Chapter 4

Extreme graphs and networks.

Schedule. Simple graphs, finite graphs, vertices, edges, isomorphism of graphs, adjacency, incidence, neighborhood of a vertex, subgraph, spanning subgraph, complete graph, subgraph generated by vertices, subgraph generated by edges, walk, degenerate and non-degenerated walks, open and closed walks, trail, path, circuit, cycle, connected graph, components of a graph, forest, tree, weighted graph, the weight of a subgraph, the weight of trail, the weight of walk, operations on graphs, union, disjoint union, intersection, difference, deleting edges, deleting vertices, quotient graphs, quotient by an edge, splitting a vertex, splitting of a vertex, graphs in metric spaces, the length of an edge, the length of a graph, the length of minimum spanning tree, minimum spanning tree, the length of Steiner minimal tree, Steiner minimal tree, the length of minimal filling, minimal filling, mst-spectrum of finite metric space, calculation of mst-spectrum in terms of partitions, graphs with boundaries, boundary (fixed) vertices, interior (movable) vertices, networks, parameterizing graphs of networks, boundary of a network, the length of a network, splitting and splitting off for networks, full Steiner tree, Steiner minimal trees existence in boundedly compact metric spaces.

In this section, we collect information about various kinds of extreme graphs and networks. We will consider two types of such graphs: minimal spanning trees and shortest trees, also called Steiner minimal trees.

4.1 Necessary information from graph theory

We will consider only simple graphs, so in what follows by a *graph* we mean a pair $G = (V, E)$ consisting of two sets V and E, respectively called *the set of vertices* and *the set of edges* of the graph G; the elements from V are called *vertices*, and from E are called *edges* of the graph G. The set E is a subset of the family of two-element subsets of V. If V and E are finite sets then the graph G is called *finite*.

It is convenient to use the following notation:

- if $\{v, w\} \in E$ is an edge of the graph G, then we will write it in the form vw or wv; we will also say that the edge *vw joins the vertices v and w*, and that v and w are *the vertices of the edge vw*;
- if the sets V and E are not explicitly indicated, and only the notation for the graph G is introduced, then the set of vertices of this graph is usually written as $V(G)$, and the set of edges is denoted by $E(G)$.

Recall some concepts from the graph theory. Graphs $G = (V, E)$ and $H = (W, F)$ are called *isomorphic* if there exists a bijective map $f : V \to W$ such that $uv \in E$ if and only if $f(u)f(v) \in F$. Such a mapping f is called an *isomorphism of the graphs G and H*. Isomorphic graphs are often identified and, therefore, are not distinguished.

Two vertices $v, w \in V(G)$ are called *adjacent* if $vw \in E(G)$. Two different edges $e_1, e_2 \in E(G)$ are called *adjacent* if they have a common vertex, i.e., if $e_1 \cap e_2 \neq \emptyset$. Each edge $vw \in E(V)$ and its vertex, i.e., v or w, *are incident* to each other. The set of vertices of a graph G adjacent to a vertex $v \in V$ is called the *neighborhood of the vertex v* and denoted by N_v. The cardinal number of edges incident to a vertex v is called the *degree of the vertex v* and is denoted by $\deg v$, so $\deg v = \#N_v$.

A *subgraph* of a graph $G = (V, E)$ is each graph $H = (W, F)$ provided that $W \subset V$ and $F \subset E$. The fact that a graph H is a subgraph of a graph G will be written as $H \subset G$. If $W = V$ then the subgraph $H \subset G$ is called *spanning*.

On the set of all graphs whose vertex sets lie in a given set V, the inclusion relation $\subset$ defines a partial order. The smallest element in this order is the empty graph $(\emptyset, \emptyset)$; the greatest one is called *the complete graph on V*, which we denote by $K(V)$: in $K(V)$ each pair of vertices are joined by an edge. This partial order induces the one on the set of all subgraphs of a graph $G = (V, E)$: now the smallest element is again the empty graph $(\emptyset, \emptyset)$, but the greatest one is G.

For each $W \subset V$ we define *the subgraph $G(W)$ of the graph G generated by W*: its set of vertices coincides with W, and its set of edges consists of all $e \in E$ that connect the vertices from W. In other words, $G(W)$ is maximal among subgraphs of G whose vertex sets coincides with W.

Now, we define a similar construction by interchanging vertices and edges. Namely, for $F \subset E$ we define *the subgraph $G(F)$ of the graph G generated by F*: its set of edges coincides with F, and its set of vertices is the collection of all vertices of G incident to edges from F. In what follows we also apply this construction.

A finite sequence $\gamma = (v_0 = v, v_1, \ldots, v_k = w)$ of vertices of a graph G is called a *walk of length k joining v and w* if for every $i = 1, \ldots, k$ the vertices v_{i-1} and v_i are adjacent, and the edges $e_i = v_{i-1}v_i$ are called the *edges of the walk γ*. A walk containing at least one edge is called *nondegenerate*, and not containing is called *degenerate*. The walt is called *closed* if $v_0 = v_n$, and it is called *open* otherwise. *A trail* is a walk with no repeated edges. *A path* is an open trail with no repeated vertices. *A circuit* is a closed trail. *A cycle* is a circuit with no repeated vertices.

A graph G is called *connected*, if each pair of its vertices are joined by a walk. Maximal (by inclusion) connected subgraphs of a graph G are called *components* of G. A graph without cycles is called *a forest*, and a connected forest is called *a tree*.

A weighted graph is a graph $G = (V, E)$ equipped with *a weight function* $\omega\colon E \to [0, \infty)$ (sometimes it is useful to consider more general weight functions, for instance, with possibility of negative values or ∞). Sometimes we denote such weighted graph as (V, E, ω) or (G, ω). *The weight $\omega(H)$ of a subgraph $H \subset G$* is the sum of the weights of edges from this subgraph: $\omega(H) = \sum_{e \in E(H)} \omega(e)$. We can extend this definition to trails, in particular, to paths, circuits and cycles, considering them as subgraphs of G. In the case of the walk $\gamma = (v_0 = v, v_1, \ldots, v_k = w)$, its weight is defined as the sum of weights of its consecutive edges: $\omega(\gamma) = \sum_{i=1}^{n} \in (v_{i-1}v_i)$. For graphs without weight functions these notions are defined as well by assigning the weight 1 to each edge by default.

Remark 4.1. As in the case of metric spaces, we sometimes won't explicitly denote the weight function. Instead of that, when we speak about weighted graph G, the weights of all the objects x related to such G we denote by $|x|$, for example, for $e \in E$ by $|e|$ we mean the weight of this edge, and $H \subset G$ by $|H|$ we mean the weight of H, etc.

4.1.1 Some operations on graphs

Let $H_1 = (W_1, F_1)$ and $H_2 = (W_2, F_2)$ be subgraphs of a graph G. Then the following subgraphs are defined:

- *the union* $H_1 \cup H_2 = (W_1 \cup W_2, F_1 \cup F_2)$;
- *the disjoint union*: if $W_1 \cap W_2 = \emptyset$, then to emphasize this fact, instead of $H_1 \cup H_2$ we write $H_1 \sqcup H_2$;
- *the intersection* $H_1 \cap H_2 = (W_1 \cap W_2, F_1 \cap F_2)$;
- *the difference* $H_1 \setminus H_2 = H_1(W_1 \setminus W_2)$.

Remark 4.2. We can define the operations described above on any graphs H_i, not only on subgraphs of a graph. To reduce these definitions to the previous ones, we consider H_i as subgraphs of the graph $K(W_1 \cup W_2)$.

Problem 4.1. Show that each forest is the disjoint union of trees that are components of this forest.

If $G = (V, E)$ is a graph, and F is a set possibly intersecting E, then the operation of *deleting the set of edges F from the graph G* produces the graph $G \setminus^e F := (V, E \setminus F)$. If $F = \{e\}$ then instead of $G \setminus^e \{e\}$ we will write $G \setminus^e e$. The operation of *deleting a set of vertices W from the graph G* produces the graph $G \setminus^v W := G(V \setminus W)$. If $W = \{w\}$ then instead of $G \setminus^v \{w\}$ we will write $G \setminus^v w$. If it is clear that F refers to edges, or W refers to vertices, we write simplified $G \setminus F$ or $G \setminus W$, respectively.

Using the operation of deleting edges, we define *the complement of a graph $G = (V, E)$* or, in other words, *the graph dual to G* to be the graph $\bar{G} = K(V) \setminus E$. Thus, the dual graph $\bar{G}$ has the same set of vertices V, and its edges are exactly those edges of the complete graph on V that were absent in the original graph G.

Another useful operation for us produces *a quotient graph $G = (V, E)$*: let $\sim$ be an equivalence relation on V, and $V = \sqcup_{i \in I} V_i$ the partition into classes of this equivalence. We put $V/\!\sim\, = \{V_i\}$, and as $E/\!\sim$ we take the set of pairs V_iV_j, $V_i \neq V_j$ for which there exist $v_i \in V_i$, $v_j \in V_j$, such that $v_iv_j \in E$. By *the quotient graph $G/\!\sim$* we call the graph $(V/\!\sim, E/\!\sim)$. An important particular case of this operation creates *the quotient of G by an edge $e = vw \in E$*: the result is the graph $G/\!\sim$ for the equivalence relation identifying the vertices v and w. We denote this quotient graph by G/e.

The following notation and concepts are also useful: the equivalence class containing a given vertex v will be denoted by $[v]$; the mapping $\pi\colon V \to V/\!\sim$, $\pi\colon v \mapsto [v]$, is called *the canonical projection*.

Problem 4.2. Let $G = (V, E)$ be a connected graph, and $\sim$ an arbitrary equivalence relation on the set V. Show that the graph $G/\!\sim$ is connected.

Problem 4.3. Let $G = (V, E)$ be an arbitrary tree, and $\sim$ be an equivalence relation on the set V such that for each class V_i of this equivalence the subgraph $G(V_i)$ is a tree. Show that then $G/\!\sim$ is a tree.

In some cases, the following operations are inverse to the quotient by an edge. We define two such operations: splitting a vertex of degree greater than or equal to 4, and splitting off some vertex of degree 1 from a vertex of degree greater than or equal to 2.

So, let $G = (V, E)$ be a graph, $v \in V$, $\deg v \ge 4$. We partition the neighborhood N_v of the vertex v into two sets V_1 and V_2, each of which contains at least two vertices. Consider the graph $G \setminus v$, add to its vertex set $V \setminus \{v\}$ two elements w_1 and w_2 not contained in $V \setminus \{v\}$, and, to the set of edges, all pairs of the form w_1v_1, $v_1 \in V_1$, w_2v_2, $v_2 \in V_2$, as well as the pair w_1w_2. We call the obtained graph *the result of splitting the vertex* v, and the edge w_1w_2 *the splitting edge*. It is clear that the graph obtained from G by the composition of splitting a vertex and the quotient by the corresponding splitting edge, is isomorphic to G (just in this sense, the splitting is inverse to the quotient operation).

To determine the splitting off a vertex of degree 1 from v, add to V an element w not contained in V, and add the edge vw to E. The obtained graph is called *the result of splitting off the vertex w from the vertex v*, and the edge wv is called *the splitting edge*. It is clear that the quotient by the splitting edge is isomorphic to the original graph (in this sense, splitting off is also inverse to the quotient operation).

4.2 Graphs and optimization problems

Let $G = (V, E)$ be an arbitrary graph. We say that the graph G is defined *in a metric space* X if $V \subset X$. For every such graph, *the length $|e|$ of its edge $e = vw$* is defined as the distance $|vw|$ between the ending vertices v and w of these edge, as well as *the length $|G|$ of the graph G* itself as the sum of the lengths of all its edges. More generally, one can replace the metric space X with a weighted complete graph $\big(K(X), \omega\big)$; another possibility — to consider some weighted graph (H, ω) with $V(H) = X$ (not necessarily the complete one), such that G is a subgraph of H. Let us note that each metric space (X, ρ) can be considered as a weighted complete graph, namely, as $\big(K(X), \rho\big)$.

4.2.1 Minimum spanning tree problem

Let M be a metric space. We consider M as a weighted complete graph $K(M)$, and denote by $\mathcal{T}(M)$ the set of all spanning trees in $K(M)$. Then we put

$$\operatorname{mst}(M) = \inf_{T \in \mathcal{T}(M)} |T|$$

and call it *the length of minimum spanning tree on M*. Each $T \in \mathcal{T}(M)$ with $|T| = \operatorname{mst}(M)$ is call *a minimum spanning tree on M*. The set of all minimum spanning trees on M is denoted by $\operatorname{MST}(M)$.

Remark 4.3. If M is finite, them $\operatorname{MST}(M) \ne \emptyset$. For infinite M the situation is rather more difficult, see [1] and [2].

Example 4.4. If all nonzero distances in M are the same, then every spanning tree in $K(M)$ is minimal, so $\operatorname{MST}(M) = \mathcal{T}(M)$.

If $\#M = 3$, then each minimum spanning tree is obtained from the complete graph $K(M)$ by deleting the longest edge (if there are several, then any of them).

Problem 4.4. Let M be a finite metric space. Partition M into nonempty subsets M_1 and M_2, and let $v_i \in M_i$ were chosen in such a way that $|v_1v_2| = |M_1M_2|$. Prove that there exists a minimum spanning tree $T \in \operatorname{MST}(M)$ such that $v_1v_2 \in E(T)$.

Remark 4.5. The problem of finding a minimum spanning tree can be naturally generalized. Let M be a set. Consider a connected weighted graph H with $V(H) = M$, and denote by $\mathcal{T}(H)$ the set of all spanning subtrees of H. Then we put

$$\operatorname{mst}(H) = \inf_{T \in \mathcal{T}(H)} |T|$$

and call it *the weight of minimum spanning tree in H*. If there exists $T \in \mathcal{T}(H)$ such that $|T| = \operatorname{mst}(H)$ then we call such T *a minimum spanning tree in H*. The set of all minimum spanning trees in H is denoted by $\operatorname{MST}(H)$. If M is a metric space, and $H = K(M)$ the corresponding weighted complete graph, then $\operatorname{mst}(H) = \operatorname{mst}(M)$ and $\operatorname{MST}(H) = \operatorname{MST}(M)$.

Note that there are a number of fast algorithms that solve the problem of finding a minimum spanning tree in a finite weighted connected graph. The most popular of them are Kruskal [5] and Prim [6] algorithms.

4.2.2 Steiner minimal tree problem

Now we generalize the notion of minimum spanning tree. To do that, we consider M as a subset of another metric spaces X, then we will minimize $\mathrm{mst}(V)$ over all $M \subset V \subset X$. Namely, we put

$$\mathrm{smt}_X(M) = \inf\bigl\{\mathrm{mst}(V) : M \subset V \subset X\bigr\}$$

and call it *the length of Steiner minimal tree on* M. Each $T \in \mathcal{T}(V)$ for $M \subset V \subset X$ is called *a shortest tree on* M or *a Steiner minimal tree on* M if $|T| = \mathrm{smt}_X(M)$. The set of all Steiner minimal trees on M is denoted by $\mathrm{SMT}_X(M)$. If it is clear or not important what X the set M belongs to, we simply write $\mathrm{smt}(M)$ and $\mathrm{SMT}(M)$ omitting X.

Remark 4.6. The following terminology is convenient when we study Steiner minimal trees or minimum spanning trees: if $G = (V, E)$ is a graph such that $V \subset X$, then we say that G is a graph *in the space* X. If $M \subset V$ then we say that G *joins* M; if $M = V$ then we say that G *spans* M. Thus, looking for minimum spanning trees we minimize the length of the trees spanning M, and for Steiner minimal trees we deal with the trees in the space X joining M.

Remark 4.7. The classical problem of finding a shortest tree is formulated for the Euclidean plane $X = \mathbb{R}^2$. The case $\#M = 3$ arose as early as 1643 in works of Fermat [7]. For an arbitrary finite number of points on the Euclidean plane, the problem was posed by Jarník and Kössler in 1934 [8]. Courant and Robbins [9] mistakenly called the problem of finding a shortest tree on the Euclidean plane the Steiner problem. Due to popularity of the book [9], this title has been fixed. The Steiner problem can be solved by Melzak's algorithm [10] and its many improvements, see for example [11] and [12]. As shown in [13], the Steiner problem is algorithmically complex (NP-complete).

Remark 4.8.

(1) Generally speaking, the set $\mathrm{SMT}(M)$ can be empty, also for finite M, however, the value $\mathrm{smt}(M)$ is always defined.

(2) The set $\mathrm{SMT}_X(M)$ and the value $\mathrm{smt}_X(M)$ depend not only on the distances between points from M, but also on the geometry of the ambient space X: isometric M lying in different metric spaces X can be joined by Steiner minimal trees of different lengths. Some details on the theory of Steiner minimal trees can be found, for example, in [3] or [4].

Problem 4.5. Find all Steiner minimal trees for 3-point boundaries in the Euclidean plane. How many such trees exist for different boundaries?

Problem 4.6. Find all Steiner minimal trees for the vertices of a square in the Euclidean plane. How many such trees exist?

Problem 4.7. Find all Steiner minimal trees for 3-point boundaries in the plane with ℓ_1-metric defined by the norm $\bigl\|(x, y)\bigr\| = |x| + |y|$. How many such trees exist for different boundaries?

Problem 4.8. Construct an example of a complete metric space and of some its finite subset M, such that there is no a Steiner minimal tree joining M.

4.2.3 One-dimensional minimal filling problem

We now fix a finite metric space M. We will embed it isometrically into various metric spaces X, and minimize $\mathrm{smt}_X(M)$ over all such embeddings. To overcome the Cantor paradox, we put

$$\mathrm{mf}(M) = \inf\Bigl\{r \mid \text{there exists an isometric embedding } \nu\colon M \to X \text{ with } \mathrm{smt}_X\bigl(\nu(M)\bigr) \le r\Bigr\}$$

and call it *the length of minimal filling of* M. Each tree $G \in \mathrm{SMT}_X\bigl(\nu(M)\bigr)$ such that $|G| = \mathrm{mf}(M)$ is called *a minimal filling of* M. The set of all minimal fillings of M is denoted by $\mathrm{MF}(M)$.

Remark 4.9. Each graph G in a metric space X can be naturally considered as a weighted graph with the weight function assigning to the edges their lengths. Thus, each minimal filling is a weighted graph. The triangle inequality in X leads to the fact that the distances between points in M are majorized by the length of pathes in G connecting these points.

All this motivates an alternative equivalent definition of minimal fillings. Namely, let $G = (V, E, \omega)$ be a weighted connected graph. Recall that in Construction 2.7 we introduced the corresponding pseudometric d_ω on V as follows: for arbitrary $v, w \in W$ we put

$$d_\omega(v,w) = \inf\bigl\{\omega(\gamma) : \gamma \text{ is a walk joining } v \text{ and } w\bigr\}.$$

Let M be a metric space. A connected weighted graph $G = (V, E, \omega)$ joining M is called *a filling of M* if for any $v, w \in M$ we have $|vw| \le d_\omega(v, w)$.

Problem 4.9. Prove that for any metric space M it holds

$$\mathrm{mf}(M) = \inf\bigl\{\omega(G) : G \text{ is a filling of } M\bigr\}.$$

The following results were obtained in [15].

Problem 4.10. Prove that for any finite metric space M there exists a minimal filling.

Problem 4.11. Let M be a finite metric space with equal non-zero distances. Describe all minimal filling of M.

Remark 4.10. The multidimensional problem on minimal fillings was formulated by M.Gromov [14]. One-dimensional minimal filling as a stratified version of the Gromov's problem was studied by Ivanov and Tuzhilin [15].

4.3 mst-spectrum of a finite metric space

In this section we consider only finite metric spaces M, i.e., $\#M < \infty$.

To start with, we note that the minimum spanning tree, generally speaking, is not uniquely defined. For $G \in \mathrm{MST}(M)$, by $\sigma(G)$ we denote the vector whose elements are the lengths of the edges of the tree G sorted in descending order. The following result is well known, however, we present its proof for completeness.

Proposition 4.11. *For any $G_1, G_2 \in \mathrm{MST}(M)$ it holds $\sigma(G_1) = \sigma(G_2)$.*

Proof. Recall the standard algorithm for converting one minimum spanning tree to another [5].

Let $G_1 \ne G_2$, $G_i = (M, E_i)$, then $E_1 \ne E_2$ and $\#E_1 = \#E_2$, therefore, there exists $e \in E_2 \setminus E_1$. The graph $G_1 \cup e$ has a cycle C containing the edge e. There is no longer edge in the C cycle than e, because otherwise $G_1 \notin \mathrm{MST}(M)$. The forest $G_2 \setminus e$ consists of two trees whose vertex sets we denote by V' and V''. Clearly, $M = V' \sqcup V''$. The cycle C contains an edge $e' \ne e$ joining a vertex from V' with a vertex from V''. This edge does not lie in E_2, otherwise G_2 would contain a loop. Therefore, $e' \in E_1 \setminus E_2$.

The graph $G_2 \cup e'$ also contains some cycle C'. By the choice of e', the cycle C' also has the edge e. Similarly to the above, the length of the edge e is less than or equal to the length of the edge e', otherwise $G_2 \notin \mathrm{MST}(M)$. Therefore, $|e| = |e'|$.

Replacing the edge e' in G_1 with e, we get a tree G_1' of the same length, i.e., it is a minimum spanning tree as well, and G_1' and G_2 have one common edge more than the trees G_1 and G_2. Thus, in a finite number of steps, we rebuild the tree G_1 into the tree G_2, passing through minimum spanning trees. It remains to notice that $\sigma(G_1') = \sigma(G_1)$, therefore, $\sigma(G_1) = \sigma(G_2)$. □

Proposition 4.11 motivates the following definition.

Definition 4.12. For any finite metric space M, by $\sigma(M)$ we denote $\sigma(G)$ for an arbitrary $G \in \mathrm{MST}(M)$ and call it *the* mst*-spectrum of the space M*.

Construction 4.1. For a set M and a cardinal number $k \le \#M$, by $\mathcal{D}_k(M)$ we denote the family of all possible partitions of the set M into k of its nonempty subsets. Now let M be a metric space and $D = \{M_i\}_{i\in I} \in \mathcal{D}_k(M)$. Put $\alpha(D) = \inf\bigl\{|M_iM_j| : i \ne j\bigr\}$.

Theorem 4.13. *Let M be a finite metric space and $\sigma(M) = (\sigma_1, \ldots, \sigma_{n-1})$. Then*

$$\sigma_k = \max\bigl\{\alpha(D) : D \in \mathcal{D}_{k+1}(M)\bigr\}.$$

Proof. Let $G = (M, E) \in \mathrm{MST}(M)$ and the set E be ordered so that $|e_i| = \sigma_i$. Denote by $D = \{M_1, \ldots, M_{k+1}\}$ the partition of the set M into the sets of vertices of the trees $G \setminus \{e_i\}_{i=1}^k$.

Lemma 4.14. *We have* $\alpha(D) = |e_k|$.

Proof. Indeed, we choose arbitrary M_i and M_j, $i \ne j$, in them we take points P_i and P_j, respectively, and let γ be the unique path in G, joining P_i and P_j. Then γ contains some edge e_p, $1 \le p \le k$. However, due to the minimality of the tree G, we have $|P_iP_j| \ge |e_p| \ge \min_i |e_i| = |e_k|$, thus $|M_iM_j| \ge |e_k|$, so $\alpha(D) \ge |e_k|$. On the other hand, if i and j are chosen so that e_k joins M_i and M_j, then we get $\alpha(D) \le |M_iM_j| = |e_k|$. □

Now consider an arbitrary partition $D' = \{M'_1, \dots, M'_{k+1}\}$.

Lemma 4.15. *We have* $\alpha(D') \le \alpha(D)$.

Proof. By virtue of Lemma 4.14, it suffices to show that $\alpha(D') \le |e_k|$. Denote by E' the set consisting of all edges $e_p \in E$, for each of which there are M'_i and M'_j, $i \ne j$, such that e_p joins M'_i and M'_j. Since G is connected, the set E' consists of at least k edges; otherwise, the set of indices $\{1, \dots, k+1\}$ is split into two nonempty subsets I and J such that the sets $\cup_{i\in I} M'_i$ and $\cup_{j\in J} M'_j$ that generate the partition M are not joined by any edge from E. On the other hand, if some M'_i and M'_j, $i \ne j$, are joined by an edge $e' \in E'$, then $|M'_iM'_j| \le |e'|$, hence $\alpha(D') = \min |M'_iM'_j| \le \min_{e'\in E'} |e'| \le |e_k|$. □

Lemma 4.15 completes the proof of the theorem. □

4.4 Networks

To study Steiner minimal trees and minimal fillings, it is sometimes more convenient to work with so-called networks instead of the graphs in metric spaces. For example, if we investigate deformations of such graphs perturbing the positions of some their vertices, it may happen that after such a perturbation some vertices coincide, however we would like to preserve the structure of the graph by considering the coinciding vertices as different ones. To achieve this, we suppose that the graph is given apart of the metric space, and "the positions of its vertices in the space" are provided by a mapping from the vertex set of the graph to this space. Such mappings are called networks.

Remark 4.16. Let us discuss three additional observations.

(1) In the optimization problems we usually deal with connected graphs, thus the domain of each network will be the vertex set of a connected graph, more often, of a tree.

(2) We usually investigate boundary-value problems, that is why we need to partition the vertices into boundary and all remaining (nonboundary) ones.

(3) We usually minimize the length of a graph. If a graph G in a metric space contains a nonboundary vertex of degree 1 or 2, we can simplify G preserving its connectedness, boundary, and not increasing its length. In the case of degree 1 nonboundary vertex, we can simply remove the edge incident to this vertex. In the case of degree 2 nonboundary vertex, we can change the both edges incident to this vertex by the unique edge joining the remaining vertices of these edges. That is why we usually assume that each boundary contains all the vertices of degree 1 and 2.

Remark 4.17. Indeed, we could also destroy possible cycles in the graph we optimize, and thus we might restrict ourselves with trees. However, in what follows we will see that such restriction leads us to some inconvenience, that is why we do not limit ourselves with trees only, but develop the corresponding theory for general connected graphs.

Now we are ready to give formal definitions.

We will assume that in each graph G there is a certain set of vertices $\partial G \subset V(G)$ containing all vertices of degree 1 and 2 (the set ∂G can be empty), which is called *the boundary of the graph* G, and the vertices from ∂G are called *boundary* ones. The remaining vertices of the graph G are called *interior* ones. Sometimes the boundary vertices are also called *fixed*, while the interior vertices are called *movable*.

Let $G = (V, E)$ be a connected graph with some boundary ∂G. *A network of the type* G *in a metric space* X is an arbitrary mapping $\Gamma\colon V \to X$. The G is also called *the parameterizing graph* of Γ. In what follows, we will transfer to the networks all the terminology from the graph theory related to their parameterizing graphs.

Let $\Gamma\colon V \to X$ be a network parameterized by a connected graph $G = (V, E)$ with a boundary $\partial G \subset V$. Then

(1) the restrictions of Γ to the vertices and the edges of G are called *the vertices* and *the edges* of Γ, respectively;

(2) the restriction of Γ to ∂G is called *the boundary of* Γ and is denoted by $\partial\Gamma$;

(3) for each $vw \in E$ the value $|\Gamma(v)\Gamma(w)|$ is called *the length of the edge* $\Gamma\colon \{v, w\} \to X$;

(4) the sum of the lengths of all edges of Γ is called *the length of* Γ and is denoted by $|\Gamma|$.

Example 4.18. Let $G = (V, E)$ be a connected graph in a metric space X joining $M \subset X$, i.e., $M \subset V \subset X$, such that M contains all the vertices of G of degree 1 and 2. Put $\partial G = M$, and define a network $\Gamma\colon V \to X$ as the inclusion mapping $\Gamma\colon v \mapsto v$. Thus, Γ is a network in X of the type G, and $|\Gamma| = |G|$.

Example 4.19. Let X be a metric space, $G = (V, E)$ a connected graph with $\partial G = M \subset X$. Let $\Gamma\colon V \to X$ be a network whose restriction to M is the inclusion: $\Gamma(v) = v$ for all $v \in M$. Suppose that the mapping Γ is injective. By identifying each vertex $v \in V$ with its image $\Gamma(v) \in X$, we can consider G as a graph in X, so the length $|G|$ of G is defined. Then, with this identification, we have $|\Gamma| = |G|$.

Definition 4.20. As in Example 4.19, given a metric space X, let $G = (V, E)$ be a connected graph with $\partial G = M \subset X$, and $\Gamma\colon V \to X$ be a network whose boundary $\partial\Gamma\colon M \to X$ is the inclusion: $\partial\Gamma(v) = v$ for all $v \in M$. For such Γ we say that Γ *joins the subset M of the space X* (we do not assume here that Γ is injective outside M).

4.4.1 Networks and quotients

Now let V' be a set, $\sim$ an equivalence relation on V', and suppose that $V = V'/\!\sim$. Denote by $\pi\colon V' \to V$ the canonical projection. Let $G' = (V', E')$ be a connected graph such that $G = G'/\!\sim$ and $\partial G = \pi(\partial G')$. Let $\Gamma\colon V \to X$ be a network of the type G with the boundary $\partial\Gamma\colon \partial G \to X$, then the composition $\Gamma' = \Gamma \circ \pi\colon V' \to X$ is correctly defined and is a network in X of the type G' with the boundary $\partial\Gamma' = \partial\Gamma \circ \pi|_{\partial G'}$.

Problem 4.12. Show that

(1) $|\Gamma'| \ge |\Gamma|$;

(2) if G' is a tree, $V = \{V'_i\}$, and $G'(V'_i) \subset G'$ is a tree for each i, then G is also a tree and $|\Gamma'| = |\Gamma|$;

(3) give an example in which G' is a tree, G is not a tree, and $|\Gamma'| = |\Gamma|$;

(4) give an example in which G' and G are trees, and $|\Gamma'| > |\Gamma|$.

4.4.2 Splitting and splitting off for networks

In this section we extend the operations of "splitting a vertex" and "splitting off from a vertex" which we defined above, to the case of graphs with boundaries and the corresponding networks. These operations enable us to simplify the structures of networks in consideration.

In Section 4.1 we defined splitting a vertex of degree greater than or equal to 4, and splitting off a vertex of degree 1 from a vertex of degree greater than or equal to 2. For graphs with boundary, we refine these definitions.

We will split only interior vertices, while the resulting vertices will again be classified as interior; we will only split off from boundary vertices, and if we split off a vertex w from the boundary vertex v, then we assign the vertex v to interior one, and w to boundary one. The both these operations can be naturally defined for networks.

A graph G with a boundary is called *non-splittable* if no vertex can be split off from any boundary vertex, and no interior vertex can be split. For a finite graph G, we define *the degree $p(G)$ of non-splitting*, setting it equal to the sum of $\deg v - 3$ over all interior vertices v of G. It is easy to see that the G, for which the degrees of all boundary vertices are 1, is not splittable if and only if $p(G) = 0$.

The next lemma will be the key point in the proof of Theorem 4.23.

Lemma 4.21. *For each finite graph $G = (V, E)$ with a boundary ∂G there exists a finite graph $G' = (V', E')$ with a boundary $\partial G'$, and an equivalence relation $\sim$ on V', such that $G = G'/\!\sim$ and the following properties hold*:

(1) *all boundary vertices of G' have degree* 1;

(2) *all interior vertices of G' have degree* 3;

(3) *for connected G, the graph G' is connected*;

(4) *for a tree G, the graph G' is a tree*;

(5) *the $\sim$-class of each boundary vertex of G' is a singleton*;

(6) *the canonical projection mapping $\pi\colon V' \to V$ corresponding to $\sim$ is a bijection between $\partial G'$ and ∂G*;

(7) *if G (and G') is a tree, then for each $v \in V$, $W = \pi^{-1}(v)$, the graph $G'(W) \subset G'$ is a tree.*

Proof. To start with, we split off a vertex of degree 1 from each boundary vertex of G. As a result, we obtain a graph with all boundary vertices of degree 1. In this graph, we will successively split all its interior vertices of degree greater than or equal to 4. It is easy to see that the degree of non-splitting of this graph decreases by 1 for each splitting, therefore, in a finite number of steps, we arrive to a non-splittable graph $G' = (V', E')$. We denote by $\sim$ the equivalence relation on V', which is obtained from the trivial equivalence relation on V (whose all equivalence classes are singletons) according to the following rule: for each splitting off and splitting, the resulting pair of vertices is equivalent to all those ones to which the original vertex was equivalent, and to each other. It is clear that $G'/{\sim} = G$. It remains to notice that those splittings preserve the connectivity and do not create cycles. □

4.5 Steiner minimal trees existence

As we already mentioned in Problem 4.8, some metric spaces, also under assumption of completeness, may contain finite subsets which can not be joined by a Steiner minimal tree. In this section we prove that Steiner minimal trees always exist in boundedly compact metric spaces. To do that, we first need to reduce this problem to minimization of a finite number of continuous functions. To construct these functions, we show that it suffices to minimize the lengths of networks whose types can be chosen from some finite collection.

A tree G, and a network Γ of the type G in a metric space X, both joining a set $M \subset X$, are called *full Steiner trees* if all their boundary vertices have degree 1 and all their interior vertices have degree 3. Let us stress that the boundary $\partial\Gamma$ of this network is the inclusion $M \subset X$.

Remark 4.22. If the graph G from Lemma 4.21 is a tree then the graph G' from this lemma is a full Steiner tree.

Theorem 4.23. *Let X be an arbitrary metric space and M be a finite subset of X. Then $\operatorname{smt}(M)$ is equal to the infimum of the lengths of all full Steiner trees Γ joining M.*

Proof. Recall that

$$\operatorname{smt}(M) = \inf\bigl\{|G| : G \text{ is a tree in } X \text{ with } \partial G = M\bigr\}.$$

As we mentioned in Remark 4.16, it suffices to consider only trees G whose vertices of degree 1 and 2 belong to M. In what follows, we will minimize over such trees only.

Problem 4.13. Prove that any finite tree G with $n \ge 2$ boundary vertices contains at most $n-2$ interior vertices. The equality holds exactly in the case when G is a full Steiner tree.

Problem 4.13 states that in calculation of $\operatorname{smt}(M)$ we can consider only finite trees G with the boundary M. This enables us to use Lemma 4.21, according to which for every G there exists a full Steiner tree G' and an equivalence relation $\sim$ on $V(G')$ such that $G = G'/{\sim}$. Let $\pi\colon V(G') \to V(G)$ be the corresponding canonical projection, then this π, being considered as a mapping from $V(G')$ to X, is a full Steiner tree joining M. Since G and G' are trees, then by Lemma 4.21 each equivalence class generates a subtree of G', thus, by Problem 4.12, we have $|\pi| = |G|$, that completes the proof. □

To list, up to a natural isomorphism, all the full Steiner trees from Theorem 4.23, we construct a model set of such trees. By *a model full Steiner tree* we mean a full Steiner tree $G = (V, E)$ with $V = \{1, 2, \ldots, 2n-2\}$ and $\partial G = \{1, \ldots, n\}$. Two model full Steiner trees are called *equivalent* if there is an isomorphism between them that is identical on the boundary $\{1, \ldots, n\}$. Thus, equivalent trees differ from each other by the numbering of their interior vertices. We denote by $\mathcal{B}_n$ the set of all model full Steiner trees with n boundary vertices considered up to the introduced equivalence. In other words, we construct $\mathcal{B}_n$ by choosing in each equivalence class an arbitrary representative.

Now let X be a metric space and M be a finite subset of X consisting of n points. We enumerate the points from M in an arbitrary way, i.e., consider some bijection $\varphi\colon \{1, \ldots, n\} \to M$. Choose an arbitrary $G \in \mathcal{B}_n$, and consider a network Γ of the type G for which $\partial\Gamma = \varphi$. Then all such networks differ only in the "positions" of their interior vertices. The set of such networks is denoted by $[G, \varphi]$.

It is clear that all networks from Theorem 4.23 are obtained from the networks just described by identification, concordant with φ, of the set $\{1,\dots,2n-2\}$ with the sets of vertices of the graphs parameterizing the former networks. Thus, we have proved the following result.

Corollary 4.24. *Let X be an arbitrary metric space and $M\subset X$ be a finite subset of X. Then*

$$\operatorname{smt}(M)=\inf\bigl\{|\Gamma|:\Gamma\in[G,\varphi],\ G\in\mathcal{B}_n\bigr\}.$$

Remark 4.25. Note that the set $\mathcal{B}_n$ by which we minimize in Corollary 4.24 is finite, and the set $[G,\varphi]$, by which minimization is also carried out, can be infinite. In some cases, it is easy to prove that the infimum in $[G,\varphi]$ is attained for every G, which immediately implies that $\operatorname{SMT}(M)\neq\emptyset$ because $\mathcal{B}_n$ is finite.

Recall that a metric space is called boundedly compact if each of its closed balls is compact. Equivalent condition: a subset is compact if and only if it is closed and bounded.

We will present a technical result that is rather simple, but necessary in the future. Let $f\colon X\to\mathbb{R}$ be some function defined on a metric space. We fix an arbitrary point $p\in X$ and for $r\ge 0$ we set $F_p(r)=\inf_{x\in X\setminus B_r(p)}f(x)$.

Problem 4.14. Suppose that $F_p(r)\to\infty$ as $r\to\infty$. Prove that at any point $q\in X$ it holds $F_q(r)\to\infty$ as $r\to\infty$.

From Problem 4.14, the correctness of the following definition immediately follows. We say that a function $f\colon X\to\mathbb{R}$, defined on a metric space, *blows up at infinity* if the corresponding function $F_p(r)$ tends to infinity as $r\to\infty$ for some and, therefore, for every choice of the point $p\in X$.

Proposition 4.26. *Let $f\colon X\to\mathbb{R}$ be a continuous function defined on a boundedly compact metric space X. Suppose that f blows up at infinity, then f is bounded below and attains its infimum.*

Proof. Indeed, since the corresponding function $F_p(r)$ blows up at infinity, for some r_0 it holds $f(x)\ge 0$ for all $x\in X\setminus B_{r_0}(p)$. On the other hand, the function f is bounded below on the closed ball $B_{r_0}(p)$ due to its compactness. Thus, the lower boundedness of the function f is proved.

Let $f_0=\inf_{x\in X}f(x)$, and r be such that for all $x\in X\setminus B_r(p)$ we have $f(x)\ge f_0+1$. This means that the infimum of the function f is attained on the ball $B_r(p)$, and, due to the compactness of this ball, there exists a point $x_0\in B_r(p)$ for which $f(x_0)=f_0$. □

Theorem 4.27. *Let X be a boundedly compact metric space. Then for every nonempty finite $M\subset X$ we have $\operatorname{SMT}(M)\neq\emptyset$.*

Proof. We use Corollary 4.24. If $\#M=n$, then we choose an arbitrary enumeration $\varphi\colon\{1,\dots,n\}\to M$, as well as an arbitrary model full Steiner tree $G\in\mathcal{B}_n$. Then each network $\Gamma\in[G,\varphi]$ is uniquely determined by the positions of its interior vertices, i.e., by the "vector" $z=\bigl(\Gamma(n+1),\dots,\Gamma(2n-2)\bigr)\in X^{n-2}$. The function $\ell(z)=|\Gamma|$ is continuous as the sum of continuous functions. In addition, this function blows up at infinity, therefore, by virtue of Proposition 4.26, it attains its infimum. Also, there are a finite number of such functions in the formula from Corollary 4.24, so the infimum from this formula is attained at a minimum point of one of these functions. □

References to Chapter 4

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Exercises to Chapter 4

Exercise 4.1. Show that each forest is a disjoint union of trees that are components of this forest.

Exercise 4.2. Let $G = (V, E)$ be a connected graph, and $\sim$ an arbitrary equivalence relation on the set V. Show that the graph $G/\!\sim$ is connected.

Exercise 4.3. Let $G = (V, E)$ be an arbitrary tree, and $\sim$ be an equivalence relation on the set V such that for each class V_i of this equivalence the subgraph $G(V_i)$ is a tree. Show that then $G/\!\sim$ is a tree.

Exercise 4.4. Let M be a finite metric space. Partition M into nonempty subsets M_1 and M_2, and let $v_i \in M_i$ were chosen in such a way that $|v_1v_2| = |M_1M_2|$. Prove that there exists a minimum spanning tree $T \in \mathrm{MST}(M)$ such that $v_1v_2 \in E(T)$.

Exercise 4.5. Find all Steiner minimal trees for 3-point boundaries in the Euclidean plane. How many such trees exist for different boundaries?

Exercise 4.6. Find all Steiner minimal trees for the vertices of a square in the Euclidean plane. How many such trees exist?

Exercise 4.7. Find all Steiner minimal trees for 3-point boundaries in the plane with ℓ_1-metric defined by the norm $\big\|(x, y)\big\| = |x| + |y|$. How many such trees exist for different boundaries?

Exercise 4.8. Construct an example of a complete metric space and of some its finite subset M, such that there is no a Steiner minimal tree joining M.

Hint. Consider on the set $X = \{0, 1, 2, \ldots\}$ the distance function $|mn| = 1 + \frac{1}{m+n}$, $m \neq n$. Prove that it is a complete metric. Consider the space X^3 with the complete metric generated by the norm $\|\cdot\|_\infty$. Put $M = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$. Prove that $\mathrm{SMT}(M) = \emptyset$.

Exercise 4.9. Prove that for any metric space M it holds

$$\mathrm{mf}(M) = \inf\big\{\omega(G) : G \text{ is a filling of } M\big\}.$$

Exercise 4.10. Prove that for any finite metric space M there exists a minimal filling.

Exercise 4.11. Let M be a finite metric space with equal non-zero distances. Describe all minimal fillings of M.

Exercise 4.12. Let $G = (V, E)$ be a connected graph in a metric space X joining $M \subset X$, i.e., $M \subset V \subset X$. Put $\partial G = M$, and define a network $\Gamma\colon V \to X$ as the embedding mapping $\Gamma\colon v \mapsto v$. Let V' be a set, $\sim$ an equivalence relation on V', and suppose that $V = V'\!\sim$. Denote by $\pi\colon V' \to V$ the canonical projection. Let $G' = (V', E')$ be a connected graph such that $G = G'/\!\sim$. Then the composition $\Gamma' = \Gamma \circ \pi\colon V' \to X$ is a network in X of the type G'. Show that

(1) $|\Gamma'| \geq |\Gamma|$;

(2) if $G' = (V', E')$ is a tree, and for each class V_i' of the equivalence $\sim$ the subgraph $G'(V_i') \subset G'$ is a tree, then G is also a tree and $|\Gamma'| = |\Gamma|$;

(3) give an example in which G' is a tree, G is not a tree, and $|\Gamma'| = |\Gamma|$;

(4) give an example in which G' and G are trees, and $|\Gamma'| > |\Gamma|$.

Exercise 4.13. Prove that any finite tree G with $n \geq 2$ boundary vertices contains at most $n - 2$ interior vertices. The equality holds exactly in the case when G is a full Steiner tree.

Exercise 4.14. Let $f\colon X \to \mathbb{R}$ be some function defined on a metric space. We fix an arbitrary point $p \in X$ and for $r \geq 0$ we set $F_p(r) = \inf_{x \in X \setminus B_r(p)} f(x)$. Suppose that $F_p(r) \to \infty$ for $r \to \infty$. Prove that at any point $q \in X$ it holds $F_q(r) \to \infty$ as $r \to \infty$.