Chapter 3

Curves in Metric Spaces.

Schedule. Curves in a topological space, parameter of a curve, reparametrization, polygonal line in a metric space, its edges, the length of the edge, the length of the polygonal line, the length of a curve in a metric space, rectifiable curves, properties of the length functional, intrinsic metric, generalized intrinsic pseudometric, maximal and minimal generalized pseudometrics, minimum of generalized intrinsic pseudometrics, quotients of generalized intrinsic pseudometric spaces, Hopf-Rinow condition, Hopf-Rinow theorem Part 1, convergence and uniform convergence in terms of the corresponding product spaces, limits of sequences of Lipschitz mappings, arc-length and uniform curves, reparametrizations, uniform reparametrizations, Arzela-Ascoli theorem, shortest curves and geodesics, existences theory for shortest curves, geodesic metric space, midpoints and ε-midpoints, existence of shortest curves in term of midpoints, intrinsic metrics and ε-midpoints.

In this section we discuss some results related to the geometry of curves in metric spaces.

Recall that *a curve* in a topological space X is any continuous mapping $\gamma\colon [a,b]\to X$ from a segment $[a,b]\subset\mathbb{R}$ with the standard topology; the variable $t\in[a,b]$ is called *the parameter of the curve* γ, and the curve γ is sometimes written in the form $\gamma(t)$.

Each homeomorphism $\varphi\colon [c,d]\to[a,b]$ generates a new curve $\gamma\circ\varphi\colon [c,d]\to X$, about which we say that it is obtained from γ *by replacement* φ *of the parameter* t *with the parameter* $s\in[c,d]$. Moreover, if there is no misunderstanding, instead of the curve $(\gamma\circ\varphi)(s)$ we simply write $\gamma(s)$. Such replacement φ is also called *a reparametrization.*

Note that each reparametrization is a strictly monotonic continuous function. If the function φ grows, then we say that φ *reverses the direction*, otherwise that it *changes the direction.*

3.1 Rectifiable curves

Let X be a metric space. A finite sequence $L=(A_0,\dots,A_n)$ of points in the space X we called *a polygonal line in* X; moreover, the pairs (A_{i-1},A_i) will be called *edges of the polygonal line* L, and the numbers $|A_{i-1}A_i|$ *the lengths* of these edges. The sum of the lengths of all these edges we call *the length of the polygonal line* L and denote by $|L|$.

Let $\gamma\colon [a,b]\to X$ be an arbitrary curve. For each partition $\xi=(a=t_0<t_1<\dots<t_m=b)$, consider the corresponding polygonal line $L_\gamma(\xi)=\big(\gamma(t_0),\dots,\gamma(t_m)\big)$ (such polygonal lines will be called *inscribed in the curve* γ), then the value

$$|\gamma|=\sup\Big\{\big|L_\gamma(\xi)\big| : \xi \text{ is a partition of the segment } [a,b]\Big\}$$

is called *the length of the curve* γ. A curve γ is called *rectifiable* if $|\gamma|<\infty$.

Let us give some examples of rectifiable curves.

Example 3.1. Each C-Lipschitz curve $\gamma\colon [a,b]\to X$ is rectifiable, because for any partition ξ of the segment $[a,b]$ we have $|L_\gamma(\xi)|\le C(b-a)$ and, therefore, $|\gamma|\le C(b-a)<\infty$.

Denote by $\Omega(X)$ the family of all curves in a metric space X, and by $\Omega_0(X)\subset\Omega(X)$ the subfamily of all rectifiable curves. Note that on $\Omega(X)$ and on $\Omega_0(X)$ there are defined

(1) *the restriction* of each curve $\gamma\colon [a,b]\to X$ to each subsegment $[c,d]\subset[a,b]$;

(2) *the gluing* $\gamma_1\cdot\gamma_2$ of those pairs of curves $\gamma_1\colon [a,b]\to X$, $\gamma_2\colon [b,c]\to X$ for which $\gamma_1(b)=\gamma_2(b)$, namely, $(\gamma_1\cdot\gamma_2)\colon [a,c]\to X$ is the curve whose restrictions to $[a,b]$ and $[b,c]$ coincide with γ_1 and γ_2, respectively;

(3) *the reparametrization* and equivalence identifying curves that differ by parameterization.

The following proposition describes some properties of the length of a curve.

Proposition 3.2. *Let X be an arbitrary metric space. Then*

(1) **generalized triangle inequality:** *if $\gamma\in\Omega(X)$ joins the points $x,y\in X$, then $|\gamma|\ge|xy|$;*

(2) **additivity:** *if $\gamma=\gamma_1\cdot\gamma_2$ is the gluing of curves $\gamma_1,\gamma_2\in\Omega(X)$ then $|\gamma|=|\gamma_1|+|\gamma_2|$;*

(3) **continuity:** *for any $\gamma\in\Omega_0(X)$, $\gamma\colon[a,b]\to X$, the function $f(t)=\big|\gamma|_{[a,t]}\big|$ is continuous;*

(4) **independence from parameter:** *for each $\gamma\in\Omega(X)$, $\gamma\colon[a,b]\to X$, and reparametrization $\varphi\colon[c,d]\to[a,b]$, it holds $|\gamma|=|\gamma\circ\varphi|$;*

(5) **consistency with topology:** *for each $x\in X$, $\varepsilon>0$, $y\in X\setminus U_\varepsilon(x)$, and a curve $\gamma\in\Omega(X)$ joining x and y, it holds $|\gamma|\ge\varepsilon$;*

(6) **lower semicontinuity:** *for any sequence $\gamma_n\in\Omega_0(X)$ that converges pointwise to some $\gamma\in\Omega_0(X)$, we have*
$$|\gamma|\le\liminf_{n\to\infty}|\gamma_n|.$$

Proof. Only the items (3) and (6) are nontrivial, we prove them.

(3) Choose an arbitrary $t\in[a,b]$ and show that for any $\varepsilon>0$ there is $\delta>0$ such that for all $s\in[a,b]\cap(t-\delta,t+\delta)$ the inequality $\big|f(t)-f(s)\big|<\varepsilon$ holds. Put $\ell=|\gamma|$. By definition, there exists a partition ξ of the segment $[a,b]$ such that $\ell-\varepsilon/2<\big|L_\gamma(\xi)\big|\le\ell$. If $t\notin\xi$, add it to ξ (we denote the resulting partition by the same letter). It is clear that for the resulting partition, $\ell-\varepsilon/2<\big|L_\gamma(\xi)\big|\le\ell$ is still satisfied.

For δ_1 we take the distance from t to the nearest element of the partition ξ, other than t. Since subdivisions of the partition ξ can change the length of the polygonal line $L_\gamma(\xi)$ only within $(\ell-\varepsilon/2,\ell]$, then for each $s\in[a,b]\cap(t-\delta_1,t+\delta_1)$ the length $\ell_{ts}=\big|f(t)-f(s)\big|$ of the fragment of the curve γ between points $\gamma(t)$ and $\gamma(s)$ differs from $\big|\gamma(t)\gamma(s)\big|$ by less than $\varepsilon/2$. On the other hand, since the map γ is continuous, there exists $\delta_2>0$ such that for all $s\in[a,b]\cap(t-\delta_2,t+\delta_2)$ we have $\big|\gamma(t)\gamma(s)\big|<\varepsilon/2$. It remains to put $\delta=\min\{\delta_1,\delta_2\}$.

(6) Choose an arbitrary $\varepsilon>0$ and show that for sufficiently large n the inequality $|\gamma|\le|\gamma_n|+\varepsilon$ holds, thus $|\gamma|\le\liminf_{n\to\infty}|\gamma_n|+\varepsilon$ and, due to the arbitrariness of ε, we get what is required.

So, let $\varepsilon>0$ be fixed. Choose a partition $\xi=(a=t_0<t_1<\cdots<t_m=b)$ of the segment $[a,b]$ such that $|\gamma|-\big|L_\gamma(\xi)\big|<\varepsilon/2$. There is N such that for any $n>N$ and all i the inequality $\big|\gamma(t_i)\gamma_n(t_i)\big|<\frac{\varepsilon}{4m}$ holds. This immediately implies that
$$\big|\gamma(t_{i-1})\gamma(t_i)\big|<\big|\gamma_n(t_{i-1})\gamma_n(t_i)\big|+\frac{\varepsilon}{2m},$$
therefore $\big|L_\gamma(\xi)\big|<\big|L_{\gamma_n}(\xi)\big|+\varepsilon/2$. Thus,
$$|\gamma|<\big|L_\gamma(\xi)\big|+\varepsilon/2<\big|L_{\gamma_n}(\xi)\big|+\varepsilon/2+\varepsilon/2\le|\gamma_n|+\varepsilon,$$
as required. □

Problem 3.1. Prove the remaining items of Proposition 3.2.

Problem 3.2. Will the items (3) and (6) of Proposition 3.2 remain true if we change $\Omega_0(X)$ to $\Omega(X)$?

Problem 3.3. Show that a piecewise smooth curve in $\mathbb{R}^n$ is Lipschitzian with a Lipschitz constant equal to the maximum modulus of the velocity vector of the curve, therefore every such curve is rectifiable.

Let X be a metric space in which any two points are connected by a rectifiable curve. Then for any $x,y\in X$ the quantity
$$d_{in}(x,y)=\inf\big\{|\gamma|:\gamma\text{ is a curve joining }x\text{ and }y\big\}$$
is finite.

Problem 3.4. Let X be a metric space in which any two points are connected by a rectifiable curve.

(1) Prove that d_{in} is a metric.

(2) Denote by τ the metric topology of X w.r.t. the initial metric on X, by τ_{in} the metric topology w.r.t. d_{in}, by X_{in} the set X with metric d_{in} and topology τ_{in}. Show that $\tau\subset\tau_{in}$. In particular, if a mapping $\gamma\colon[a,b]\to X_{in}$ is continuous, then the mapping $\gamma\colon[a,b]\to X$ is continuous as well.

(3) Construct an example when $\tau \neq \tau_{in}$.

(4) Prove that for each rectifiable curve $\gamma\colon [a,b] \to X$ the mapping $\gamma\colon [a,b] \to X_{in}$ is continuous.

(5) Denote by $|\gamma|_{in}$ the length of a curve $\gamma\colon [a,b] \to X_{in}$. Show that for each curve $\gamma\colon [a,b] \to X$ which is also a curve in X_{in}, it holds $|\gamma| = |\gamma|_{in}$. Thus, the sets of rectifiable curves for X and X_{in} coincide, and each non-rectifiable curve in X is either a non-rectifiable one in X_{in}, or the mapping $\gamma\colon [a,b] \to X_{in}$ is discontinuous.

(6) Construct an example of continuous mapping $\gamma\colon [a,b] \to X$ such that the mapping $\gamma\colon [a,b] \to X_{in}$ is not continuous. Notice that the curve $\gamma\colon [a,b] \to X$ can not be rectifiable.

Definition 3.3. If d_{in} coincides with the original metric, then the original metric is called *intrinsic*. A metric space with an intrinsic metric is also called *intrinsic*.

Problem 3.5. Let X be a metric space in which any two points are connected by a rectifiable curve. Prove that the metric d_{in} is intrinsic.

Example 3.4. Let S^1 be the standard circle on the Euclidean plane $\mathbb{R}^2$.

(1) If for $x, y \in S^1$ we put $d(x,y)$ equal to the distance in $\mathbb{R}^2$ between these points, then the metric d on S^1 is not intrinsic.

(2) If, for $d(x,y)$, we choose the length of the smaller of the two arcs of the circle S^1 into which x and y divide it, then the resulting metric will be intrinsic.

Remark 3.5. If we allow generalized pseudometrics, then we can define d_{in} and intrinsic metric not only for those metric spaces where each pair of points is connected by a rectifiable curve, but for generalized pseudometric spaces too. To do that, we need only put $\inf \emptyset = \infty$, where inf is applied to subsets of $[0,\infty]$. Thus, if $x, y \in X$ cannot be joined by a curve, then we get $d_{in}(x,y) = \infty$. The same holds for x and y joined by non-rectifiable curves only.

Recall that in the previous chapter we introduced the equivalence relation $\sim_1$ such that $x \sim_1 y$ if and only if the generalized distance between these points equals ∞. Applying this equivalence to the generalized d_{in}, we get a partition of the space X into metrics subspaces with finite d_{in}.

3.2 Maximal and minimal pseudometrics, quotients

In Chapter 2 we defined maximal pseudometric, see Construction 2.12, and we demonstrated its relation with quotient distance. Now, let us define maximal and minimal pseudometrics for arbitrary families of generalized pseudometrics. Also, we apply these generalizations to investigation of intrinsic distances.

Construction 3.1 (Maximal and minimal pseudometric for arbitrary family of pseudometrics)**.** Let X be a set, and $\mathcal{D}$ an arbitrary nonempty family of generalized pseudometrics on X. Then we consider the standard partial order on the set of all generalized pseudometrics on X, namely, $d_1 \le d_2$ if $d_1(x,y) \le d_2(x,y)$ for all $x, y \in X$. By means of this partial order, we define $\inf \mathcal{D}$ and $\sup \mathcal{D}$ in the standard way.

Put $\overline{d}_{\mathcal{D}}(x,y) = \sup_{d\in\mathcal{D}} d(x,y)$, then the same proof as for Lemma 2.16 can be carried out to obtain that $\overline{d}_{\mathcal{D}}$ is a generalized pseudometric and $\overline{d}_{\mathcal{D}} = \sup \mathcal{D}$, thus, $\sup \mathcal{D}$ exists for any nonempty $\mathcal{D}$. We call $\overline{d}_{\mathcal{D}}$ *the maximal generalized pseudometric for the family* $\mathcal{D}$. Notice that Construction 2.12 is a particular case of the present one: $d_b = \sup \mathcal{D}_b$.

Now, let us put $b(x,y) = \inf_{d\in\mathcal{D}} d(x,y)$. This b is not necessarily a generalized pseudometric (construct an example), however, the $d_b = \sup \mathcal{D}_b$ is, and we denote it by $\underline{d}_{\mathcal{D}}$.

Proposition 3.6. *Under above notations,* $\underline{d}_{\mathcal{D}} = \inf \mathcal{D}$*, and, thus,* $\inf \mathcal{D}$ *exists for any nonempty* $\mathcal{D}$*.*

Proof. Since for any $d' \in \mathcal{D}_b$ and $d \in \mathcal{D}$ we have $d' \le b \le d$, then $\underline{d}_{\mathcal{D}} \le \mathcal{D}$, i.e., $\underline{d}_{\mathcal{D}}$ is a lower bound for $\mathcal{D}$.

Now, let d' be an arbitrary lower bound for $\mathcal{D}$, then $d' \le b$ and, therefore, $d' \in \mathcal{D}_b$. Thus $d' \le \underline{d}_{\mathcal{D}}$. □

We call $\overline{d}_{\mathcal{D}}$ *the minimal generalized pseudometric for the family* $\mathcal{D}$.

Also, we can define $\sup \mathcal{D}$ and $\inf \mathcal{D}$ for empty family $\mathcal{D}$: the first is the zero pseudometric, and the last is the generalized pseudometric equal ∞ for any pair of distinct points.

Below we will use the following result.

Problem 3.6. Let $\rho_1 \le \rho_2$ be generalized pseudometrics on a set X, and Y be a topological space. Prove that each mapping $f\colon Y \to X$, continuous w.r.t. ρ_2, is also continuous w.r.t. ρ_1, in particular, if γ is a curve in (X, ρ_2), then γ is also a curve in (X, ρ_1); moreover, if ρ_1' and ρ_2' denote the corresponding generalized intrinsic pseudometrics, then $\rho_1' \le \rho_2'$.

Proposition 3.7. *Let X be a set, and $\mathcal{D}$ an arbitrary family of intrinsic generalized pseudometrics on X. Then $\underline{d}_{\mathcal{D}} = \inf \mathcal{D}$ is intrinsic.*

Proof. We put $b(x, y) = \inf_{d\in\mathcal{D}} d(x, y)$, $\rho = \underline{d}_{\mathcal{D}} = d_b$, and denote by $\rho' \ge \rho$ the generalized intrinsic pseudometric corresponding to ρ'. Since $\rho \le d$ for all $d \in \mathcal{D}$, and all d are intrinsic, then $\rho' \le d$ for all $d \in \mathcal{D}$ due to Problem 3.6. Thus, $\rho' \le b$ and, therefore, $\rho' \in \mathcal{D}_b$. However, ρ is maximal for the class $\mathcal{D}_b$, so $\rho \ge \rho'$ and we get $\rho = \rho'$. □

In the previous chapter we considered the construction of quotient spaces. What can we say about the quotient space if the distance function of the initial one is intrinsic?

Proposition 3.8. *Let X be a generalized pseudometric space whose distance function is intrinsic, and $\sim$ an arbitrary equivalence on X. Then the generalized pseudometric of $X/{\sim}$ is intrinsic as well.*

Proof. Define $d\colon X \times X \to [0, \infty]$ by setting $d(x, y) = 0$ if $x \sim y$, and $d(x, y) = \infty$ otherwise. It is easy to see that d is a generalized pseudometric. Thus, the space (X, d) is partitioned into subspaces X_i such that $|X_iX_j| = \infty$ for $i \ne j$, and the distance between points of each X_i vanishes. Thus, the topology on each X_i is anti-discrete, so each mapping $[a, b] \to X_i$ is continuous, i.e., it is a curve, and such curve has zero length. This implies that the distance function of X is intrinsic.

Let ρ be the original distance function of X. Let us put $\mathcal{D} = \{\rho, d\}$, then $\rho_\sim = \inf \mathcal{D}$, because $\min\{\rho, d\}$ equals the function $b_\sim$ from the definition of the quotient space. It remains to apply Proposition 3.7. □

Problem 3.7. Let X be an arbitrary set covered by a family $\{X_i\}_{i\in I}$ of generalized pseudometric spaces. Denote the distance function on X_i by ρ_i, and consider the set $\mathcal{D}$ of all generalized pseudometrics d on X such that for any i and $x, y \in X_i$ it holds $d(x, y) \le \rho_i(x, y)$. Extend each ρ_i to the whole X by setting $\rho_i'(x, y) = \infty$ if at least one of x, y does not belong to X_i, and $\rho_i'(x, y) = \rho(x, y)$ otherwise (it is easy to see that each ρ_i' is a generalized pseudometric). Denote by $\mathcal{D}'$ the set of all such ρ_i'. Prove that $\sup \mathcal{D} = \inf \mathcal{D}'$, and if all ρ_i are intrinsic, then $\sup \mathcal{D}$ is intrinsic as well.

Problem 3.8. Let $\mathcal{D}$ be a collection of generalized pseudometrics defined on the same set X, and X_d for $d \in \mathcal{D}$ denote the generalized pseudometric space (X, d). Put $W = \sqcup_{d\in\mathcal{D}} X_d$ and denote by ρ the generalized pseudometric of W. Define on W an equivalence relation $\sim$ by identifying those points $x_d \in X_d$ and $x_{d'} \in X_{d'}$ which correspond to the same point x of the set X. The equivalence class of these points x_d and $x_{d'}$ we denote by $[x]$. Denote by $\rho_\sim$ the quotient generalized pseudometric on $W/{\sim}$. Define the mapping $\varphi\colon W/{\sim} \to X$ as $\varphi\colon [x] \to x$, then φ is bijective, and $\rho_\sim$ can be considered as a generalized pseudometric on X. Prove that $\rho_\sim = \inf \mathcal{D}$.

Problem 3.9. Let ρ_1 and ρ_2 be intrinsic metrics on a set X. Suppose that these metrics generate the same topology, and that each $x \in X$ has a neighborhood U^x such that the restrictions of ρ_1 and ρ_2 to U^x coincide. Prove that $\rho_1 = \rho_2$. Show that the condition "ρ_1 and ρ_2 are intrinsic" is essential.

3.3 Hopf–Rinow condition

General metric spaces can be geometrically very different from $\mathbb{R}^n$. For example, in discrete spaces, balls of nonzero radius can coincide with their centers. In particular, the distance from an arbitrary point to such a ball will be equal to the distance from this point to the center. In spaces with an internal metric, this does not occur.

Theorem 3.9 (Hopf–Rinow condition)**.** *Let X be a space with an intrinsic metric, x, $y \in X$, $x \ne y$, and $0 < r \le |xy|$. Then*

$$\big|yU_r(x)\big| = |xy| - r.$$

Remark 3.10. For general metric spaces X, Theorem 3.9 does not hold. For example, if $X = \{x, y\}$, $|xy| = 1$, and $r = 0.5$, then $U_r(x) = \{x\}$, $\big|yU_r(x)\big| = 1 \ne |xy| - r = 0.5$.

Proof of Theorem 3.9. For any point $z \in U_r(x)$ we have $|yz| \ge |yx| - |zx| > |xy| - r$, therefore $\big|yU_r(x)\big| \ge |xy| - r$. Let us prove that the converse inequality also holds.

For each $0 < \varepsilon < r$ we consider a rectifiable curve $\gamma\colon [0,1] \to X$, $x = \gamma(0)$ and $y = \gamma(1)$, for which $|\gamma| \le |xy| + \varepsilon$. We define a continuous function $f(t) = \big|x\gamma(t)\big|$, $f(0) = 0$, $f(1) = |xy|$, and choose an arbitrary t_0 such that $f(t_0) = r - \varepsilon$. We denote by γ_1 the part of the curve γ between 0 and t_0, and by γ_2 the remaining part of the curve γ. Then $|\gamma_1| \ge r - \varepsilon$ by Item (1) of Proposition 3.2, so $|\gamma_2| \le |xy| - r + 2\varepsilon$ and, by the same proposition, $\big|\gamma(t_0)y\big| \le |\gamma_2| \le |xy| - r + 2\varepsilon$. However, $\gamma(t_0) \in U_r(x)$, therefore $\big|yU_r(x)\big| \le |xy| - r + 2\varepsilon$. Since ε is arbitrary, we obtain what is required. □

Remark 3.11. The Hopf–Rinow condition can also be satisfied in spaces whose metric is not intrinsic, for example, in the metric space $\mathbb{Q}$ of all rational numbers (with the standard distance function).

We give some corollaries from Theorem 3.9. First we give a necessary definition.

Let X be a metric space, $x \in X$, $r \ge 0$. Note that a closed ball $B_r(x)$ is a closed set, but, generally speaking, different from the closure of the open ball $U_r(x)$: if, as in the above example, X consists of two points x and y at the distance 1, then $U_1(x) = \{x\}$, $B_1(x) = \{x, y\}$, $\overline{U_1(x)} = \{x\} \ne B_1(x)$. However, if the metric of the space X is intrinsic, then Theorem 3.9 immediately implies the following result.

Corollary 3.12. *Let X be a space with an intrinsic metric. Then $B_r(x) = \overline{U_r(x)}$.*

Proof. A point y is adherent for a ball $U_r(x)$ if and only if $|yU_r(x)| = 0$, thus $|xy| \le r$, i.e., $y \in B_x(r)$ and, therefore, $\overline{U_r(x)} \subset B_r(x)$. Let us prove the reverse inclusion.

Let $y \in B_r(x)$. If $|xy| < r$, then $y \in U_r(x) \subset \overline{U_r(x)}$. If $|xy| = r$, then, by Theorem 3.9, we have $|yU_r(x)| = |xy| - r = 0$, thus $y \in \overline{U_r(x)}$. □

The following result will be used in the proof of the first part of Hopf–Rinow Theorem.

Corollary 3.13. *Let X be a space with intrinsic metric and $\varepsilon > 0$. Then for each ε-net S in the ball $B_r(x) \subset X$ and any $\delta' > \delta > 0$ we have $B_{r+\delta}(x) \subset \cup_{s\in S} U_{\varepsilon+\delta'}(s)$, i.e., S is $(\varepsilon + \delta')$-net for $B_{r+\delta}(x)$.*

Proof. For any point $y \in B_{r+\delta}(x)$ we have $|xy| \le r + \delta$, therefore either $y \in U_r(x)$ and, thus, $|yU_r(x)| = 0$, or, by Theorem 3.9, $\big|yU_r(x)\big| \le \delta$ holds. Thus, for any $\delta' > \delta$ there exists $z \in U_r(x) \subset B_r(x)$ such that $|yz| < \delta'$. On the other hand, there exists $s \in S$ for which $U_\varepsilon(s) \ni z$, whence $|sy| \le |sz| + |zy| < \varepsilon + \delta'$, therefore $y \in U_{\varepsilon+\delta'}(s)$, as required. □

3.4 Local compactness

Definition 3.14. A metric space X is called *locally compact* if for every point $x \in X$ there exists $\varepsilon > 0$ such that the closed ball $B_\varepsilon(x)$ is compact.

Problem 3.10. Prove that a metric space X is locally compact if and only if for each point $x \in X$ there exists a neighborhood with compact closure.

Remark 3.15. Unlike compactness, local compactness, even in combination with the intrinsic metric, does not guarantee the completeness of the metric space. An obvious example is an open ball in Euclidean space. Another example is the Euclidean space with a point removed.

Theorem 3.16 (Hopf-Rinow, Part 1)**.** *Let X be a locally compact space with intrinsic metric. Then the space X is complete if and only if every closed ball in X is compact.*

Proof. Suppose first that each closed ball is compact. We prove the completeness. Consider an arbitrary fundamental sequence $x_1, x_2, \ldots$. Then there exists r such that all x_n are contained in $B_r(x_1)$. By Theorem 2.24, the ball $B_r(x_1)$ is complete, therefore the sequence $x_1, x_2, \ldots$ converges to some point $x \in B_r(x) \subset X$, as required.

Now let the space X be complete. On X we define a function $\rho\colon X \to [0, \infty]$ as follows:

$$\rho(x) = \sup\{r > 0 : \text{the ball } B_r(x) \text{ is compact}\}.$$

Lemma 3.17. *Suppose that there exists a point $x_0 \in X$ such that $\rho(x_0) = \infty$. Then each ball $B_r(x)$ is compact and, therefore, ρ is identically equal to ∞.*

Proof. For every x and $r > 0$, the ball $B_r(x)$ is contained in some compact ball $B_{r'}(x_0)$, therefore, since the set $B_r(x)$ is closed, the ball $B_r(x)$ is also compact. $\square$

Thus, it suffice to prove that there exists a point $x_0 \in X$ such that $\rho(x_0) = \infty$. Assume the contrary, i.e., that the function ρ is everywhere finite.

Lemma 3.18. *The function ρ is 1-Lipschitz and, therefore, continuous.*

Proof. Otherwise, there exists $x, y \in X$ such that $\big|\rho(x) - \rho(y)\big| > |xy|$. To be definite, assume that $\rho(x) \ge \rho(y)$, thus $\rho(x) > \rho(y) + |xy|$, and if $\varepsilon > 0$ is chosen in such a way that $\rho(x) > \rho(y) + 2\varepsilon + |xy|$, then $B_{\rho(y)+\varepsilon}(y) \subset B_{\rho(x)-\varepsilon}(x)$, however, $B_{\rho(x)-\varepsilon}(x)$ is compact, thus $B_{\rho(y)+\varepsilon}(y)$ is compact as well, a contradiction with definition of $\rho(y)$. $\square$

Lemma 3.19. *Under the assumptions made, the ball $B_{\rho(x)}(x)$ is compact for every x.*

Proof. Since the ball $B_{\rho(x)}(x)$ is a closed subset of the complete space X, this ball is also complete. Therefore, by Theorem 2.24, it suffices to prove that for each $\varepsilon > 0$ this ball contains a finite ε-net.

To do that, we choose $0 < r < \rho(x)$ such that $\delta := \rho(x) - r < \varepsilon/2$, then the ball $B_r(x)$ is compact, thus it contains a finite $(\varepsilon/2)$-net S. By Corollary 3.13, the set S is $(\varepsilon/2 + \varepsilon/2)$-net for the ball $B_{r+\delta}(x) = B_{\rho(x)}(x)$. $\square$

Since ρ is a continuous function, its restriction to the compact set $B_{\rho(x)}(x)$ attains its minimum and, thus, this minimum is positive. We denote this minimum by ε, then, by Lemma 3.19, all the balls $B_\varepsilon(y)$, $y \in B_{\rho(x)}(x)$ are compact. Let S be a finite $(\varepsilon/2)$-net in $B_{\rho(x)}(x)$, and $0 < \delta < \varepsilon/2$, then, by Corollary 3.13, the set S is $(\varepsilon/2+\varepsilon/2)$-net for the ball $B_{\rho(x)+\delta}(x)$. In particular, $B_{\rho(x)+\delta}(x)$ is contained in the set $\cup_{s\in S} B_\varepsilon(s)$ which is compact as a finite union of compact sets. Therefore, $B_{\rho(x)+\delta}(x)$ is compact that contradicts to definition of the function ρ. $\square$

Definition 3.20. A metric space in which every closed ball is compact is called *proper* or *boundedly compact.*

Corollary 3.21. *A metric space with an intrinsic metric is boundedly compact if and only if it is locally compact and complete.*

Problem 3.11. Show that a metric spaces is boundedly compact if and only if its compact subsets are exactly those subsets that are closed and bounded.

3.5 Lipschitz, convergence and uniform convergence

In this section, we state and prove a few useful technical results regarding the convergence of Lipschitz mappings.

Let $f_n\colon X \to Y$ be a family of arbitrary mappings from a set X to a metric space Y. We say that the sequence f_n *converges pointwise* to a mapping $f\colon X \to Y$ if for each $x \in X$ the sequence $f_n(x)$ converges to $f(x)$. The sequence f_n *converges uniformly* to a mapping $f\colon X \to Y$ if for any $\varepsilon > 0$ there exists N such that for every $n \ge N$ the inequality $\big|f(x)f_n(x)\big| < \varepsilon$ holds for all $x \in X$.

Remark 3.22. Let us introduce the convergences described above by representing the mappings f_n as points of the set $\prod_{x\in X} Y$, which, recall, we defined as the family Y^X of all mappings from X to Y. Now we model the pointwise and uniform convergences by means of some topologies.

We start with the case of pointwise convergence. We define on $Y^X = \prod_{x\in X} Y$ the Tychonoff's topology, see Construction 1.3.

Problem 3.12. Show that convergence in the Tychonoff topology of points f_n to a point f is equivalent to pointwise convergence of the mappings f_n to the mapping f.

To model uniform convergence, we first give some definitions. A mapping $f\colon X \to Y$ is called *bounded* if its image $f(X)$ is a bounded subset of Y. The family of all bounded mappings from X to Y we denote by $\mathcal{B}(X, Y)$. We define the following distance function on $\mathcal{B}(X, Y)$: $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$.

Problem 3.13. Prove that the distance function defined above on $\mathcal{B}(X, Y)$ is a metric, and that the convergence in this metric of a sequence $f_n \in \mathcal{B}(X, Y)$ to a point $f \in \mathcal{B}(X, Y)$ is equivalent to the uniform convergence of the mappings f_n to the mapping f.

For arbitrary f_n and f we can also define such convergence by considering $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$ as generalized metric.

Problem 3.14. Prove that the generalized distance function $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$ defined on Y^x is a generalized metric, and that the convergence in this generalized metric of a sequence $f_n \in Y^X$ to a point $f \in Y^X$ is equivalent to the uniform convergence of the mappings f_n to the mapping f.

Proposition 3.23. *Let X be compact, and Y be arbitrary metric spaces, and let $f_n\colon X \to Y$ be a sequence of C-Lipschitz mappings converging pointwise to some mapping $f\colon X \to Y$. Then f is a C-Lipschitz mapping, and the sequence f_n converges to f uniformly.*

Proof. To verify that the mapping f is C-Lipschitz, it is sufficient to pass to the limit in the inequality $\big|f_n(x)f_n(x')\big| \le C\cdot|xx'|$ for arbitrary fixed $x,\,x' \in X$.

We now prove uniform convergence. Choose an arbitrary $\varepsilon > 0$ and show that there exists N such that for all $n > N$ and all $x \in X$ we have $\big|f(x)f_n(x)\big| < \varepsilon$.

Put $\delta = \varepsilon/(3C)$, and let $\{x_i\} \subset X$ be a finite δ-net. We choose N such that for all $n > N$ and all i the inequality $\big|f(x_i)f_n(x_i)\big| < \varepsilon/3$ holds.

Fix an arbitrary $x \in X$. There is i such that $|xx_i| < \delta$. Since f_n and f are C-Lipschitz, we conclude that $\big|f_n(x)f_n(x_i)\big| \le C\cdot|xx_i| < \varepsilon/3$ and, similarly, $\big|f(x)f(x_i)\big| < \varepsilon/3$, therefore

$$\big|f(x)f_n(x)\big| \le \big|f(x)f(x_i)\big| + \big|f(x_i)f_n(x_i)\big| + \big|f_n(x_i)f_n(x)\big| < \varepsilon/3+\varepsilon/3+\varepsilon/3 = \varepsilon,$$

as required. □

The following version of the previous statement is useful in studying curves.

Corollary 3.24. *Let X be a metric space, and $\gamma_n\colon [a,b] \to X$ be a sequence of C-Lipschitz curves converging pointwise to a mapping $\gamma\colon [a,b] \to X$. Then γ is a C-Lipschitz curve, and the sequence γ_n converges to γ uniformly.*

The proposition below can be proved similarly to Proposition 3.23.

Proposition 3.25. *Let X be compact, Y be an arbitrary metric spaces, and $f_n\colon X \to Y$ be a sequence of C-Lipschitz mappings. Suppose that for some everywhere dense subset $Z \subset X$ the sequence $f_n|_Z$ converges pointwise. Then the sequence f_n converges pointwise to some mapping $f\colon X \to Y$ and, therefore, by Proposition 3.23, this convergence is uniform, and the mapping f is C-Lipschitz.*

Corollary 3.26. *Let $\gamma_n\colon [a,b] \to X$ be a sequence of C-Lipschitz curves in a metric space X. Suppose that for some everywhere dense subset $Z \subset [a,b]$ the sequence of mappings $\gamma_n|_Z$ converges pointwise. Then the sequence of curves γ_n converges pointwise to some curve $\gamma\colon [a,b] \to X$ and, therefore, by virtue of Corollary 3.24, this convergence is uniform, and the curve γ is C-Lipschitz.*

3.6 Arc-length and uniform curves

Definition 3.27. A curve $\gamma(s)$ and its parameter $s \in [a,b]$ are called *natural* or *arc-length*, if for any $a \le s_1 \le s_2 \le b$ it holds $\big|\gamma|_{[s_1,s_2]}\big| = s_2 - s_1$. A curve $\gamma(t)$ and its parameter $t \in [a,b]$ are called *uniform*, if there exists $\lambda \ge 0$ such that for any $a \le t_1 \le t_2 \le b$ it holds $\big|\gamma|_{[t_1,t_2]}\big| = \lambda(t_2 - t_1)$; the value λ is called *the velocity* or *the speed of uniform* γ.

Remark 3.28. Let $\gamma(t)$, $t \in [a,b]$, be uniform curve.

(1) If $a \ne b$, then its velocity λ is uniquely determined by the equation $|\gamma| = \lambda(b-a)$. In particular, $|\gamma| = 0$, i.e., γ is a constant mapping if and only if $\lambda = 0$.

(2) If $a = b$, then λ can be arbitrary. However, it is natural to consider this case as the limiting one for constant mappings γ, where $\lambda = 0$. So, to be definite, we make the following agreement: **if** $a = b$ **then** $\lambda = 0$.

Thus, under the above agreement, the velocity of a uniform curve γ vanishes if and only if γ is a constant mapping. We call such curves *degenerate*, and all the remaining curves *nondegenerate*. So, each degenerate curve is uniform.

Remark 3.29. The following simple observations concern the relations between arc-length and uniform curves.

(1) Each arc-length curve is uniform.

(2) A degenerate curve $\gamma(s)$, $s \in [a,b]$, is arc-length if and only if $a = b$; in this case the arc-length curve has zero velocity.

(3) A degenerate curve $\gamma(t)$, $t \in [a, b]$, with $a < b$ is uniform but not arc-length.

(4) A nondegenerate uniform curve is arc-length if and only if its velocity equals 1.

Proposition 3.30. *If a curve $\gamma(t)$, $t \in [a, b]$, is uniform with the velocity λ, then the mapping γ is λ-Lipschitz.*

Proof. For any $a \le t_1 \le t_2 \le b$ we have

$$\big|\gamma(t_1)\gamma(t_2)\big| \le \big|\gamma|_{[t_1,t_2]}\big| = \lambda(t_2 - t_1).$$

□

Remark 3.31. The following types of curves $\gamma\colon [a, b] \to X$ cannot be reparameterized to arc-length curves:

(1) not rectifiable γ (otherwise, $b = \infty$);

(2) $\gamma = \text{const}$ when $a \ne b$;

(3) more general, γ containing *stops*, i.e., when there exists $[\alpha, \beta] \subset [a, b]$, $\alpha \ne \beta$, such that $\gamma|_{[\alpha,\beta]} = \text{const}$.

Curves that do not contain stops are called *non-stop* ones.

Problem 3.15. Let γ be a curve in a metric space. Prove that

(1) nondegenerate γ can be reparameterized to an arc-length or, more generally, to a uniform one if and only if γ is rectifiable and non-stop;

(2) degenerate γ can be reparameterized to an arc-length one if and only if its domain is singleton (indeed, such γ is arc-length itself and, thus, it need not a reparametrization);

(3) degenerate γ is always uniform.

In [1] it is proposed to extend the class of reparametrizations, namely, to consider monotone (not necessarily strictly monotone) surjective mappings between the domains of the curves. It turns out that with this definition of reparametrization, it is possible to introduce an arc-length parameter on any rectifiable curve.

Definition 3.32. We say that curves $\gamma\colon [a, b] \to X$ and $\bar\gamma\colon [c, d] \to X$ are obtained from each other by *a monotone reparametrization* if either there exists a monotone surjective mapping $\varphi\colon [c, d] \to [a, b]$ such that $\bar\gamma = \gamma \circ \varphi$, or there exists a monotone surjective mapping $\psi\colon [a, b] \to [c, d]$ such that $\gamma = \bar\gamma \circ \psi$.

Remark 3.33. It is easy to see that monotone reparametrization does not change the length.

Problem 3.16. Prove that a curve γ in a metric space can be monotonically reparameterized to an arc-length or, more generally, a uniform one if and only if γ is rectifiable.

The reparameterized curve is unique upto the choice of its domain and direction. In arc-length case one can choose any segment of the length $|\gamma|$. In the uniform case the domain can be arbitrary nondegenerate segment for nondegenerate γ, and arbitrary segment for degenerate γ.

Remark 3.34. An instructive example is a curve that is a parametrization of the segment $[0, 1] \subset \mathbb{R}$ by the Cantor staircase. The Cantor staircase is a graph of a function $f\colon [0, 1] \to [0, 1]$, the construction of which we will now describe.

At the points 0 and 1, we set the value of the function f equal to 0 and 1, respectively. Next, we divide the segment $[0, 1]$ into three equal parts and on the middle interval we set f equal to $1/2$. The remaining two segments are again divided into three equal parts each, and on the middle intervals we assume that the function f is equal to the arithmetic mean of its values at the nearest intervals where it is defined. Thus, on the left-most interval, the function f is equal to $1/4$, and on the right-most interval it is equal to $3/4$. Continuing this process to infinity, we define a function f on an everywhere dense subset of the segment $[0, 1]$, which is the complement to the Cantor set. Extend f to the remaining points of the segment $[0, 1]$ by continuity (make sure that this can be done).

We now consider the Cantor staircase as a curve $f\colon [0, 1] \to \mathbb{R}$ on the Euclidean line. Note that the subset of the segment $[0, 1]$, on which the point of this curve changes its position, is the Cantor set that has measure zero. So, this curve stops almost everywhere, however, its length equals 1 and it can be reparameterized to an arc-length curve.

3.7 Arzela-Ascoli Theorem

Developing the ideas from Section 3.5, we formulate and prove a variant of the famous Arzela–Ascoli theorem. First we give necessary definitions.

Definition 3.35. Let $\gamma_n\colon [a_n, b_n] \to X$ be a sequence of curves in a metric space X. We say that this sequence *converges* (*uniformly converges*) to a curve $\gamma\colon [a, b] \to X$ if there exist curves $\bar\gamma_n\colon [c, d] \to X$ and $\bar\gamma\colon [c, d] \to X$ obtained from γ_n and γ, respectively, by monotone reparametrization, such that the mappings $\bar\gamma_n$ converge (uniformly converge) to the mapping $\bar\gamma$.

Theorem 3.36 (Arzela–Ascoli)**.** *Let X be a compact metric space, and γ_n be a sequence of curves in X. Suppose that the lengths of the curves γ_n are uniformly bounded, i.e., there exists a real number C such that $|\gamma_n| \le C$ for all n. Then in this sequence there is a subsequence that converges uniformly to a curve whose length is at most C.*

Proof. In virtue of Problem 3.16, the curves γ_n can be monotonically reparameterized to uniform curves $\bar\gamma_n\colon [0, 1] \to X$ with speeds at most C. It follows from Remark 3.33 and Proposition 3.30 that all the curves $\bar\gamma_n$ are C-Lipschitz.

Choose a countable everywhere dense subset $Z \subset [0, 1]$, $Z = \{z_i\}_{i=1}^\infty$. The sequence $\big(\bar\gamma_n(z_1)\big)_{n=1}^\infty$ has a convergent subsequence $\big(\gamma_n^1(z_1)\big)_{n=1}^\infty$; the sequence $\big(\gamma_n^1(z_2)\big)_{n=1}^\infty$ has a convergent subsequence $\big(\gamma_n^2(z_2)\big)_{n=1}^\infty$, etc. Then the sequence $\big(\gamma_n^n(z_k)\big)_{n=1}^\infty = \big(\bar\gamma_{n_i}(z_k)\big)_{i=1}^\infty$ convergence for any k (Cantor diagonal process). Let us put $f(z_k) = \lim_{i\to\infty} \bar\gamma_{n_i}(z_k)$, then $\bar\gamma_{n_i}|_Z \to f$.

By Corollary 3.26, the mappings $\bar\gamma_{n_i}\colon [0, 1] \to X$ converge uniformly to some C-Lipschitz curve $\bar\gamma\colon [0, 1] \to X$. By Item (6) of Proposition 3.2, we have $|\bar\gamma| \le \liminf_{n_i\to\infty} |\bar\gamma_{n_i}| \le C$, as required. □

3.8 Existence of shortest curves

We apply the previous results to investigation of curves of smallest length.

Definition 3.37. A rectifiable curve in a metric space is called *shortest* if its length is equal to the infimum of the lengths of all the curves joining its ends.

Remark 3.38. If X is a space with an intrinsic metric, then a curve γ in X joining x and y is *shortest* if and only if $|xy| = |\gamma|$.

The following proposition is obvious.

Proposition 3.39. *A curve in a metric space is shortest if and only if each of its parts is a shortest curve.*

Problem 3.17. Prove that an arc-length curve $\gamma\colon [a, b] \to X$ in a space X with an intrinsic metric is shortest if and only if γ is an isometric embedding.

Definition 3.40. A curve $\gamma\colon [a, b] \to X$ in a metric space X is called *locally shortest* if for each $t \in [a, b]$ there exists an interval $(\alpha, \beta) \subset \mathbb{R}$ containing t such that $\gamma|_{[a,b]\cap[\alpha,\beta]}$ is a shortest curve.

Definition 3.41. A uniform locally shortest curve is called *a geodesic*.

Arzela-Ascoli theorem, together with a few other previous propositions, implies the following result.

Corollary 3.42. *Any two points x and y of a compact metric space X that are joined by a rectifiable curve are also joined by a shortest curve.*

Proof. Let ℓ be the infimum of the lengths of the curves joining x and y. There is a sequence γ_n for which $|\gamma_n| \to \ell$ and, thus, the lengths of γ_n are uniformly bounded. Theorem 3.36 implies that the sequence γ_n contains a subsequence γ_{n_i} which uniformly converging to some curve γ. By Item (6) of Proposition 3.2, we have $|\gamma| \le \liminf_{i\to\infty} |\gamma_{n_i}| = \ell$, however, by the minimality of ℓ, it holds $|\gamma| \ge \ell$, therefore $|\gamma| = \ell$ and, thus, γ is a shortest curve. □

Remark 3.43. Corollary 3.42 remains true if we change compact X to a boundedly compact one (verify this).

Definition 3.44. A metric on X is called *strictly intrinsic* if any two points in X are joined by a curve whose length is equal to the distance between these points. A metric space with strictly intrinsic metrics is called *strictly intrinsic* or *geodesic*.

Taking into account the Hopf–Rinow theorem, we obtain the following

Corollary 3.45. *Each complete locally compact space with an intrinsic metric is a geodesic space.*

3.9 Shortest curves and midpoints

Definition 3.46. A point z of a metric space is called *a midpoint between or for points x and y* of this space if $|xz| = |yz| = \frac{1}{2}|xy|$.

Theorem 3.47. *Let X be a complete metric space. Suppose that for each pair of points x, $y \in X$ there is a midpoint. Then X is a geodesic space.*

Proof. Choose two arbitrary points x and y from X. We show that these points can be joined by a curve $\gamma\colon [0,1] \to X$, for which $|\gamma| = |xy|$.

We will sequentially determine the map γ for various points of the segment $[0,1]$. Put $\gamma(0) = x$ and $\gamma(1) = y$. Next, let $\gamma(1/2)$ be a midpoint between x and y; $\gamma(1/4)$ be a midpoint between $\gamma(0)$ and $\gamma(1/2)$, and $\gamma(3/4)$ be a midpoint between $\gamma(1/2)$ and $\gamma(1)$. Continuing this process, we define γ at all binary rational points of the segment $[0,1]$, i.e., at all points of the form $m/2^n$, where $0 \le m \le 2^n$ is an integer, and $n = 0, 1, \ldots$. Note that the set of all binary rational points of the segment $[0,1]$ is everywhere dense in $[0,1]$. In addition, it is easy to show that the constructed mapping γ is $|xy|$-Lipschitz. The proof of the following technical lemma is left as an exercise.

Lemma 3.48. *Let Z be an everywhere dense subset of a metric space X, and $f\colon Z \to Y$ be some C-Lipschitz mapping into a complete metric space Y. Then there exists a unique continuous mapping $F\colon X \to Y$ extending f. Moreover, the mapping F is also C-Lipschitz.*

Problem 3.18. Prove Lemma 3.48.

So, using Lemma 3.48, we extend by continuity the mapping γ onto the entire segment $[0,1]$, and we again denote the resulting $|xy|$-Lipschitz curve by γ. As noted in Example 3.1, it holds $|\gamma| \le |xy|(1-0) = |xy|$, from where, by virtue of Item (1) of Proposition 3.2, we have $|\gamma| = |xy|$ and, therefore, γ is a shortest curve. □

Remark 3.49. In a complete metric space, the property of a metric to be intrinsic is not sufficient for midpoints and shortest curves between any points to exist. Consider a countable family of segments $[0, 1+1/n]$, $n \in \mathbb{N}$, each with the standard metric, and glue all their zeros at one point A, and at another point B we glue all the other ends $1+1/n$. If x and y belong to different segments, say to $[0,1+1/n]$ and $[0,1+1/m]$, then we set the distance between x and y equal to $\min(x+y, 1-x+1/n+1-y+1/n)$ (i.e., the intrinsic circle metric is considered on each pair of glued segments). Then the distance between A and B is 1 and is not reached on any curve. In addition, there is no midpoints between A and B.

Definition 3.50. A point z of a metric space X is called *an ε-midpoint between or for points x and y* of this space if $\big||xz| - \frac{1}{2}|xy|\big| \le \varepsilon$ and $\big||yz| - \frac{1}{2}|xy|\big| \le \varepsilon$.

Theorem 3.51. *Let X be a complete metric space. Suppose that for each pair of points x, $y \in X$ and each $\varepsilon > 0$, there is an ε-midpoint. Then the metric of X is intrinsic.*

Proof. The proof is similar with the one of Theorem 3.47, however, now we find not strict midpoints, but approximate ones, making sure that the total "spread" is not large (we use the fact that $\sum_{i=1}^\infty \varepsilon/2^i = \varepsilon$). □

There are also converse obvious statements, even without assuming the completeness of the ambient space.

Proposition 3.52. *In a space with an intrinsic (strictly intrinsic) metric, for any two points and any $\varepsilon > 0$ there is an ε-midpoint (a midpoint), respectively.*

Problem 3.19. Prove Proposition 3.52.

References to Chapter 3

[1] D.Burago, Yu.Burago, S.Ivanov, *A Course in Metric Geometry.* Graduate Studies in Mathematics, vol.33, A.M.S., Providence, RI, 2001.

Exercises to Chapter 3

Exercise 3.1. Let X be an arbitrary metric space and $\Omega(X)$ the family of all curves in X. Verify that

(1) if $\gamma \in \Omega(X)$ joins the points $x, y \in X$, then $|\gamma| \geq |xy|$;

(2) if $\gamma = \gamma_1 \cdot \gamma_2$ is the gluing of curves $\gamma_1, \gamma_2 \in \Omega(X)$ then $|\gamma| = |\gamma_1| + |\gamma_2|$;

(3) for each $\gamma \in \Omega(X)$, $\gamma\colon [a,b] \to X$, and reparametrization $\varphi\colon [c,d] \to [a,b]$, it holds $|\gamma| = |\gamma \circ \psi|$;

(4) for each $x \in X$, $\varepsilon > 0$, $y \in X \setminus U_\varepsilon(x)$ and the curve $\gamma \in \Omega(X)$ joining x and y, $|\gamma| \geq \varepsilon$ holds;

(5) is it true that for any $\gamma \in \Omega(X)$, $\gamma\colon [a,b] \to X$, the function $f(t) = \bigl|\gamma|_{[a,t]}\bigr|$ is continuous?

(6) is it true that for any sequence $\gamma_n \in \Omega(X)$ converging pointwise to some $\gamma \in \Omega(X)$, we have

$$|\gamma| \leq \liminf_{n\to\infty} |\gamma_n|?$$

Exercise 3.2. Show that the piecewise smooth curve in $\mathbb{R}^n$ is Lipschitzian with a Lipschitz constant equal to the maximum modulus of the velocity vector of the curve, therefore each such curve is rectifiable.

Exercise 3.3. Let X be a metric space in which any two points are connected by a rectifiable curve.

(1) Prove that d_{in} is a metric.

(2) Denote by τ the metric topology of X w.r.t. the initial metric on X, by τ_{in} the metric topology w.r.t. d_{in}, by $X_\in$ the set X with metric d_{in} and topology τ_{in}. Show that $\tau \subset \tau_{in}$. In particular, if a mapping $\gamma\colon [a,b] \to X_{in}$ is continuous, then the mapping $\gamma\colon [a,b] \to X$ is continuous as well.

(3) Construct an example when $\tau \neq \tau_{in}$.

(4) Prove that for each rectifiable curve $\gamma\colon [a,b] \to X$ the mapping $\gamma\colon [a,b] \to X_{in}$ is continuous.

(5) Denote by $|\gamma|_{in}$ the length of a curve $\gamma\colon [a,b] \to X_{in}$. Show that for each curve $\gamma\colon [a,b] \to X$ which is also a curve in X_{in}, it holds $|\gamma| = |\gamma|_{in}$. Thus, the sets of rectifiable curves for X and X_{in} coincide, and each non-rectifiable curve in X is either a non-rectifiable one in X_{in}, or the mapping $\gamma\colon [a,b] \to X_{in}$ is discontinuous.

(6) Construct an example of continuous mapping $\gamma\colon [a,b] \to X$ such that the mapping $\gamma\colon [a,b] \to X_{in}$ is not continuous. Notice that the curve $\gamma\colon [a,b] \to X$ can not be rectifiable.

Exercise 3.4. Let X be a metric space in which any two points are connected by a rectifiable curve. Prove that the metric d_{in} is intrinsic.

Exercise 3.5. Let $\rho_1 \leq \rho_2$ be generalized pseudometrics on a set X, and Y be a topological space. Prove that each mapping $f\colon Y \to X$, continuous w.r.t. ρ_2, is also continuous w.r.t. ρ_1, in particular, if γ is a curve in (X, ρ_2), then γ is also a curve in (X, ρ_1); moreover, if ρ_1' and ρ_2' denote the corresponding generalized intrinsic pseudometrics, then $\rho_1' \leq \rho_2'$.

Exercise 3.6. Let X be an arbitrary set covered by a family $\{X_i\}_{i\in I}$ of generalized pseudometric spaces. Denote the distance function on X_i by ρ_i, and consider the set $\mathcal{D}$ of all generalized pseudometrics d on X such that for any i and $x, y \in X_i$ it holds $d(x,y) \leq \rho_i(x,y)$. Extend each ρ_i to the whole X by setting $\rho_i'(x,y) = \infty$ if at least one of x, y does not belong to X_i, and $\rho_i'(x,y) = \rho(x,y)$ otherwise (it is easy to see that each ρ_i' is a generalized pseudometric). Denote by $\mathcal{D}'$ the set of all such ρ_i'. Prove that $\sup \mathcal{D} = \inf \mathcal{D}'$, and if all ρ_i are intrinsic, then $\sup \mathcal{D}$ is intrinsic as well.

Exercise 3.7. Let $\mathcal{D}$ be a collection of generalized pseudometrics defined on the same set X, and X_d for $d \in \mathcal{D}$ denote the generalized pseudometric space (X, d). Put $W = \sqcup_{d\in\mathcal{D}} X_d$ and denote by ρ the generalized pseudometric of W. Define on W an equivalence relation $\sim$ by identifying those points $x_d \in X_d$ and $x_{d'} \in X_{d'}$ which correspond to the same point x of the set X. The equivalence class of these points x_d and $x_{d'}$ we denote by $[x]$. Denote by $\rho_\sim$ the quotient generalized pseudometric on $W/\!\sim$. Define the mapping $\varphi\colon W/\!\sim \to X$ as $\varphi\colon [x] \to x$, then φ is bijective, and $\rho_\sim$ can be considered as a generalized pseudometric on X. Prove that $\rho_\sim = \inf \mathcal{D}$.

Exercise 3.8. Let ρ_1 and ρ_2 be intrinsic metrics on a set X. Suppose that these metrics generate the same topology, and that each $x \in X$ has a neighborhood U^x such that the restrictions of ρ_1 and ρ_2 to U^x coincide. Prove that $\rho_1 = \rho_2$. Show that the condition "ρ_1 and ρ_2 are intrinsic" is essential.

Exercise 3.9. Prove that a metric space X is locally compact if and only if for each point $x \in X$ there exists a neighborhood with compact closure.

Exercise 3.10. Show that a metric spaces is boundedly compact if and only if its compact subsets are exactly those subsets that are closed and bounded.

Exercise 3.11. Let X be an arbitrary set and Y an arbitrary metric space. Consider the collection of sets of the form $\prod_{x\in X} V(x) \subset \prod_{x\in X} Y$, where $\big\{V(x)\big\}_{x\in X}$ is the family of nonempty open subsets of Y such that for all $x \in X$, except for their finite number, $V(x) = Y$. Show that the family defined in this way forms a basis of a topology, and the convergence in this topology of points f_n to a point f is equivalent to pointwise convergence of the mappings f_n to the mapping f.

Exercise 3.12. Let X be an arbitrary set and Y an arbitrary metric space. A mapping $f\colon X \to Y$ is called *bounded* if its image $f(X)$ is a bounded subset of Y. The family of all bounded mappings from X to Y we denote by $\mathcal{B}(X, Y)$. We define the following distance function on $\mathcal{B}(X, Y)$: $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$. Prove that the distance function defined above is a metric, and that the convergence in this metric of a sequence $f_n \in \mathcal{B}(X, Y)$ to some $f \in \mathcal{B}(X, Y)$ is equivalent to uniform convergence of the mappings f_n to the mapping f.

Exercise 3.13. Let X be an arbitrary set and Y an arbitrary metric space. Define the following generalized distance function on Y^X: $|fg| = \sup_{x\in X}\big|f(x)g(x)\big|$. Prove that the generalized distance function defined above is a generalized metric, and that the convergence in this generalized metric of a sequence $f_n \in Y^X$ to some $f \in Y^X$ is equivalent to uniform convergence of the mappings f_n to the mapping f.

Exercise 3.14. Let γ be a curve in a metric space. Prove that

(1) nondegenerate γ can be reparameterized to an arc-length or, more generally, to a uniform one if and only if γ is rectifiable and non-stop;

(2) degenerate γ can be reparameterized to an arc-length one if and only if its domain is singleton (indeed, such γ is arc-length itself and, thus, it need not a reparametrization);

(3) degenerate γ is always uniform.

Exercise 3.15. Prove that a curve γ in a metric space can be monotonically reparameterized to an arc-length or, more generally, a uniform one if and only if γ is rectifiable.

The reparameterized curve is unique upto the choice of its domain and direction. In arc-length case one can choose any segment of the length $|\gamma|$. In the uniform case the domain can be arbitrary nondegenerate segment for nondegenerate γ, and arbitrary segment for degenerate γ.

Exercise 3.16. Prove that an arc-length curve $\gamma\colon [a, b] \to X$ in a space X with an intrinsic metric is shortest if and only if γ is an isometric embedding.

Exercise 3.17. Let Z be an everywhere dense subset of a metric space X, and $f\colon Z \to Y$ be some C-Lipschitz map into a complete metric space Y. Then there exists a unique continuous mapping $F\colon X \to Y$ extending f. Moreover, the mapping F is also C-Lipschitz.

Exercise 3.18. Show that in a space with an intrinsic (strictly intrinsic) metric, for any two points and any $\varepsilon > 0$ there is an ε-midpoint (a midpoint), respectively.