Chapter 5

Hausdorff distance.

Schedule. Hausdorff distance, equivalence of three definitions, triangle inequality for Hausdorff distance, Hausdorff distance is a metric on the set of all closed bounded nonempty subsets, coincidence of Vietoris topology and metric topology generated by Hausdorff distance on the set of all compact subsets, limits theory for nonempty subsets, definition of lim sup and some its properties (equivalent definitions), definition of lim inf and some its properties (equivalent definitions), convergence w.r.t. Hausdorff distance (Hausdorff convergence) and calculating lim inf, Hausdorff convergence of singletons, definition of lim, Hausdorff convergence implies existence of lim, the cases of decreasing and increasing sequences that are Hausdorff converging, equivalence of Hausdorff convergence and existence of lim in compact spaces, the cases of decreasing and increasing sequences in compact spaces, convergence in complete metric spaces, simultaneous completeness (total boundness, compactness) of the original space and the hyperspace of all closed bounded nonempty subsets, inheritance of the property to be geodesic for compact space and the hyperspace of all its closed nonempty subsets.

Let X be an arbitrary metric space. For arbitrary nonempty sets $A, B \subset X$ we put

$$d_H^1(A,B) = \max\Big(\sup\big\{|aB| : a \in A\big\}, \sup\big\{|Ab| : b \in B\big\}\Big), \tag{5.1}$$

$$d_H^2(A,B) = \inf\big\{r \in [0,\infty] : A \subset B_r(B) \ \&\ B_r(A) \supset B\big\}, \tag{5.2}$$

$$d_H^3(A,B) = \inf\big\{r \in [0,\infty] : A \subset U_r(B) \ \&\ U_r(A) \supset B\big\}. \tag{5.3}$$

Proposition 5.1. *For nonempty subsets $A, B \subset X$ of a metric space X we have $d_H^1(A,B) = d_H^2(AB) = d_H^3(A,B)$.*

Proof. Put $r_i = d_H^i(A,B)$. First, let $r_1 = \infty$. Without loss of generality, we assume that $\sup\big\{|aB| : a \in A\big\} = \infty$, but then neither $A \subset U_r(B)$, nor $A \subset B_r(B)$ holds for any finite $r > 0$, therefore $r_2 = r_3 = \infty$. Now suppose that $r_1 < \infty$, then for any $r > r_1$ all inclusions in the definitions of r_2 and r_3 take place, therefore r_2 and r_3 are also finite. So, we have shown that either all three r_i are infinite at the same time, or all of them are finite.

It remains to analyze the case of finite r_i. First, we show that $r_1 = r_2$ and $r_1 = r_3$. Let us note that $r_2 \le r_3$ because $A \subset U_r(B)$ implies $A \subset B_r(B)$ (and the same for A and B swaped).

By definition of r_1, we have $|aB| \le r_1$ for all $a \in A$, and $|Ab| \le r_1$ for all $b \in B$, therefore for all $r > r_1$ it holds $A \subset U_r(B)$ and $B \subset U_r(A)$, hence $r_3 \le r$. Since $r > r_1$ is arbitrary, we have $r_2 \le r_3 \le r_1$. On the other hand, for any $r > r_2$ we have $A \subset B_r(B)$ and $B \subset B_r(A)$, so for any $a \in A$ and $b \in B$ it holds $|aB| \le r$ and $|Ab| \le r$, therefore, $r_1 \le r$ and since $r > r_2$ is arbitrary, we get $r_1 \le r_2 \le r_3$. □

The value $d_H^i(A,B)$ from Proposition 5.1 is denoted by $d_H(A,B)$. It is easy to see that d_H is non-negative, symmetric, and $d_H(A,A) = 0$ for any nonempty $A \subset X$, thus, d_H is a generalized distance on the family $\mathcal{P}_0(X)$ of all nonempty subsets of a metric space X. The function d_H is called *the Hausdorff distance.*

Proposition 5.2. *For an arbitrary metric space X, the function d_H is a generalized pseudometric on $\mathcal{P}_0(X)$.*

Proof. It remains to prove the triangle inequality. Choose arbitrary $A, B, C \in \mathcal{P}_0(X)$ and set $c = d_H(A,B)$, $a = d_H(B,C)$, $b = d_H(A,C)$. We have to show that $b \le c + a$.

If either $c = \infty$, or $a = \infty$, then the equality holds. Suppose now that the both c and a are finite. Choose arbitrary finite $r > c$ and $s > a$, then $A \subset U_r(B)$ and $B \subset U_s(C)$ implies, by virtue of Item (4) of Problem 2.2, that $A \subset U_r\big(U_s(C)\big) \subset U_{r+s}(C)$. Similarly, $U_{r+s}(A) \supset C$. Thus, $b \le r + s$. Since $r > c$ and $s > a$ are arbitrary, we obtain what is required. □

Denote by $\mathcal{H}(X) \subset \mathcal{P}_0(X)$ the set of all nonempty closed bounded subsets of a metric space X.

Theorem 5.3. *For an arbitrary metric space X, the generalized pseudometric d_H is a metric on $\mathcal{H}(X)$.*

Proof. Choose arbitrary $A, B \in \mathcal{H}(A)$. Since they are bounded, for some $r > 0$ we have $A \subset U_r(B)$ and $U_r(A) \supset B$, hence $d_H(A, B) < \infty$. Thus, d_H is finite.

If $A \neq B$, then without loss of generality we can assume that there exists $a \in A \setminus B$, but since the set $X \setminus B$ is open, there exists $r > 0$ such that $U_r(a) \cap B = \emptyset$, in particular, $|aB| \geq r$ and, therefore, $d_H(A, B) \geq r$. Thus, d_H is non-degenerate and, therefore, positively defined. □

Recall that by $\mathcal{K}(X)$ we denoted the collection of all nonempty compact subsets of a topological space X. Since each compact subset of a metric space is closed and bounded, we have $\mathcal{K}(X) \subset \mathcal{H}(X)$ and, thus, we get

Corollary 5.4. *For an arbitrary metric space X, the function d_H is a metric on $\mathcal{K}(X)$.*

In what follows, when speaking about the distance in $\mathcal{H}(X)$, we will always have in mind the Hausdorff metric, and for topology consider the corresponding metric one. Note that different authors use different notations for this hyperspace. We introduced the notation $\mathcal{H}(X)$ by virtue of the fact that this is the largest natural set of subsets of a metric space on which the Hausdorff distance is defined.

We present a few Hausdorff distance properties in the next exercise.

Problem 5.1. Prove the following statements for an arbitrary metric space X.

(1) Let $f\colon X \to \mathcal{P}_0(X)$ be given by the formula $f\colon x \mapsto \{x\}$, then f is an isometric embedding.

(2) For any $A, B \in \mathcal{P}_0(X)$ we have $d_H(A, B) = d_H(A, \bar{B}) = d_H(\bar{A}, B) = d_H(\bar{A}, \bar{B})$.

(3) For any $A, B \in \mathcal{P}_0(X)$ we have $d_H(A, B) = 0$ if and only if $\bar{A} = \bar{B}$.

(4) If $Y \subset X$ is an ε-net in $A \subset X$, then $d_H(A, Y) \leq \varepsilon$.

Proposition 5.5. *Let X be an arbitrary metric space, and $A, B \in \mathcal{P}_0(X)$, $r = d_H(A, B)$. Then $A \subset B_r(B)$, $B \subset B_r(A)$, and for less r one of these inclusions fails. Thus, for $A, B \in \mathcal{P}_0(X)$ we can change* inf *to* max *in equality* (5.2).

Proof. The fact that smaller r do not fit follows directly from the definition of the Hausdorff distance. Let us prove the first part. Suppose the contrary, and let, say, $B \not\subset B_r(A)$. This means that there exists $b \in B$ for which $R := |bA| > r$, thus for $r < s < R$ we have $B \not\subset B_s(A)$, which contradicts the definition of $d_H(A, B)$. □

Problem 5.2. Prove that for $A, B \in \mathcal{K}(X)$ there exist $a \in A$ and $b \in B$ such that $d_H(A, B) = |ab|$. Is it possible to change $\mathcal{K}(X)$ with $\mathcal{H}(X)$?

Problem 5.3. Let X be an arbitrary metric space and $A, B, A', B' \in \mathcal{H}(X)$ such that $A' \subset A$ and $B' \subset B$. Prove that $d_H(A \cup B', B \cup A') \leq d_H(A, B)$.

Problem 5.4. Let X be an arbitrary metric space and $A, B, C \in \mathcal{H}(X)$ such that $C \subset B$. Prove that $d_H(A, A \cup C) \leq d_H(A, B)$.

5.1 Vietoris topology and Hausdorff metric

In Construction 1.5 we defined the Vietoris topology on the set of all nonempty subsets of a topological space X. Recall that a base of this topology is the family of sets

$$\langle U_1, \ldots, U_n \rangle = \{Y \subset X : Y \subset \cup_{i=1}^n U_i, \text{ and } Y \cap U_i \neq \emptyset \text{ for all } i = 1, \ldots, n\}$$

over all possible finite families $U_1, \ldots, U_n$ of open subsets of X.

Theorem 5.6. *Let (X, d) be an arbitrary metric space, then the metric topology on $\mathcal{K}(X)$ defined by the Hausdorff metric d_H coincides with the Vietoris topology. In particular, for a boundedly compact X we have $\mathcal{H}(X) = \mathcal{K}(X)$, thus the above result holds if we change $\mathcal{K}(X)$ with $\mathcal{H}(X)$.*

Proof. We use Problem 1.5 to prove that each open set in metric topology generated by d_H is also open in Vietoris topology. To do that, it suffices to take arbitrary $A \in \mathcal{K}(X)$, $r > 0$, and to construct an open neighborhood of A in Vietoris topology that belongs to $U_r^{d_H}(A)$. Consider the family $\mathcal{C} = \big\{U_{r/3}(a)\big\}_{a \in A}$, then $\mathcal{C}$ is an open cover of the compact set A, thus we can extract from $\mathcal{C}$ a finite subcover $\{U_1, \dots, U_n\}$. Notice that $A \in \langle U_1, \dots, U_n\rangle$ because $A \subset \cup_{i=1}^n U_i$ and $A \cap U_i \neq \emptyset$ for each $i = 1, \dots, n$. Further, we claim that $\langle U_1, \dots, U_n\rangle \subset U_r^{d_H}(A)$. Indeed, take an arbitrary $A' \in \langle U_1, \dots, U_n\rangle$, then $A' \subset U_{r/3}(A)$. Further, for every $a \in A$ there exist $U_i \ni a$ and $a' \in U_i \cap A'$, hence $|aa'| < 2r/3$, so $A \subset U_{2r/3}(A')$, hence $d_H(A', A) \le 2r/3 < r$.

Now, consider an arbitrary $\langle U_1, \dots, U_n\rangle$ and any $A \in \langle U_1, \dots, U_n\rangle$. Let $C = X \setminus \cup_{i=1}^n U_i$, then C is a closed set that does not intersect A. By Item (2) of Problem 2.2, the function $x \mapsto |xC|$ is continuous, thus its restriction onto the compact A is bounded and attains its minimal value at some point $a \in A$. Since C is closed and $a \notin C$, we have $|aC| > 0$ and, therefore, $r := |AC| > 0$.

Further, since $A \cap U_i \neq \emptyset$ for each $i = 1, \dots, n$, we can choose some point a_i in it. By the definition of the topology on X, for every i there is r_i such that $U_{r_i}(a_i) \subset U_i$. Put $\rho = \min\{r, r_1, \dots, r_n\}$. Then for each $B \in U_\rho^{d_H}(A)$, that is, for each $B \in \mathcal{K}(X)$ satisfying $d_H(A, B) < \rho$, we have

(1) $B \subset \cup_{a \in A} U_\rho(a) \subset \cup_{i=1}^n U_i$, because $\rho \le r$, and

(2) for each $i = 1, \dots, n$ there is $b_i \in B$ such that $b_i \in U_\rho(a_i) \subset U_i$ (because $\rho \le r_i$), therefore B intersects all U_i.

It follows that $U_\rho^{d_H}(A) \subset \langle U_1, \dots, U_n\rangle$, thus, $< U_1, \dots, U_n\rangle$ is open in the metric topology generated by d_H. The proof is over. □

Corollary 5.7. *If two metrics on a set X induce the same topology on X, then also the same metric topology is induced on the corresponding spaces $\mathcal{K}(X)$. In other words, the metric topology of the space $\mathcal{K}(X)$ does not depend on the specific form of the metric on X, but only on the topology defined by this metric on X. For boundedly compact X the same is true for $\mathcal{H}(X)$.*

The next example shows that the Vietoris and the metric topologies on $\mathcal{H}(X)$ can be different also if X is bounded and locally compact metric space.

Example 5.8. Let $X = (-1, 0) \cup (0, 1) \subset \mathbb{R}$ endowed with the metric induced from the standard metric on $\mathbb{R}$. Then $A = (-1, 0)$ is a closed bounded subset of X, thus $A \in \mathcal{H}(X)$. Since A is also an open set, then $\mathcal{U} = \langle A\rangle$ is a neighborhood of A in Vietoris topology consisting of all closed nonempty subsets of X which belong to A. However, for any $\varepsilon > 0$ the ball $U_\varepsilon^{d_H}(A) \subset \mathcal{H}(X)$ contains the closed subset $B = A \cup \{\varepsilon/2\}$ such that $B \notin \mathcal{U}$, therefore $U_\varepsilon^{d_H}(A) \not\subset \mathcal{U}$ and, thus, $\mathcal{U}$ is not open in the metric topology generated by the Hausdorff distance.

Example 5.9. In Example 5.8 the space X was not complete. Now we present an example of complete metric space where the Vietoris topology and the metric topology on $\mathcal{H}(X)$ are different.

Let $X = \ell_2$ be the space of all sequences $(x_1, x_2, \dots)$ of real numbers such that $\sum_{i=1}^\infty x^2 < \infty$. For $\xi = (x_1, x_2, \dots) \in \ell_2$ and $\xi' = (x_1', x_2', \dots) \in \ell_2$ we put $d(\xi, \xi')^2 = \sum_{i=1}^\infty (x_i - x_i')^2$.

Problem 5.5. Prove that d is a metric on ℓ_2, and that (ℓ_2, d) is a complete metric space.

Denote by $e_i \in \ell_2$ the sequence $(x_1, x_2, \dots)$ such that $x_i = 1$ and $x_j = 0$ for all $j \neq i$.

Problem 5.6. Prove that $A = \{e_i\}_{i=1}^\infty$ is a closed subset of ℓ_2, thus $A \in \mathcal{H}(\ell_2)$.

Now, we put $U_i = U_{\frac{1}{2i}}(e_i)$ and let $U = \cup_{i=1}^\infty U_i$, then U is an open neighborhood of A. Let $\mathcal{U} = \langle U\rangle \subset \mathcal{H}(\ell_2)$ be the corresponding neighborhood of $A \in \mathcal{H}(\ell_2)$ in the Vietoris topology. Choose an arbitrary $\varepsilon > 0$. We show that $U_\varepsilon^{d_H}(A) \not\subset \mathcal{U}$, thus, $\mathcal{U}$ is not open in the metric topology generated by d_H. Consider an arbitrary i such that $\frac{1}{2i} < \varepsilon$, and choose any $\frac{1}{2i} < \delta < \min\{1, \varepsilon\}$. Put $\xi = (1 - \delta)e_i$ and $B = A \cup \{\xi\} \in \mathcal{H}(\ell_2)$, then $d(\xi, e_i) = \delta < \varepsilon$, therefore, $d_H(A, B) < \varepsilon$. On the other hand, since $\frac{1}{2i} < \delta$, then $\xi \notin U_i$; also, for any $j \neq i$ we have $d(\xi, e_j) = \sqrt{1 + (1-\delta)^2} > 1 > \frac{1}{2j}$, thus $\xi \notin U_j$, and, in account, $\xi \notin \mathcal{U}$.

Problem 5.7. Let $X = \mathbb{N}$, and define two metrics on X: $d^1(x, y) = 1$ for any $x \neq y$, and $d^2(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ for any x, y. Then the corresponding metric topologies are both discrete. Prove that the corresponding Hausdorff metric generates non-homeomorphic topologies on $\mathrm{CL}(X)$.

5.2 Limits Theory

For the future, we need one technical result. Let $A_1, A_2, \ldots$ be a sequence of nonempty subsets of a metric space X.

Definition 5.10. Put

$$\limsup A_k = \cap_{n=1}^{\infty} \overline{A_n \cup A_{n+1} \cup \cdots}$$

and call it *the upper limit* of the sequence A_k.

Remark 5.11. Since $\limsup A_k$ is equal to the intersection of closed sets, it is always closed (possibly empty).

Proposition 5.12. *We have*

$$\limsup A_k = \big\{x \in X : \forall \varepsilon > 0 \text{ it holds } \#\{k : U_\varepsilon(x) \cap A_k \neq \emptyset\} = \infty\big\}.$$

Proof. Put $B_n = \cup_{k=n}^{\infty} A_k$ and $A = \limsup A_k = \cap_{n=1}^{\infty} \bar{B}_n$.

Choose an arbitrary $\varepsilon > 0$. Let $x \in A$, then x is adherent point of each set B_n, so $B_n \cap U_\varepsilon(x) \neq \emptyset$. If the condition $A_k \cap U_\varepsilon(x) \neq \emptyset$ were satisfied only for a finite number k, then for some n they would have $B_n \cap U_\varepsilon(x) = \emptyset$, so x would not be the adherent point of the set B_n, a contradiction.

Conversely, suppose that for every $\varepsilon > 0$ it holds $\#\{k : U_\varepsilon(x) \cap A_k \neq \emptyset\} = \infty$. Then for every $n \in \mathbb{N}$ we have $U_\varepsilon(x) \cap B_n \neq \emptyset$, so $x \in \bar{B}_n$ for any n, therefore, $x \in A$. □

Corollary 5.13. *We have*

$$\limsup A_k = \{x \in X : \text{there exists a sequence } a_{i_k} \in A_{i_k} \text{ that converges to } x\}.$$

Proof. Let $x \in \limsup A_k$. By Proposition 5.12, for each $k \in \mathbb{N}$ one can find $a_{i_k} \in A_{i_k}$ such that $|a_{i_k} x| < 1/k$, and there are infinitely many such i_k (for fixed k). Thus, we can compose an increasing sequence $i_1 < i_2 < \cdots$ for which $a_{i_k} \in A_{i_k}$, $|a_{i_k} x| < 1/k$, and therefore, $a_{i_k} \to x$.

Conversely, let some sequence $a_{i_k} \in A_{i_k}$ converge to some $x \in X$. This means that for every $\varepsilon > 0$ there is n such that for any $k \geq n$ we have $U_\varepsilon(x) \cap A_{i_k} \neq \emptyset$, therefore $x \in \limsup A_k$ by virtue of Proposition 5.12. □

Remark 5.14. The upper limit may be empty.

Example 5.15. Take as X the interval $(0, 1)$ with the standard metric, and put $A_k = \{1/k\}$, $k \in N$, then one cannot select a convergent subsequence a_{i_k}; therefore, $\limsup = \emptyset$. A similar situation concerns the space $X = \mathbb{R}$ and the sequence $A_k = \{k\}$. Thus, the space X can be totally bounded or complete, but it can contain sequences of subsets $A_k \subset X$ for which $\limsup A_k$ is empty.

The construction of upper limit can be extended as follows.

Definition 5.16. Put

$$\liminf A_k = \big\{x \in X : \forall \varepsilon > 0 \text{ it holds } \#\{k : U_\varepsilon(x) \cap A_k = \emptyset\} < \infty\big\}$$

and call it *the lower limit* of the sequence A_k.

Remark 5.17. Note that $\liminf A_k \subset \limsup A_k$.

Proposition 5.18. *We have*

$$\liminf A_k = \{x \in X : \text{there exists a sequence } a_k \in A_k \text{ that converges to } x\}.$$

Proof. Let $x \in \liminf A_k$. By Definition 5.16, for each $n \in \mathbb{N}$ one can find $k_n \in \mathbb{N}$ such that for all $k \geq k_n$ there exists $a_k^n \in A_k$ for which $|a_k^n x| < 1/n$. Without loss of generality, we assume that the sequence k_n is strictly monotone, and construct a sequence $a'_k \in A_k$, starting with arbitrary $a_1, \ldots, a_{k_1-1}$, and then adding $a_{k_1}^1, \ldots, a_{k_2-1}^1$, $a_{k_2}^2, \ldots, a_{k_3-1}^2$, etc. It is clear that the sequence $a'_k \in A_k$ constructed in this way converges to x.

Conversely, let some sequence $a_k \in A_k$ converge to some $x \in X$. This means that for every $\varepsilon > 0$ there is n such that for any $k \geq n$ we have $U_\varepsilon(x) \cap A_k \neq \emptyset$, therefore $x \in \liminf A_k$ by Definition 5.16. □

Problem 5.8. Prove that $\liminf A_i$ is a closed subset of X.

Definition 5.19. If X is a metric space, $A_k \in \mathcal{P}_0(X)$, $k \in \mathbb{N}$, $A \in \mathcal{P}_0(X)$, and A_k converges to A w.r.t. the Hausdorff distance, then we write this as $A_k \xrightarrow{d_H} A$.

Proposition 5.20. *If $A_k, A \in \mathcal{P}_0(X)$ and $A_k \xrightarrow{d_H} A$, then $\bar{A} = \liminf A_k$, where $\bar{A}$ is the closure of A.*

Proof. By Item (2) of Problem 5.1, we have $A_k \to \bar{A}$, thus, without loss of generality, we can assume that A is a closed subset of X.

Now, we show that A always contains the limit point of each convergent sequence $a_k \in A_k$, i.e., that $\liminf A_k \subset A$. Let $a_k \to a$. By definition, for any $\varepsilon > 0$ there exists n such that for all $k \ge n$ we have $d_H(A, A_k) < \varepsilon/2$ and $|aa_k| < \varepsilon/2$. Hence, for such k we have $A_k \subset U_{\varepsilon/2}(A)$, in particular, $a_k \in U_{\varepsilon/2}(A)$. The latter means that for such k there exist $a'_k \in A$ for which $|a'_k a_k| < \varepsilon/2$ and, therefore, $|a'_k a| < \varepsilon$. Thus, $a'_k \to a$ and therefore, since A is closed, we have $a \in A$.

To complete the proof, it remains to verify that each point $a \in A$ is the limit point for some sequence $a_k \in A_k$, i.e., that $\liminf A_k \supset A$. For each n there exists k_n such that for all $k \ge k_n$ it holds $d_H(A_k, A) < 1/n$. For such k we have $A \subset U_{1/n}(A_k)$, in particular, $a \in U_{1/n}(A_k)$. It is clear that the sequence k_n can be chosen strictly monotonic. For such k_n, we take a_k as follows: for $1 \le k \le k_1 - 1$ we choose $a_k \in A_k$ arbitrarily; for $k_1 \le k < k_2$ we select $a_k \in A_k$ so that $|a_k a| < 1$; for $k_2 \le k < k_3$, select $a_k \in A_k$ so that $|a_k a| < 1/2$; etc. Such sequence converges to a, as required. □

Corollary 5.21. *Let $A_k = \{a_k\}$ for all k. Then the sequence A_k converges w.r.t. d_H if and only if the sequence a_k converges. Moreover, if $a_k \to a$, then $A_k \xrightarrow{d_H} \{a\}$.*

Proof. By virtue of Item (1) of Problem 5.1, the mapping $f\colon X \to \mathcal{H}(X)$, $f\colon x \mapsto \{x\}$, is isometric, therefore the convergence of $a_k \to a$ implies the convergence of $A_k \xrightarrow{d_H} \{a\}$.

Conversely, if $A_k \xrightarrow{d_H} A$, then $A \ne \emptyset$ by definition. Also, A cannot contain two different points. Indeed, if $a, a' \in A$, $a \ne a'$, then for $0 < \varepsilon < |aa'|/2$ and sufficiently large k we have $\{a, a'\} \subset A \subset U_\varepsilon(A_k) = U_\varepsilon(a_k)$, that is impossible. Thus, $A = \{a\}$ and for any $\varepsilon > 0$ there exists n such that for any $k \ge n$ we have $A_k \in U_\varepsilon(A)$, i.e., $a_k \in U_\varepsilon(a)$, thus $a_k \to a$. □

Definition 5.22. If the upper and lower limits of a sequence $A_1, A_2, \ldots$ are nonempty and equal to each other, then the sequence $A_1, A_2, \ldots$ is said to have *a limit*, which is denoted by $\lim A_k$. We write this as $A_k \to A$.

Let us discuss what the existence of the limit $\lim A_k$ means.

Theorem 5.23. *If $A_k, A \in \mathcal{P}_0(X)$ and $A_k \xrightarrow{d_H} A$, then there exists $\lim A_k$ and $\bar{A} = \lim A_k$.*

Proof. By Proposition 5.20 and Remark 5.17, we have $\bar{A} = \liminf A_k \subset \limsup A_k$. Therefore, it suffices to show that $\limsup A_k \subset \bar{A}$.

Let $x \in \limsup A_k$. We show that x is an adherent point for A and, therefore, $x \in \bar{A}$. To do this, in turn, it is enough to show that for an arbitrary $\varepsilon > 0$ the ball $U_\varepsilon(x)$ intersects A.

Since $A_k \xrightarrow{d_H} A$, there exists n such that for every $k \ge n$ we have $A_k \subset U(A, \varepsilon/2)$. By Proposition 5.18, there exists $k \ge n$ for which $U_{\varepsilon/2}(x) \cap A_k \ne \emptyset$, i.e., there exists $a_k \in A_k$ such that $|xa_k| < \varepsilon/2$. Since $A_k \subset U(A, \varepsilon/2)$, there exists $a \in A$ for which $|a_k a| < \varepsilon/2$, thus $|xa| < \varepsilon$ and, hence $U_\varepsilon(x) \cap A \ne \emptyset$. □

Is the converse statement to that of Theorem 5.23 true as well? The following example demonstrates that this is not the case in general, also when $A_k, A \in \mathcal{H}(X)$.

Example 5.24. Let X be the interval $(0, 3) \subset \mathbb{R}$ with the standard distance function. Put $A_k = \{2, 1/k\}$. Then $\limsup A_k = \liminf A_k = \{2\}$, so that $\lim A_k$ exists and is equal to $\{2\}$. If for some $A \in \mathcal{H}(X)$ we have $A_k \xrightarrow{d_H} A$, then, by virtue of Theorem 5.23, we would have $A = \{2\}$. However, for $\varepsilon = 1$ there is no A_k that belongs to $U_\varepsilon(A)$, so the sequence A_k diverges.

The case of $X = \mathbb{R}$ and $A_k = \{0, k\}$, for which $\liminf A_k = \limsup A_k = \{0\}$, but the sequence A_k diverges, is similarly analyzed. Thus, the existence of a limit does not imply the convergence of the sequence $A_k \in \mathcal{H}(X)$ for either totally bounded or complete X.

We give some corollaries.

Corollary 5.25. *If $A_k, A \in \mathcal{P}_0(X)$, $A_k \xrightarrow{d_H} A$, and for all k we have $A_k \supset A_{k+1}$, then $\bar{A} = \cap_k \bar{A}_k$.*

Proof. By Theorem 5.23, we have

$$\bar{A} = \lim A_k = \limsup A_k = \cap_{k=1}^{\infty} \overline{A_k \cup A_{k+1} \cup \cdots} = \cap_{k=1}^{\infty} \bar{A}_k.$$

□

Corollary 5.26. *If $A_k, A \in \mathcal{P}_0(X)$, $A_k \xrightarrow{d_H} A$, and for all k we have $A_k \subset A_{k+1}$, then $\bar{A} = \overline{\cup_k A_k}$.*

Proof. By Theorem 5.23, we have

$$\bar{A} = \limsup A_k = \overline{A_1 \cup A_2 \cup \cdots},$$

since all the sets $B_k = A_k \cup A_{k+1} \cup \cdots$ coincide. □

5.2.1 Limits for compact X

Example 5.15 shows that a upper limit can be empty. However, for compact X this is no longer the case.

Proposition 5.27. *For any compact metric space X and any sequence of nonempty $A_k \subset X$ we have* $\limsup A_k \neq \emptyset$.

Proof. Choose an arbitrary sequence $a_k \in A_k$, then it contains a subsequence that converges to some $x \in X$. By Corollary 5.13, we have $x \in \limsup A_k$, so the upper limit of this sequence is not empty. □

We will need the following technical result in the future.

Proposition 5.28. *For any compact metric space X and any sequence of nonempty $A_k \subset X$ the following statement holds: for any $\varepsilon > 0$ there exists n such that for any $k \geq n$*

$$A_k \subset U_\varepsilon(\limsup A_k) \quad \textit{and} \quad \liminf A_k \subset U_\varepsilon(A_k).$$

Proof. To start with, we prove the first inclusion. Put $A = \limsup A_k$, then, by Proposition 5.27, we have $A \neq \emptyset$. Now, we suppose to the contrary that for some sequence A_k there exists an $\varepsilon > 0$ and a subsequence A_{i_k} such that $A_{i_k} \not\subset U_\varepsilon(A)$. The latter means that in every A_{i_k} there is a point a_{i_k} for which $|a_{i_k}A| \geq \varepsilon$. Since X is compact, there exists a subsequence in the sequence a_{i_k} that converges to some $x \in X$. Then, by Corollary 5.13, we have $x \in A$, which implies that for some sufficiently large k it holds $|a_{i_k}A| < \varepsilon$, a contradiction.

We now prove the second inclusion. Put $A = \liminf A_k$ and assume the contrary, i.e., that there exists $\varepsilon > 0$ and a sequence $i_1 < i_2 < \cdots$ for which $A \not\subset U_\varepsilon(A_{i_k})$, in particular, $A \neq \emptyset$. The latter is equivalent to the existence of $a'_{i_k} \in A$ such that $a'_{i_k} \notin U_\varepsilon(A_{i_k})$. By Problem 5.8, the set A is closed, thus A is compact as a closed subset of a compact metric space, therefore, in the sequence a'_{i_k} one can choose a subsequence converging to some $a \in A$. Without loss of generality, we assume that this subsequence coincides with the entire sequence a'_{i_k}.

Since $A = \liminf A_k$, by Proposition 5.18 there exists a sequence $a_k \in A_k$ converging to a, thus $|a'_{i_k}a_{i_k}| \to 0$, so for sufficiently large k we have $a'_{i_k} \in U_\varepsilon(A_{i_k})$, a contradiction. □

Corollary 5.29. *For any compact metric space X and any sequence of nonempty $A_k \subset X$, the A_k converges w.r.t. d_H if and only if there exists* $\lim A_k$*. Moreover, if $A_k \xrightarrow{d_H} A$, then $\bar{A} = \lim A_k$.*

Proof. By Theorem 5.23, the convergence of the sequence A_k to some A implies the existence of $\lim A_k$ and that $\bar{A} = \lim A_k$. Thus, it remains to prove the converse.

So, let $\lim A_k =: A$ exist. We will show that $A_k \xrightarrow{d_H} A$. By Proposition 5.28, for any $\varepsilon > 0$ there exists n such that for any $k \geq n$ we have $A_k \subset U_\varepsilon(\limsup A_k) = U_\varepsilon(A)$ and $A = \liminf A_k \subset U_\varepsilon(A_k)$, hence $A_k \xrightarrow{d_H} A$, as required. □

Let us demonstrate how the theory of limits works in the compact case.

Corollary 5.30. *For an arbitrary, not necessarily compact metric space X, and each decreasing sequence of nonempty sets $A_1 \supset A_2 \supset \cdots$ starting from compact A_1, we have $A := \cap_k \bar{A}_k \neq \emptyset$ and $A_k \xrightarrow{d_H} A$.*

Proof. Without loss of generality, we assume that $X = A_1$, then X is a compact space. By Proposition 5.27, we have $\limsup A_k \neq \emptyset$, however, by definition of lim sup, we have $\limsup A_k = A$, thus A is not empty. On the other hand, each point $a \in A$ is the limit of the constant sequence $a_k = a \in \bar{A}_k$, and for each k there exists $a'_k \in A_k$ such that $|a_k a'_k| < 1/k$, thus $a'_k \to a$ as well and, hence, $a \in \liminf A_k$. Since $a \in A$ is arbitrary then $\limsup A_k \subset \liminf A_k$. However, $\liminf A_k \subset \limsup A_k$, thus $\liminf A_k = A = \limsup A_k$, and it remains to use Corollary 5.29. □

Remark 5.31. In Corollary 5.30 it may happen that $\cap_{k=1}^{\infty} A_k = \emptyset$. Let $X = \mathbb{R}$, $A_1 = [0,1]$, $A_k = (0, 1/k)$ for $k > 1$, then $\cap_{k=1}^{\infty} A_k = \emptyset$, however,

$$\limsup A_k = \cap_{k=1}^{\infty} \bar{A}_k = \cap_{k=1}^{\infty} [0, 1/k] = \{0\} \neq \emptyset.$$

Corollary 5.32. *For an arbitrary, not necessarily compact, space X, if for an increasing sequence of nonempty sets $A_1 \subset A_2 \subset \cdots$ the set $A = \overline{\cup_{k=1}^{\infty} A_k}$ is compact, then $A_k \xrightarrow{d_H} A$.*

Proof. Again, without loss of generality, we assume that $X = A$. By definition of lim sup, we have $\limsup A_k = A$, therefore, by Corollary 5.13, for every $a \in A$ there exists a sequence $a_{i_k} \in A_{i_k}$, $i_1 < i_2 < \cdots$, converging to a. Notice that each a_{i_k} also belongs to all A_p for $p > i_k$, therefore this sequence can be extended to a sequence $a_i \in A_i$ converging to a, so, by Proposition 5.18, we have $a \in \liminf A_k$, hence $\liminf A_k = \limsup A_k = A$, and it remains to use Corollary 5.29. □

5.2.2 Limits for complete X

We give a series of results for fundamental sequences of subsets of complete metric spaces. It will be convenient for us to use the following construction.

Construction 5.1. Let Y be an arbitrary generalized pseudometric space and $\omega = (y_1, y_2, \ldots)$ be some sequence in it. Put

$$d_n(\omega) = \sup_{p,q \geq n} |y_p y_q|.$$

Note that $d_n(\omega)$ is a non-negative non-increasing sequence, therefore, it has a limit $d(\omega)$.

Proposition 5.33. *The sequence $\omega = (y_1, y_2, \ldots)$ is fundamental if and only if $d(\omega) = 0$.*

Proof. If y is fundamental, then for every $\varepsilon > 0$ there exists n such that for any $p, q \geq n$ we have $|y_p y_q| < \varepsilon$, so $d_n(\omega) \leq \varepsilon$ and, therefore, $d(\omega) = 0$. Conversely, if $d(\omega) = 0$, then for any $\varepsilon > 0$ starting from some n it holds $d_n(\omega) < \varepsilon$, so for such n and any $p, q \geq n$ we have $|y_p y_q| < \varepsilon$, which means that the sequence ω is fundamental. □

Now we present an important technical result on the fundamental sequences of an arbitrary metric space X.

Proposition 5.34. *Let $\Omega = (A_1, A_2, \ldots)$, $A_k \in \mathcal{P}_0(X)$, be a fundamental sequence. Choose arbitrary $a_1 \in A_1$ and $\varepsilon > 0$. Then there exists a sequence $i_1 = 1 < i_2 < i_3 < \cdots$ of natural numbers and a fundamental sequence $\omega = (a_{i_1}, a_{i_2}, a_{i_3}, \ldots)$, $a_{i_k} \in A_{i_k}$, such that $d_1(\omega) < d_1(\Omega) + \varepsilon$.*

Proof. Proposition 5.33 implies $d_k(\Omega) \to 0$, so there exists $i_2 \in \mathbb{N}$, $i_2 > i_1$, such that $d_{i_2}(\Omega) < \varepsilon/8$. Since $d_H(A_{i_1}, A_{i_2}) \leq d_1(\Omega)$, there exists $a_{i_2} \in A_{i_2}$ for which $|a_{i_1} a_{i_2}| < d_1(\Omega) + \varepsilon/4$. Next, we find a natural $i_3 > i_2$ such that $d_{i_3}(\Omega) < \varepsilon/16$, as well as $a_{i_3} \in A_{i_3}$ for which $|a_{i_2} a_{i_3}| < \varepsilon/8$. Continuing this process, we construct a sequence $\omega = (a_{i_1} = a_1, a_{i_2}, a_{i_3}, \ldots)$, $a_{i_k} \in A_{i_k}$, for which $|a_{i_{k-1}} a_{i_k}| < \varepsilon/2^k$ for $k \geq 3$. By the triangle inequality, for any $p, q \in \mathbb{N}$, $2 \leq p \leq q$, we have $|a_{i_p} a_{i_q}| < \varepsilon/2^p$, therefore, the sequence ω is fundamental. On the other hand, for any $p, q \in \mathbb{N}$ we have $|a_{i_p} a_{i_q}| < d_1(\Omega) + \varepsilon/2$, hence $d_1(\omega) \leq d_1(\Omega) + \varepsilon/2 < d_1(\Omega) + \varepsilon$. □

Example 5.15 demonstrates that for a complete X a upper limit can be empty. However, for a fundamental sequence A_k this is no longer the case.

Proposition 5.35. *Let X be a complete space and $A_k \in \mathcal{P}_0(X)$ be a fundamental sequence, then* $\limsup A_k \neq \emptyset$.

Proof. We choose an arbitrary $a_1 \in A_1$, then, by Proposition 5.34, there exists a fundamental sequence $(a_{i_1} = a_1, a_{i_2}, a_{i_3}, \ldots)$ such that $a_{i_k} \in A_{i_k}$. Since X is a complete space, this sequence converges to some $x \in X$. Proposition 5.12 implies that $x \in \limsup A_k$, therefore $\limsup A_k \neq \emptyset$. □

Now we will give an analogue of Proposition 5.28.

Theorem 5.36. *Let X be an arbitrary metric space, $A_k \in \mathcal{P}_0(X)$ a fundamental sequence, and $A = \limsup A_k$ (possibly empty). Then for any $\varepsilon > 0$ there exists n such that for any $k \geq n$*

(1) *$A \subset U_\varepsilon(A_k)$;*

(2) *if the space X is complete, then $A \neq \emptyset$ and $A_k \subset U_\varepsilon(A)$.*

Therefore, in complete X we have $A_k \xrightarrow{d_H} A$ and, hence, there exists $\lim A_k$.

Proof. Since the sequence $\Omega' = (A_1, A_2, \ldots)$ is fundamental, Proposition 5.33 implies that $d_k(\Omega') \to 0$. Choose n such that $d_n(\Omega') < \varepsilon/2$. Consider an arbitrary $k \ge n$ and put $\Omega = (A_k, A_{k+1}, \ldots)$. Since the sequence $d_i(\Omega')$ is monotonic, we have $d_1(\Omega) < \varepsilon/2$.

(1) If $A = \emptyset$, then the inclusion is proved. Now let $A \ne \emptyset$. We choose an arbitrary $a \in A$, then, by Corollary 5.13, there exists a sequence $\omega = (a_{i_1}, a_{i_2}, \ldots)$ converging to a so that for some $p \in \mathbb{N}$ we have $|a_{i_p}a| < \varepsilon/2$. On the other hand, since $d_1(\Omega) < \varepsilon/2$, then $d_H(A_k, A_{i_p}) < \varepsilon/2$, therefore there exists $a_k \in A_k$ for which $|a_k a_{i_p}| < \varepsilon/2$. By the triangle inequality, $|a_k a| < \varepsilon$, therefore, because a is arbitrary, we have $A \subset U_\varepsilon(A_k)$.

(2) Let us apply Proposition 5.34 to the sequence Ω. By this proposition, each $a_k \in A_k$ is included in some fundamental sequence ω for which $d_1(\omega) < d_1(\Omega) + \varepsilon/2 < \varepsilon$. Due to the completeness of the space X and Proposition 5.12, the sequence ω converges to some element $a \in A$, hence, taking into account the previous estimate on $d_1(\omega)$, we get $|a_k a| < \varepsilon$, therefore $A_k \subset U_\varepsilon(A)$. The existence of the limit follows from Theorem 5.23. □

Corollary 5.37. *A metric space X is complete if and only if $\mathcal{H}(X)$ is complete.*

Proof. The completeness of $\mathcal{H}(X)$ for complete X is proved in Theorem 5.36. Conversely, let $\mathcal{H}(X)$ be a complete space. Consider an arbitrary fundamental sequence a_k in X; then, due to the isometry of the mapping $x \mapsto \{x\}$, the sequence $A_k = \{a_k\} \in \mathcal{H}(X)$ is also fundamental. Since $\mathcal{H}(X)$ is complete, the sequence A_k converges to some $A \in \mathcal{H}(X)$. However, by Corollary 5.21, the sequence a_k also converges. □

5.2.3 Inheritance of total boundedness and compactness

The purpose of this section is to prove the following theorem.

Theorem 5.38. *Let X be an arbitrary metric space. Then the following properties are simultaneously present or not in both X and $\mathcal{H}(X)$:*

(1) *completeness (Khan),*

(2) *total boundedness,*

(3) *compactness (Hausdorff, Blaschke).*

Proof. (1) This is Corollary 5.37.

(2) Let X be totally bounded. The total boundedness of $\mathcal{H}(X)$ immediately follows from the following lemma and problem.

Lemma 5.39. *Let W be an arbitrary metric space, $\varepsilon > 0$, and $Y \subset W$ be some ε-net, then for any $\delta > \varepsilon$ the set $\mathcal{P}_0(Y)$ is a δ-net in $\mathcal{P}_0(W)$.*

Proof. Indeed, we choose an arbitrary $F \in \mathcal{P}_0(W)$ and put $M = \{y \in Y : |yF| < \varepsilon\}$, thus for every $y \in M$ there exists $w \in F$ such that $|yw| < \varepsilon$, therefore, $M \subset U_\varepsilon(F)$.

Further, since Y is an ε-net for W, then for every $w \in F$ there exists $y \in Y$ for which $|yw| < \varepsilon$, hence $|yF| \le |yw| < \varepsilon$ and, therefore, $y \in M$. In particular, $M \ne \emptyset$, i.e., $M \in \mathcal{P}_0(W)$. Since w is arbitrary, we get $F \subset U_\varepsilon(M)$, therefore, $d_H(F, M) \le \varepsilon < \delta$. □

Problem 5.9. If W is a metric space, $Y \subset W$ is an ε-net, and $Z \subset W$ is not empty. Then Z contains an (2ε)-net S such that $\#S \le \#Y$.

Conversely, let $\mathcal{H}(X)$ be totally bounded. The total boundedness of X follows from the following lemma.

Lemma 5.40. *Let W be an arbitrary metric space, $\varepsilon > 0$, and $\mathcal{Y} \subset \mathcal{H}(W)$ be some ε-net. Denote by $M \subset W$ the set obtained by choosing at each element $Y \in \mathcal{Y}$ any one point $p(Y)$. Then M is an ε-net in W (of the same cardinality as $\mathcal{Y}$).*

Proof. We choose an arbitrary $w \in W$, then there exists $Y \in \mathcal{Y}$ such that $d_H\big(\{w\}, Y\big) < \varepsilon$. Hence $Y \subset U_\varepsilon(w)$ and, therefore, $\big|p(Y)w\big| < \varepsilon$. □

(3) This follows from the previous section and Theorem 2.24. □

5.3 Inheritance of geodesic

In this section we present the results from [4].

Theorem 5.41. *Let X be a compact metric space, then $\mathcal{H}(X)$ is geodesic if and only if X is geodesic.*

Proof. First, let X be a geodesic space. To prove that $\mathcal{H}(X)$ is geodesic, we use Theorems 5.38 and 3.47. The first of them claims that $\mathcal{H}(X)$ is compact and, therefore, complete; the second is that $\mathcal{H}(X)$ is geodesic if for any its elements there exists a midpoints. That is what we will prove.

Consider arbitrary $A, B \in \mathcal{H}(X)$, $r := d_H(A, B)$, put $C = B_{r/2}(A) \cap B_{r/2}(B)$, and show that C is a midpoint between A and B.

Indeed, C is closed as the intersection of closed sets. We show that $C \neq \emptyset$. Choose an arbitrary $a \in A$. By Proposition 5.5, $A \subset B_r(B)$, therefore, since B is compact, there exists $b \in B$ such that $|ab| \le r$. Since the space X is geodesic, for the points a and b there is a midpoint c, then $c \in C$, which proves the nonemptiness of C and, at the same time, that $A \subset B_{r/2}(C)$. In addition, $C \subset B_{r/2}(A)$ by construction, hence $d_H(A, C) \le r/2$. Similarly, $d_H(B, C) \le r/2$. Since $r = d_H(A, B) \le d_H(A, C) + d_H(C, B)$, then $d_H(A, C) = d_H(C, B) = r/2$, so C is a midpoint between A and B. We again use Theorem 3.47 and conclude that the space $\mathcal{H}(X)$ is geodesic.

Conversely, let the space $\mathcal{H}(X)$ be geodesic. We choose arbitrary points $a, b \in X$, $r := |ab|$, then, due to the geodesicity of $\mathcal{H}(X)$, it contains C, which is a midpoint between $\{a\}$ and $\{b\}$. By Items (1) of Problem 5.1, we have $d_H\big(\{a\}, \{b\}\big) = r$, so $d_H\big(\{a\}, C\big) = d_H\big(C, \{b\}\big) = r/2$, hence $C \subset B_{r/2}(a) \cap B_{r/2}(b)$. We choose an arbitrary point $c \in C$, then $|ac| \le r/2$ and $|cb| \le r/2$; since $r = |ab| \le |ac| + |cb|$, then $|ac| = |cb| = r/2$ and, therefore, $c \in X$ is a midpoint between a and b. It remains to use Theorem 3.47. □

We give a few examples to demonstrate that for the space $\mathcal{H}(X)$ was geodesic, neither the total boundedness of X, nor its completeness is sufficient individually.

Problem 5.10. Denote by X a subset of the space $\mathbb{R}^3$, which in the standard Cartesian coordinates has the form

$$X = \{x^2 + y^2 < 1,\ z \le 1\} \cup \{x^2 + y^2 = 1,\ z = -1,\ x \in \mathbb{Q}\} \cup \{x^2 + y^2 = 1,\ z = 1,\ x \in \mathbb{R} \setminus \mathbb{Q}\}.$$

Next, put $A = \{x^2 + y^2 = 1,\ z = -1,\ x \in \mathbb{Q}\}$ and $B = \{x^2 + y^2 = 1,\ z = 1,\ x \in \mathbb{R} \setminus \mathbb{Q}\}$. Show that X is a geodesic space, but A and B are not connected by a shortest curve (there is no midpoint between them), so $\mathcal{H}(X)$ is not geodesic.

Problem 5.11. Consider the square $X = [-1, 1] \times [-1, 1]$ and introduce the following distance function on X:

$$d\big((x, y), (x', y')\big) = |x - x'| + \min\{(1 - y) + (1 - y'), (y + 1) + (y' + 1)\}.$$

Show that (X, d) is a complete geodesic space, $A = [-1, 1] \times -1$ and $B = [-1, 1] \times \{1\}$ are compacts such that there is midpoint between them in $\mathcal{K}(X)$, i.e., $\mathcal{K}(X)$ is not geodesic.

Problem 5.12. Consider two segments $I_0 = [0, 1] \times 0$ and $I_1 = [0, 1] \times \{1\}$ in the Euclidean plane $\mathbb{R}^2$, and let $A \subset I_0$ be the subset of all points with rational abscissae, and $B \subset I_1$ the subset of all points with irrational abscissae. For each $a \in A$ and $b \in B$ we denote by $r(a, b)$ the distance between a and b in $\mathbb{R}^2$. Now, for each $a \in A$, $b \in B$ consider a circle $S(a, b)$ or the length 3, and denote by the same a and b some points of $S(a, b)$ bounding an arc $\alpha(a.b)$ of the length $r(a, b)$. Let $C(a, b) \in S(a, b)$ be the middle point of the act complement to $\alpha(a, b)$. Denote by X the quotient space obtained from all $S(a, b)$ by identifying their points $C(a, b)$. Also denote by A and B the subsets of X consisting of all points a and all points b, respectively. Prove that X is complete and geodesic, $A, B \in \mathcal{H}(X)$, however, there is no a midpoint between A and B, thus, $\mathcal{H}(X)$ is complete but not geodesic.

References to Chapter 5

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Exercises to Chapter 5

Exercise 5.1. Prove the following statements for an arbitrary metric space X.

(1) Let $f\colon X \to \mathcal{H}(X)$ be given by the formula $f\colon x \mapsto \{x\}$, then f is an isometric embedding.

(2) For any $A, B \in \mathcal{P}_0(X)$ we have $d_H(A,B) = d_H(A,\bar B) = d_H(\bar A, B) = d_H(\bar A,\bar B)$.

(3) For any $A, B \in \mathcal{P}_0(X)$ we have $d_H(A,B) = 0$ if and only if $\bar A = \bar B$.

(4) If $Y \subset X$ is an ε-net in $A \subset X$, then $d_H(A,Y) \le \varepsilon$.

Exercise 5.2. Prove that for $A, B \in \mathcal{K}(X)$ there exist $a \in A$ and $b \in B$ such that $d_H(A,B) = |ab|$. Is it possible to change $\mathcal{K}(X)$ with $\mathcal{H}(X)$?

Exercise 5.3. Let X be an arbitrary metric space and $A, B, A', B' \in \mathcal{H}(X)$ such that $A' \subset A$ and $B' \subset B$. Prove that $d_H(A \cup B', B \cup A') \le d_H(A,B)$.

Exercise 5.4. Let X be an arbitrary metric space and $A, B, C \in \mathcal{H}(X)$ such that $C \subset B$. Prove that $d_H(A, A\cup C) \le d_H(A,B)$.

Exercise 5.5. Prove that $\liminf A_i$ is a closed subset of X.

Exercise 5.6. Let ℓ_2 denote the space of all sequences $(x_1, x_2, \ldots)$ of real numbers such that $\sum_{i=1}^\infty x^2 < \infty$. For $\xi = (x_1, x_2, \ldots) \in \ell_2$ and $\xi' = (x_1', x_2', \ldots) \in \ell_2$ we put $d(\xi,\xi')^2 = \sum_{i=1}^\infty (x_i - x_i')^2$. Prove that d is a metric on ℓ_2, and that (ℓ_2, d) is a complete metric space.

Exercise 5.7. Denote by $e_i \in \ell_2$ the sequence $(x_1, x_2, \ldots)$ such that $x_i = 1$ and $x_j = 0$ for all $j \ne i$. Prove that $A = \{e_i\}_{i=1}^\infty$ is a closed subset of ℓ_2, thus $A \in \mathcal{H}(\ell_2)$.

Exercise 5.8. Let $X = \mathbb{N}$, and define two metrics on X: $d^1(x,y) = 1$ for any $x \ne y$, and $d^2(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ for any x, y. Then the corresponding metric topologies are both discrete. Prove that the corresponding Hausdorff metric generates non-homeomorphic topologies on $\mathrm{CL}(X)$.

Exercise 5.9. Suppose that in a sequence $A_i \in \mathcal{H}(X)$ all A_i are connected, $A_i \to A \in \mathcal{H}(X)$, and $\liminf A_i \ne \emptyset$. Prove that $\limsup A_i$ is connected.

Exercise 5.10. Denote by X a subset of the space $\mathbb{R}^3$, which in the standard Cartesian coordinates has the form

$$X = \{x^2 + y^2 < 1,\ z \le 1\} \cup \{x^2 + y^2 = 1,\ z = -1,\ x \in \mathbb{Q}\} \cup \{x^2 + y^2 = 1,\ z = 1,\ x \in \mathbb{R} \setminus \mathbb{Q}\}.$$

Next, put $A = \{x^2 + y^2 = 1,\ z = -1,\ x \in \mathbb{Q}\}$ and $B = \{x^2 + y^2 = 1,\ z = 1,\ x \in \mathbb{R} \setminus \mathbb{Q}\}$. Show that X is a geodesic space, but A and B are not connected by a shortest curve (there is no midpoint between them), so $\mathcal{H}(X)$ is not geodesic.

Exercise 5.11. Consider the square $X = [-1,1] \times [-1,1]$ and introduce the following distance function on X:

$$d\big((x,y),(x',y')\big) = |x - x'| + \min\{(1-y) + (1-y'), (y+1) + (y'+1)\}.$$

Show that (X,d) is a complete geodesic space, $A = [-1,1] \times -1$ and $B = [-1,1] \times \{1\}$ are compacts such that there is no midpoint between them in $\mathcal{K}(X)$, i.e., $\mathcal{K}(X)$ is not geodesic.

Exercise 5.12. Consider two segments $I_0 = [0,1] \times 0$ and $I_1 = [0,1] \times \{1\}$ in the Euclidean plane $\mathbb{R}^2$, and let $A \subset I_0$ be the subset of all points with rational abscissae, and $B \subset I_1$ the subset of all points with irrational abscissae. For each $a \in A$ and $b \in B$ we denote by $r(a,b)$ the distance between a and b in $\mathbb{R}^2$. Now, for each $a \in A$, $b \in B$ consider a circle $S(a,b)$ or the length 3, and denote by the same a and b some points of $S(a,b)$ bounding an arc $\alpha(a.b)$ of the length $r(a,b)$. Let $C(a,b) \in S(a,b)$ be the middle point of the act complement to $\alpha(a,b)$. Denote by X the quotient space obtained from all $S(a,b)$ by identifying their points $C(a,b)$. Also denote by A and B the subsets of X consisting of all points a and all points b, respectively. Prove that X is complete and geodesic, $A, B \in \mathcal{H}(X)$, however, there is no a midpoint between A and B, thus, $\mathcal{H}(X)$ is complete but not geodesic.