rho_k$ and $\nu_{ij} = |y_iy_j| =: \nu_k$, where $\{i,j,k\} = \{1,2,3\}$. Without loss of generality, we assume that $\rho_1 \le \rho_2 \le \rho_3$ and $\nu_1 \le \nu_2 \le \nu_3$. We will show that

$$d_{GH}(X,Y) = \frac{1}{2}\max\bigl\{|\rho_1 - \nu_1|, |\rho_2 - \nu_2|, |\rho_3 - \nu_3|\bigr\}.$$

By Corollary 6.21, it suffices to describe all irreducible correspondences $R_m \in \mathcal{R}^0(X,Y)$. We will use Proposition 6.22. There are three types of partitions of a three-point space: (1) into one-point subsets, (2) into one two-point subset and one-point subset, and (3) into one three-point subset. By Proposition 6.22, an irreducible correspondence defines a bijection between partitions, therefore both spaces have to be partitioned in the same way. Case (3) is not realized by Corollary 6.24, in case (1) we have a bijection, and in case (2) one-point and two-point subsets have to correspond to each other. Thus, we have two types of correspondences: if $\{i,j,k\} = \{p,q,r\} = \{1,2,3\}$, then

(1) $R_1 = \bigl\{(x_i, y_p), (x_j, y_q), (x_k, y_r)\bigr\}$,

(2) $R_2 = \bigl\{(x_i, y_q), (x_i, y_r), (x_j, y_p), (x_k, y_p)\bigr\}$.

We have

$$\operatorname{dis} R_1 = \max\bigl\{|\rho_{ij}-\nu_{pq}|, |\rho_{ik}-\nu_{pr}|, |\rho_{jk}-\nu_{qr}|\bigr\},$$
$$\operatorname{dis} R_2 = \max\bigl\{|\rho_{ij}-\nu_{pq}|, |\rho_{ik}-\nu_{pr}|, \rho_{jk}, \nu_{qr}, |\rho_{ij}-\nu_{pr}|, |\rho_{ik}-\nu_{pq}|\bigr\}.$$

Note that $\max\{\rho_{jk},\nu_{qr}\} \ge |\rho_{jk}-\nu_{qr}|$, therefore $\operatorname{dis} R_2 \ge \operatorname{dis} R_1$, so it is enough to consider only the bijections R_1.

We also note that each bijection R_1 defines a bijection between the sets $\{\rho_a\}$ and $\{\nu_a\}$. We show that the bijection $\cup_{i=1}^3\bigl\{(\rho_i,\nu_i)\bigr\}$ is optimal, i.e., if we replace this bijection with any other bijection ψ, we cannot get a lower value than $M = \max_i|\rho_i-\nu_i|$. Put $M' = \max_i\bigl|\rho_i-\psi(\rho_i)\bigr|$. We need to show that $M' \ge M$.

Let $M = |\rho_1-\nu_1|$, and suppose, without loss of generality, that $\rho_1 \le \nu_1$, thus $M = \nu_1-\rho_1$. Since $\psi(\rho_1) \ge \nu_1$, then $M' \ge \psi(\rho_1)-\rho_1 \ge M$. Similarly we deal with the case $M = |\rho_3-\nu_3|$.

Now, let $M = |\rho_2-\nu_2|$, and $\rho_2 \le \nu_2$, i.e., $M = \nu_2-\rho_2$. If $\psi(\rho_2) \ne \nu_1$, them $M' \ge M$. Suppose now that $\psi(\rho_2) = \nu_1$, then $\psi(\rho_1)$ equal either ν_2, or ν_3. Thus $M' \ge \psi(\rho_1)-\rho_1 \ge \nu_2-\rho_1 \ge \nu_2-\rho_2 = M$.

Thus, the set of isometry classes of three-point metric spaces endowed with the Gromov–Hausdorff distance is a metric space isometric to the polyhedral cone

$$\bigl\{(x,y,z) : 0 < x \le y \le z \le x+y\bigr\}$$

in the space $\mathbb{R}^3_\infty$, where the latter denotes the space $\mathbb{R}^3$ with the metric generated by the norm $\bigl\|(x,y,z)\bigr\|_\infty = \max\bigl\{|x|,|y|,|z|\bigr\}$. The isometry is given by the formula

$$\{x_1,x_2,x_3\} \mapsto \frac{1}{2}(\rho_1,\rho_2,\rho_3).$$

6.2 A few more examples

The following statement immediately follows from the definition of the Gromov–Hausdorff distance and Item (4) of Problem 5.1.

Example 6.27. Let Y be an arbitrary ε-net of a metric space X. Then $d_{GH}(X,Y) \le d_H(X,Y) \le \varepsilon$. Thus, every compact metric space is approximated (according to the Gromov-Hausdorff metric) with any accuracy by finite metric spaces.

Example 6.28. Denote by Δ_1 a one-point metric space. Then for any metric space X we have

$$d_{GH}(\Delta_1,X) = \frac{1}{2}\operatorname{diam} X.$$

Indeed, $\mathcal{R}(\Delta_1,X)$ consists of exactly one correspondence R, namely, of $R = \Delta_1\times X$, thus, $\operatorname{dis} R = \operatorname{diam} X$, and it remains to use Theorem 6.12.

Example 6.29. Let X and Y be some metric spaces, and the diameter of one of them is finite. Then

$$d_{GH}(X,Y) \ge \frac{1}{2}|\operatorname{diam} X - \operatorname{diam} Y|.$$

Indeed, if $\operatorname{diam} X < \infty$ and $\operatorname{diam} Y = \infty$, then for any $\rho \in \mathcal{D}(X,Y)$ we have $\rho_H(X,Y) = \infty$, otherwise $Y \subset U_r(X)$ for some finite r and, therefore, $\operatorname{diam} Y < \infty$.

Now let the both $\operatorname{diam} X$ and $\operatorname{diam} Y$ are finite. Then it suffices to use the triangle inequality (Proposition 6.6) for the triple X, Y, Δ_1, and Example 6.28.

Example 6.30. Let X and Y be some metric spaces, then

$$d_{GH}(X,Y) \le \frac{1}{2}\max\{\operatorname{diam} X, \operatorname{diam} Y\},$$

in particular, if X and Y are bounded metric spaces, then $d_{GH}(X,Y) < \infty$.

Indeed, if the diameter of one of the spaces X, Y is infinite, then the inequality holds. If both spaces are singletons, then everything is also obvious. Now let $0 < d := \max\{\operatorname{diam} X, \operatorname{diam} Y\} < \infty$. Then for $R = X\times Y \in \mathcal{R}(X,Y)$ we have $\operatorname{dis} R = d$, thus, by Proposition 6.11, it holds $\rho_H^R(X,Y) = \frac{1}{2}\operatorname{dis} R = d/2$, therefore, $d_{GH}(X,Y) \le d/2$.

Recall that for an arbitrary metric space X and a real number $\lambda > 0$, by λX we denote the metric space obtained from X by multiplying all distances by λ. For $\lambda = 0$ we set $\lambda X = \Delta_1$.

Example 6.31. For any bounded metric space X and any $\lambda \geq 0$, $\mu \geq 0$, we have $d_{GH}(\lambda X, \mu X) = \frac{1}{2}|\lambda - \mu| \operatorname{diam} X$, in particularly, for any $0 \leq a < b$ the curve $\gamma(t) := t\,X$, $t \in [a, b]$, is shortest.

Indeed, for the identity correspondence $R \in \mathcal{R}(\lambda X, \mu X)$ we have $\operatorname{dis} R = |\lambda - \mu| \operatorname{diam} X$, hence $d_{GH}(\lambda X, \mu X) \leq \frac{1}{2}|\lambda - \mu| \operatorname{diam} X$ by Theorem 6.12. On the other hand, by Example 6.29 we have

$$d_{GH}(\lambda X, \mu X) \geq \frac{1}{2}\big|\operatorname{diam}(\lambda X) - \operatorname{diam}(\mu X)\big| = \frac{1}{2}|\lambda - \mu| \operatorname{diam} X.$$

It remains to note that

$$|\gamma| = \sup_{a=t_0<\cdots<t_n=b} \sum_{i=1}^{n} d_{GH}(t_{i-1}\,X,\, t_i\,X) = \frac{1}{2}(b - a) \operatorname{diam} X = d_{GH}(a\,X,\, b\,X),$$

hence γ is shortest.

Example 6.32. Let X and Y be metric spaces, then for any $\lambda > 0$ we have $d_{GH}(\lambda X, \lambda Y) = \lambda\, d_{GH}(X, Y)$. If, in addition, $d_{GH}(X, Y) < \infty$, then the equality holds for all $\lambda \geq 0$.

Indeed, let $\lambda > 0$. Then for each correspondence $R \in \mathcal{R}(X, Y)$ and the correspondence $R_\lambda \in \mathcal{R}(\lambda X, \lambda Y)$, which coincides with R as a set, we have $\operatorname{dis} R_\lambda = \lambda \operatorname{dis} R$. It remains to use Theorem 6.12. We now verify that under the condition $d_{GH}(X, Y) < \infty$ the equality holds for $\lambda = 0$ as well. With this λ, we have $\lambda X = \lambda Y = \Delta_1$, therefore $d_{GH}(\lambda X, \lambda Y) = 0$. Since $d_{GH}(X, Y) < \infty$, the value $\lambda\, d_{GH}(X, Y)$ vanishes too.

6.3 GH-convergence and GH-limits

For a sequence X_k of metric spaces that converges w.r.t. the Gromov–Hausdorff distance to a metric space X, we say for short that the sequence GH-*converges* and write this as $X_k \xrightarrow{\text{GH}} X$. We call X *the* GH-*limit* and denote it by $\text{GH-}\lim_{k\to\infty} X_k$.

Let us present a few simple observations.

(1) Let $X = \text{GH-}\lim_{k\to\infty} X_k$, $Y = \text{GH-}\lim_{k\to\infty} X_k$, one of X and Y be compact, and the other one be complete, then X and Y are isometric. Indeed, due to the triangle inequality, we have $d_{GH}(X, Y) = 0$. It remains to use Problem 6.3.

(2) Hausdorff convergence implies GH-convergence.

(3) Each compact metric space is a GH-limit of finite metric spaces (of its finite $1/k$-nets).

Theorem 6.33. *Let $X_k \xrightarrow{\text{GH}} Y$. If all X_k, starting from some k, have one of the properties listed below, then this property is inherited by the space Y:*

(1) *the diameter equals infinity*;

(2) *the diameter is bounded by a certain number D (in fact,* $\operatorname{diam} X_k \to \operatorname{diam} Y$*)*;

(3) *separability*;

(4) *total boundedness*;

(5) *if Y is complete, then bounded compactness*;

(6) *if Y is complete, then the intrinsicness.*

Proof. Without loss of generality, we assume that each of there properties holds for all k, and that $d_{GH}(X_k, Y) < 1/k$. By Theorem 6.12, for each $k \in \mathbb{N}$ there exists $R_k \in \mathcal{R}(X_k, Y)$ such that $\operatorname{dis} R_k < 2/k$. In addition, by Theorem 6.14, for each $k \in \mathbb{N}$ there exist $(2/k)$-isometries $f_k \colon X_k \to Y$ and $g_k \colon Y \to X_k$.

(1) If $\operatorname{diam} Y < \infty$, then, by Example 6.29, we have $d_{GH}(X_k, Y) = \infty$, a contradiction.

(2) Since g_k is $(2/k)$-isometry, then $\operatorname{dis} g_k < 2/k$ and, hence, for any $y, y' \in Y$ we have $\big||yy'| - |g_k(y)g_k(y')|\big| < 2/k$, thus

$$|yy'| < |g_k(y)g_k(y')| + 2/k \le \operatorname{diam} X_k + 2/k.$$

Since $y, y' \in Y$ are arbitrary, we get $\operatorname{diam} Y \le \operatorname{diam} X_k + 2/k$. Swapping g_k and f_k, we get $\operatorname{diam} X_k \le \operatorname{diam} Y + 2/k$, i.e., $|\operatorname{diam} X_k - \operatorname{diam} Y| < 2/k$, therefore, $\operatorname{diam} X_k \to \operatorname{diam} Y$ and, thus, $\operatorname{diam} Y \le D$.

(3) In each X_k we choose a countable everywhere dense subset S_k. Since f_k is $(2/k)$-isometry, for any $y \in Y$ there exists $x_k \in X_k$ such that $|yf_k(x_k)| < 2/k$. Since S_k is everywhere dense in X_k, there exists $s_k \in S_k$ such that $|x_k s_k| < 1/k$. Now the condition $\operatorname{dis} f_k < 2/k$ implies that $|f_k(s_k)f_k(x_k)| < 3/k$, therefore, $|f_k(s_k)y| < 5/k$, and hence $f_k(S_k)$ is a countable $(5/k)$-net in Y. It remains to note that $\cup_{k=1}^{\infty} f_k(S_k)$ is a countable everywhere dense subset of Y.

(4) In each X_k we choose a finite $(1/k)$-net S_k. Repeating word-by-word the reasoning from Item (3), we obtain that $f_k(S_k)$ is a finite $(5/k)$-net. Since k is arbitrary, we get what is required.

(5) Choose an arbitrary closed bounded $Z \subset Y$. We have to show that it is compact. Since $\operatorname{dis} f_k < 2/k$, then $f_k^{-1}(Z)$ is bounded. Let W be the closure of $f_k^{-1}(Z)$, then W is compact, in particular, it contains a finite $(1/k)$-net S_k. As was discussed above, $f_k(S_k)$ is a finite $(3/k)$-net in $f(W) \supset Z$ and, thus, there exists a finite $(6/k)$-net Z_k in Z. Since k is arbitrary, we get that Z is totally bounded. Since Z is closed, it is compact.

(6) By Theorem 3.51, it suffices to show that for any $y, y' \in Y$ and any $\varepsilon > 0$ there exists an ε-midpoint between y and y'. Choose k such that $4/k < \varepsilon$, and arbitrary $x_k \in R_k^{-1}(y)$, $x'_k \in R_k^{-1}(y')$. Since X_k is intrinsic, there exists a $(1/k)$-midpoint s between x_k and x'_k. Choose arbitrary $w \in R_k(s)$. Since s is $(1/k)$-midpoint, and $\operatorname{dis} R_k < 2/k$, we have $\big||x_k s| - |x_k x'_k|/2\big| < 1/k$, $\big||x'_k s| - |x_k x'_k|/2\big| < 1/k$, $\big||x_k s| - |yw|\big| < 2/k$, $\big||x'_k s| - |y'w|\big| < 2/k$, $\big||x_k x'_k| - |yy|\big| < 2/k$, thus $\big||yw| - |yy'|/2\big| < 4/k < \varepsilon$ and $\big||y'w| - |yy'|/2\big| < 4/k < \varepsilon$, what is required. ☐

Corollary 6.34. *Let $X_k \xrightarrow{\text{GH}} Y$, all X_k are boundedly compact and intrinsic (such X_k are strictly intrinsic by Corollary 3.45), Y is complete. Then Y is strictly intrinsic. Moreover, if we abandon the bounded compactness property, then Y may be not strictly intrinsic, namely, the GH-limit of strictly intrinsic metric spaces may be not strictly intrinsic.*

Proof. By Item (5) and Item (6) of Theorem6.33, the space Y is boundedly compact and intrinsic. It remains to apply Corollary 3.45.

To verify the second statement of the corollary, let us consider a metric graph Y obtained by gluing the ends of the segments $[0, 1 + 1/n]$, $n \in \mathbb{N}$ (all 0 are glued together, and all ends $1 + 1/n$ are also glued together). Then Y is complete, intrinsic, but not strictly intrinsic. For X_k we take the space obtained from Y by replacing the segment $[0, 1+1/k]$ with the segment $[0, 1]$. Then all X_k are complete and strictly intrinsic, however, all they are not boundedly compact. ☐

Problem 6.8. Let $X = \{x^1, \dots, x^n\}$ be a finite metric space. Prove that a sequence X_k of metric spaces GH-converges to X if and only if for sufficiently large k there exist partitions $\{X_k^i\}_{i=1}^n \in \mathcal{D}_n(X_k)$ such that $|X_k^i X_k^j| \to |x^i x^j|$ for any $1 \le i, j \le n$.

Let d_k be a sequence of metrics on a nonempty set X. We say that the metric spaces $X_k = (X, d_k)$ *uniformly converges* to a metric space $Y = (X, d)$ if the functions d_k uniformly converges to d, i.e., if

$$\sup_{x, x' \in X} \big|d_k(x, x') - d(x, x')\big| \to 0.$$

Problem 6.9 (Uniform convergence)**.** Prove that if metric spaces X_k uniformly converges to a metric space Y then $X_k \xrightarrow{\text{GH}} Y$.

Recall that for any Lipschitz mapping $f\colon X \to Y$ of metric spaces we defined the dilatation $\operatorname{dil} f$ as the minimal Lipschitz constant for f, see Section 2.2. Also, we defined *bi-Lipschitz mapping* between metric spaces as a bijective mapping such that it and its inverse as Lipschitz ones. Now we define *the Lipschitz distance* d_L between metric spaces X and Y as the following value:

$$d_L(X, Y) = \inf_{f\colon X \to Y} \log\big(\max\{\operatorname{dil} f, \operatorname{dil} f^{-1}\}\big),$$

where infimum is taken over all bi-Lipschitz mappings (if there is no such mappings then $d_L(X, Y) = \infty$).

Problem 6.10 (Lipschitz convergence)**.** Prove that if metric spaces X_k converges to a bounded metric space Y w.r.t. the Lipschitz distance, then $X_k \xrightarrow{\text{GH}} Y$. Does it remain true without the boundedness assumption?

Problem 6.11. Let the metric space X_k be obtained from the standard sphere $S^2 \subset \mathbb{R}^3$ by removing a ball of radius $1/k$. Prove that $X_k \xrightarrow{\text{GH}} S^2$. Prove that if we change S^2 to the circle S^1, then the similar statement does not hold. In the both cases we consider the interior metrics.

References to Chapter 6

[1] Burago D., Burago Yu., Ivanov S. *A Course in Metric Geometry*. Graduate Studies in Mathematics, vol.33, A.M.S., Providence, RI, 2001.

[2] Ghanaat P. "Gromov-Hausdorff distance and applications". In: Summer school "Metric Geometry", Les Diablerets, August 25–30, 2013, `https://math.cuso.ch/fileadmin/math/document/gromov-hausdorff.pdf`

Exercises to Chapter 6

Exercise 6.1. Prove that for any metric spaces X and Y there exists a realization of (X, Y).

Exercise 6.2. Prove that for any metric spaces X and Y there exists at least one admissible metric, i.e., the set $\mathcal{D}(X, Y)$ is not empty.

Exercise 6.3. Prove that for arbitrary metric spaces X and Y, any $\sigma_1, \sigma_2 \in \mathcal{P}_0(X \times Y)$ such that $\sigma_1 \subset \sigma_2$, we have $\operatorname{dis}\sigma_1 \le \operatorname{dis}\sigma_2$.

Exercise 6.4. Prove that for arbitrary metric spaces X and Y, any $R \in \mathcal{R}(X, Y)$ it holds $\operatorname{dis} R = 0$ if and only if R is an isometry.

Exercise 6.5. Let X and Y be arbitrary sets, $R \in \mathcal{R}^0(X, Y)$, $x \in X$, $\deg_R(x) > 1$. Prove that for each $x' \in X$, $x' \ne x$, it holds $R(x) \cap R(x') = \emptyset$.

Exercise 6.6. Let $X = \{x^1, \dots, x^n\}$ be a finite metric space. Prove that a sequence X_k of metric spaces GH-converges to X if and only if for sufficiently large k there exist partitions $\{X_k^i\}_{i=1}^n \in \mathcal{D}_n(X_k)$ such that $|X_k^i X_k^j| \to |x^i x^j|$ for any $1 \le i,\, j \le n$.

Exercise 6.7 (Uniform convergence)**.** Prove that if metric spaces X_k uniformly converges to a metric space Y then $X_k \xrightarrow{\text{GH}} Y$.

Exercise 6.8 (Lipschitz convergence)**.** Prove that if metric spaces X_k converges to a bounded metric space Y w.r.t. the Lipschitz distance, then $X_k \xrightarrow{\text{GH}} Y$. Does it remain true without the boundedness assumption?

Exercise 6.9. Let the metric space X_k be obtained from the standard sphere $S^2 \subset \mathbb{R}^3$ by removing a ball of radius $1/k$. Prove that $X_k \xrightarrow{\text{GH}} S^2$. Prove that if we change S^2 to the circle S^1, then the similar statement does not hold. In the both cases we consider the interior metrics.