Chapter 2

Introduction to Metric Spaces.

Schedule. Distance, pseudometric or semimetric, metric, open ball, closed ball, sphere, distance to a nonempty subset, open and closed r-neighborhoods of a nonempty subset, diameter of a subset, natural quotient of pseudometric space, Lipschitz mappings, Lipschitz constant, dilatation of a Lipschitz mapping, uniform continuity, bi-Lipschitz mappings, isometric mapping, isometry, isometry group, standard constructions of metrics, multiplying a metric by a number, adding to metric a constant, induced distance, semidirect product of metric spaces, examples, Levenshtein distance, elements of graph theory, metric construction for graphs, distance on a connected graph, distance on a connected weighted graph, Cayley graph of a group, quotient pseudometric and metric, generalized distance, disjoint union of generalized spaces, maximal pseudometric, the relation between quotient and maximal pseudometrics and metrics, isometries groups actions and quotient spaces, metrized graphs, polyhedron spaces, convergence of sequences and completeness, fundamental sequences, completion of a metric space, equivalence of compactness and sequential compactness for metric spaces, completeness and total boundedness equivalent to compactness for metric spaces, canonical isometric embeddings of metric spaces, Frechet-Kuratowski embedding to the space of bounded continuous functions, Frechet embedding of a separable metric space to the space of bounded sequences.

In this section, we discuss some basic facts from metric spaces theory.

2.1 Distance

Let X be an arbitrary set. A function $\rho\colon X\times X\to\mathbb{R}$ is called *a distance*, if it is non-negative, it is equal to zero on all pairs of the form (x,x), and it is *symmetric*: $\rho(x,y)=\rho(y,x)$ for any $x,y\in X$.

If the distance function ρ satisfies *the triangle inequality*, namely, $\rho(x,y)+\rho(y,z)\ge\rho(x,z)$, then such ρ is called *a pseudometric* or *a semimetric*, and the set X with the introduced pseudometric on it is *a pseudometric space* or *a semimetric space*.

Thus, the metric defined above is a pseudometric not equal to zero on the pairs (x,y) with $x\ne y$.

Remark 2.1. It will be convenient for us to introduce a universal notation for the distance function defined on an arbitrary set, namely, the distance between points x and y from this set will be denoted by $|xy|$. Even if several sets are considered at the same time with the distances given on them, we will denote these distances in the same way, while understanding what distance is used according to which set the corresponding points belong to.

For each distance given on the set X, a number of subsets arise that play an important role in the study of geometry:

- *an open ball of radius $r>0$ and center $x\in X$*: $U_r(x):=\big\{y\in X:|xy|<r\big\}$ (above we defined an open ball for a metric);
- *a closed ball of radius $r\ge0$ and center $x\in X$*: $B_r(x):=\big\{y\in X:|xy|\le r\big\}$;
- *a sphere of radius $r\ge0$ and center $x\in X$*: $S_r(x):=\big\{y\in X:|xy|=r\big\}$;

Note that in metric space, for $r=0$, both a closed ball and a sphere of radius r degenerate to a point and are called *degenerate*.

For any point $x\in X$ and a nonempty set A, define *the distance from x to A* by setting

$$|xA|=|Ax|=\inf\big\{|xa|:a\in A\big\}.$$

This concept gives rise to a number of objects:

- for $r>0$ *an open r-neighborhood of the set A* is $U_r(A)=\big\{y\in X:|Ay|<r\big\}$;

- for $r \ge 0$ *a closed r-neighborhood of the set A* is $B_r(A) = \big\{y \in X : |Ay| \le r\big\}$.

In addition, for a nonempty $A \subset X$, a numerical characteristic of the set A is defined — its *diameter*

$$\operatorname{diam} A = \sup\big\{|aa'| : a, a' \in A\big\}.$$

Moreover, the diameter can also be naturally determined for the empty set by setting $\operatorname{diam} \emptyset = 0$.

Remark 2.2. Sometimes an object A under consideration can simultaneously belong to different metric spaces, for example, in one space it is a subset, and in another it is a point. Then, in the notation introduced above, $U_r(A)$, $B_r(A)$, etc., as an upper index, we will add either the name of the space to which A belongs or the metric of this space. A typical case: A can be considered both as a nonempty subset of the space X with the metric d, and as a point in the space $\mathcal{H}(X)$ of all nonempty closed bounded subsets of X with some metric d_H. In this situation, instead of $U_r(A)$ we will write $U_r^X(A)$ or $U_r^d(A)$ in the first case, and $U_r^{\mathcal{H}(X)}(A)$ or $U_r^{d_H}(A)$ in the second one.

Each pseudometric on the set X defines a natural equivalence relation: $x \sim y$ if and only if $|xy| = 0$. Let $X/\!\sim$ be the set of classes of this equivalence, and for each $x \in X$ denote by $[x]$ the equivalence class containing x.

Problem 2.1. Prove that for any $x, y \in X$ and $x' \in [x]$ and $y' \in [y]$ it is true that $|x'y'| = |xy|$. Thus, on the set $X/\!\sim$ the corresponding distance function is correctly defined: $\big|[x][y]\big| = |xy|$. Show that this distance function is a metric.

Remark 2.3. If ρ is the pseudometric on X, then the quotient space $X/\!\sim$ from Problem 2.1 is sometimes denoted by X/ρ.

Problem 2.2. Let X be an arbitrary metric space, $x, y \in X$, $r \ge 0$, $s, t > 0$, and $A \subset X$ be nonempty. Verify that

(1) $U_s\big(\{x\}\big) = U_s(x)$ and $B_r\big(\{x\}\big) = B_r(x)$;

(2) the functions $y \mapsto |xy|$, $y \mapsto |yA|$ are continuous;

(3) an open neighborhood $U_s(A)$ is an open subset of X, and a closed neighborhood $B_r(A)$ is a closed subset of X;

(4) $U_t\big(U_s(A)\big) \subset U_{s+t}(A)$ and construct an example demonstrating that the left-hand side can be different from the right-hand side;

(5) $B_t\big(B_s(A)\big) \subset B_{s+t}(A)$ and construct an example demonstrating that the left-hand side can be different from the right-hand side;

(6) $\partial U_s(x)$, $\partial B_s(x)$ are not related by any inclusion; $\partial U_s(x) \subset S_s(x)$ and $\partial B_r(x) \subset S_r(x)$; the both previous inclusions can be strict;

(7) $\operatorname{diam} U_s(x) \le \operatorname{diam} B_s(x) \le 2s$;

(8) $\operatorname{diam} U_s(A) \le \operatorname{diam} B_s(A) \le \operatorname{diam} A + 2s$.

2.2 Lipschitz mappings and isometries

Mappings of metric spaces that distort distances no more than a certain finite number of times are called Lipschitz. More formally, a mapping $f\colon X \to Y$ of metric spaces is called *Lipschitz*, if there exists $C \ge 0$ such that for any $x, x' \in X$ the inequality $\big|f(x)f(x')\big| \le C|xx'|$ holds. Each such C is called *a Lipschitz constant*. Sometimes, for brevity, a Lipschitz mapping with Lipschitz constant C is called *C-Lipschitz*. For 1-Lipschitz mapping is reserved the term *nonexpanding*.

Problem 2.3. Let $\mathcal{L}(f) \subset \mathbb{R}$ be the set of all Lipschitz constants for a Lipschitz mapping f. Prove that $\inf \mathcal{L}(f)$ is also a Lipschitz constant.

The smallest Lipschitz constant for a Lipschitz mapping f is called *the dilatation of the mapping f* and is denoted by $\operatorname{dil} f$.

A mapping $f\colon X \to Y$ of metric spaces is called *uniformly continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any points $x, x' \in X$, $|xx'| < \delta$, we have $|f(x)f(x')| < \varepsilon$.

Problem 2.4. Show that each Lipschitz mapping is uniformly continuous, and each uniformly continuous mapping is continuous.

A bijective mapping f between metric spaces such that f and f^{-1} are Lipschitz is called *bi-Lipschitz*. It is clear that each bi-Lipschitz mapping is a homeomorphism.

A mapping of metric spaces $f\colon X\to Y$ is called *isometric* if it preserves distances: $\big|f(x)f(x')\big|=|xx'|$ for any $x,x'\in X$. A bijective isometric mapping is called *an isometry*.

Problem 2.5. Show that each isometry is a homeomorphism, in particular, each isometric mapping from one metric space into another one is an embedding, i.e., we recall, it is a homeomorphism with its image.

Problem 2.6. Verify that the identity map, the composition of isometries, and the inverse mapping to an isometry are also isometries, i.e., the set of all isometries of an arbitrary metric space forms a group.

Definition 2.4. The set of all isometries $f\colon X\to X$ of the metric space X endowed with the composition operation is called *the isometry group of the space* X and is denoted by $\mathrm{Iso}(X)$.

Problem 2.7. Let X be an arbitrary metric space, $x,y\in X$ and $A\subset X$ be nonempty. Prove that $|Ax|+|xy|\ge|Ay|$, so that the function $\rho_A(x)=|Ax|$ is 1-Lipschitz and, therefore, uniformly continuous.

2.3 Standard constructions of metrics

This section provides examples of standard constructions that build various metric spaces.

Construction 2.1 (Multiplying a metric by a number)**.** If d is a distance function on the set X, then for each real $\lambda>0$ the function $\lambda\,d$ is also a distance on X; moreover, if d was a metric (pseudometric), then $\lambda\,d$ is the distance of the same type. The corresponding space will be denoted by $\lambda\,X$.

Construction 2.2 (Adding a constant)**.** Let d be the distance function on X. We define an analogue of the Kronecker symbols for $x,y\in X$ by setting $\delta_{xy}=0$ for any $x\ne y$, and $\delta_{xx}=1$ for any x. Then for every real $c\ge0$ the function $(d+c)(x,y)=d(x,y)+c\,(1-\delta_{xy})$ is a distance. Moreover, if d is a pseudometric (metric) and $c>0$, then $d+c$ is a metric. Note that this construction can also be extended to some negative numbers c.

Problem 2.8. Let d be a metric. Find the least possible c such that $d+c$ is a pseudometric. Verify that for such c and any $c'>c$ the function $d+c'$ is a metric.

Construction 2.3 (Induced distance)**.** In fact, we have already implicitly used this obvious construction. Let d be a distance function on the set X, and Y be a nonempty subset of X. We define a distance function on Y by setting $|yy'|=d(y,y')$ for any $y,y'\in Y$, and call it *induced from* d or *the restriction of* d *to* Y. Note that the restriction of a metric (pseudometric) is always a metric (pseudometric). To emphasize that Y is equipped with the induce metric, we call such Y *a subspace of* X. However, in what follows, each subset of a metric space will be considered as a subspace, unless otherwise stated.

Construction 2.4 (Product)**.** Let $\{X_i\}_{i\in I}$ be some family of sets, and d_i be a distance function on X_i. We call each X_i equipped with d_i *a space*. We set $X=\prod_{i\in I}X_i$, and for any $x,x'\in X$ we consider the function $d_I(x,x')\colon I\to\mathbb{R}$, $d_I(x,x')(i)=d_i\big(x_i,x'_i\big)$. Put

$$D_I=\cup_{x,x'\in X}\{d_I(x,x')\},$$

and let $\mathcal{V}_I\supset D_I$ be some linear subspace of the linear space $\mathcal{F}_I$ consisting of all real-valued functions defined on the set I, and $\mathcal{V}_I^+\subset\mathcal{V}_I$ consists of all functions with non-negative values. We call the subset $\mathcal{V}_+\subset\mathcal{V}$ by *the first orthant of* $\mathcal{V}$. Denote by $0\in\mathcal{F}_I$ the zero function: $0(i)=0$ for any $i\in I$. Let $\rho\colon\mathcal{V}_I^+\to[0,\infty)$ be an arbitrary function such that $\rho(0)=0$. This function ρ generates a distance function d_ρ on X: $d_\rho(x,x')=\rho\big(d_I(x,x')\big)$. The function ρ is sometimes called *a binder*, and the function d_ρ is called *the distance generated by* ρ; the space X with the distance function d_ρ is called *the semidirect product of the spaces* X_i *w.r.t.* ρ. Below we will show a few examples in which all d_i are metrics, and the resulting d_ρ is a metric as well.

Example 2.5. Consider the standard partial ordering on the first orthant $\mathcal{V}_+$ of $\mathcal{V}$: $v\le w$ if and only if $v(i)\le w(i)$ for all $i\in I$.

We call the function ρ from Construction 2.4

- *positively definite* if it vanishes only at the origin;
- *subadditive* if for any $v, w \in \mathcal{V}_+$ it holds $\rho(v+w) \le \rho(v) + \rho(w)$;
- *monotone* if it is monotone w.r.t. the standard partial order described above.

Proposition 2.6. *Under notations of Construction 2.4, let all X_i be metric spaces, and ρ be positively definite, subadditive and monotone. Then the distance function d_ρ on $\prod_{i\in I} X_i$ is a metric.*

Proof. Positive definiteness of d_ρ follows from the one of metrics and of the function ρ. It remains to verify the triangle inequality. Take arbitrary $x, x', x'' \in X$, then

$$d_\rho(x,x'') = \rho\big(d_I(x,x'')\big) \le \rho\big(d_I(x,x') + d_I(x',x'')\big) \le \rho\big(d_I(x,x')\big) + \rho\big(d_I(x',x'')\big) = d_\rho(x,x') + d_\rho(x',x'').$$

Here the first inequality follows from monotonicity of ρ and triangle inequalities for metrics, and the second one from subadditivity of ρ. □

Example 2.7. Let I be a finite set, say, $I = \{1,\dots,n\}$, and $\mathcal{V}_I = \mathcal{F}_I = \mathbb{R}^n$. For ρ we choose the corresponding restriction of one of the following standard norms on $\mathcal{V}_I$:

$$\big\|(v_1,\dots,v_n)\big\|_p = \sqrt[p]{\sum_{i=1}^n |v_i|^p} \text{ for } 1 \le p < \infty \text{ and } \big\|(v_1,\dots,v_n)\big\|_\infty = \max\big\{|v_i| : i = 1,\dots,n\big\}.$$

Then the distance function d_ρ is a metric because ρ is positively definite, subadditive and monotone (verify), thus we can apply Proposition 2.6. For $p = 2$ (when the norm is Euclidean) we get *the direct product* of the spaces X_i.

Example 2.8. On the first orthant $\mathbb{R}^2_+$ of the plane $\mathbb{R}^2$ with coordinates (x,y) consider the following function $\rho(x,y) = x + \sqrt{y}$. This function is obviously positively definite and monotonic. To prove it is subadditive, make the following calculation:

$$\rho(x+x', y+y') = x + x' + \sqrt{y+y'} \le x + \sqrt{y} + x' + \sqrt{y'} = \rho(x,y) + \rho(x',y').$$

Let $X_1 = X_2 = \mathbb{R}$, then, by Proposition 2.6, the distance function d_ρ is a metric. In explicit form

$$d_\rho\big((x,y),(x',y')\big) = |x-x'| + \sqrt{|y-y'|}.$$

Example 2.9. Let I be an arbitrary set, and the family $\{X_i\}_{i\in I}$ consists of spaces with diameters bounded by the same constant, i.e., the set of numbers $\{\operatorname{diam} X_i\}_{i\in I}$ is bounded. Then, for $\mathcal{V}$, we can choose the linear space of all bounded functions $v\colon I \to \mathbb{R}$, and for ρ we can choose the norm $\|v\|_\infty = \sup_{i\in I}\Big\{\big|v(i)\big|\Big\}$. If all X_i are metric spaces, then d_ρ is a metric.

The previous construction can be generalized if we consider not all the Cartesian product $X = \prod_{i\in I} X_i$, but only a part of it extracted by some condition.

Example 2.10. We choose the set of natural numbers $\mathbb{N}$ as I, and the real line $\mathbb{R}$ as X_i, then each element of $\prod_{i\in\mathbb{N}} X_i$ can be represented as a sequence $(x_1, x_2, \dots)$, where $x_i \in X_i = \mathbb{R}$.

Consider a subset $\ell_p \subset \prod_{i\in\mathbb{N}} X_i$ consisting of all sequences $x = (x_1, x_2, \dots)$ for which

$$\|x\|_p := \begin{cases} \sqrt[p]{\sum_{i=1}^\infty |x_i|^p} & \text{for } 1 \le p < \infty, \text{ and} \\ \sup\big\{|x_i| : i \in \mathbb{N}\big\} & \text{for } p = \infty \end{cases}$$

is finite. Then $\|\cdot\|_p$ is a norm on ℓ_p. The resulting metric space is denoted by ℓ_p. In particular, the space ℓ_∞ consists of all bounded sequences.

Example 2.11. We choose an arbitrary set of indices I, and consider I as topological space with discrete topology. We put all X_i equal to $\mathbb{R}$, and as a subset of $\prod_{i\in I} X_i$ we take the family Fin_I of all maps $x\colon I \to \mathbb{R}$ with compact support, i.e., such that $x(i) \ne 0$ only for a finite number of indices i. It is clear that Fin_I forms a linear space, and each function

$$\|x\|_p = \sqrt[p]{\sum_{i\in I} |x_i|^p} \text{ for } 1 \le p < \infty \quad \text{and} \quad \|x\|_\infty = \max_{i\in I}\big\{|x_i|\big\}$$

is a norm on the space Fin_I. If all X_i are metric space, and ρ is a norm described above, then d_ρ is a metric.

Construction 2.5 (Levenshtein distance)**.** Let A be some set, and A^* be the family of all finite sequences of elements from A, as well as an empty sequence λ. We will interpret A as *an alphabet* of some language. Then elements from A are naturally called *letters*, and the elements from A^* are *words*. *An editorial operation on* A^* is a word transformation consisting either in the exclusion of one of the letters from the word (*deletion*), or in the insertion of a letter into the word (*insertion*), or in replacing one letter with another one (*substitution*). The smallest number of editorial operations needed to move from one word to another is called *the Levenshtein distance* and generates a metric on A^*. This distance plays an important role in linguistics and bioinformatics.

Now we need to recall the notion of graph. We consider more general graphs permitting infinite numbers of vertices and edges.

2.3.1 Elements of graph theory, and metric constructions for graphs

Consider a triple $G = (V, E, i)$, where V and E are arbitrary sets, and $i\colon E \to \mathcal{F}_2(V)$ a mapping to the set of all at most 2-point nonempty subsets of V, see Section 1.12. The set V is also denoted by $V(G)$ and called *the set of vertices*; the set E is also denoted by $E(G)$ and called *the set of edges*; the mapping $i\colon E \to \mathcal{F}_2(V)$ is also denoted by i_G and called *the incidence mapping*, and it defines how edges are glued to vertices; the triple G is called *a non-oriented graph*. An edge e is called *a loop*, if $\#i(e) = 1$. The number $\#i^{-1}\big(i(e)\big)$ is called *multiplicity of the edge* e. An edge e is called *multiple*, if its multiplicity is more that 1. Graph without loops and multiple edges is called *simple*. In a simple graph each edge e can be considered as the pair $\{v, w\} = i(e)$ of different vertices v and w, that is why we can reduce the definition by setting $G = (V, E)$, where E is a subset of $V^{(2)}$, where $V^{(2)}$ is the set of all 2-point subsets of V. *A subgraph H of the graph G* is a triple (V', E', i') such that $V' \subset V$, $E' \subset E$, and $i' = i|_{E'}$.

If instead $\mathcal{F}_2(V)$ we consider $V \times V$, then we obtain definition of *an oriented graph*.

Let $G = (V, E, i)$ be an arbitrary non-oriented graph. For each edge $e \in E$ and each vertex $v \in i(e)$ we say that e and v are *incident*; if $i(e) = \{v, w\}$, the vertices v and w are called *adjacent*, and we say that *e joins v and w*; if $i(e) = \{v\}$, we say that v is *adjacent to itself*, that *e joins v with v*. The number of non-loop edges plus the double number of loops, incident to a vertex $v \in V$, is called *the degree of* v and is denoted by $\deg v$ or $\deg_G v$.

A walk of length n joining some vertices v and w is a sequence $v = v_0,\, e_1,\, v_1,\, e_2, \ldots, e_n,\, v_n = w$ of alternating vertices and edges such that each edge e_k joins v_{k-1} and v_k. If G is a simple graph, we do not need to indicate the edges since each pair of adjacent vertices defines uniquely the edge incident to these vertices. The walt is called *closed* if $v_0 = v_n$, and it is called *open* otherwise. *A trail* is a walk with no repeated edges. *A path* is an open trail with no repeated vertices. *A circuit* is a closed trail. *A cycle* is a circuit with no repeated vertices.

In the case of an oriented graph G, if $i(e) = (v, w)$, then we say that *e starts at v and ends at w*. We redefine the walk just demanding that each e_k starts at v_{k-1} and ends at v_k. All other definitions remain the same.

A graph G is called *connected*, if each pair of its vertices are joined by a walk. A graph without cycles is called *a forest*, and a connected forest is called *a tree*. Clearly that each forest is a simple graph.

A weighted graph is a graph $G = (V, E, i)$ equipped with *a weight function* $\omega\colon E \to [0, \infty)$ (sometimes it is useful to consider more general weight functions, for instance, with possibility of negative values). Sometimes we denote the weighted graph as (V, E, i, ω) or (G, ω), and, in the case of simple graph G, by (V, E, ω). *The weight of a subgraph* of a weighted graph is the sum of the weights of edges from this subgraph. We can extend this definition to paths and cycles considering them as subgraphs of G. In the case of the walk, its weight is defined as the sum of weights of its consecutive edges. For graphs without weight functions these notions are defined as well by assigning the weight 1 to each edge by default.

Construction 2.6 (Distance on a connected graph)**.** Let $\Gamma = (V, E, i)$ be a connected graph. We define a distance function on V, setting it equal to the infimum of length of walks joining a given pair of vertices (we can always change the walks to paths with the same result). This function is a metric on V (verify).

Construction 2.7 (Distance on a connected weighted graph)**.** If $\Gamma = (V, E, i, \omega)$ is a weighted connected graph with the weight function $\omega\colon E \to [0, \infty)$, then we define the distance function by setting it equal to the infimum of the weights of all walks (or paths) joining a given pair of vertices. The resulting function is a pseudometric on V. Note that even if ω is everywhere positive, the constructed distance function does not have to be a metric (consider a graph with a positive weight function in which some pair of vertices are connected by an infinite number of paths with weights tending to zero).

Construction 2.8 (Cayley graph of a group)**.** Let G be an arbitrary group with a set S of generators. *The Cayley graph of the pair* (G, S) is the directed graph $\Gamma(G, S) = (G, E, i)$, in which $(g, h) = i(e)$ for some $e \in E$ if and only if $h = gs$ for some $s \in S$.

We note that in geometric group theory the set S usually satisfies the following properties: the neutral element is not contained in S, and $S = S^{-1}$, i.e., if $s \in S$, then $s^{-1} \in S$. In this case, the graph $\Gamma(G, S)$ has no loops, and if $(g, h) = i(e)$ for some $e \in E$, then $(h, g) = i(e')$ for some $e' \in E$, since $h = gs$ implies $g = hs^{-1}$. Under these assumptions, by Cayley graph we mean a simple non-oriented graph in which each pair of mutually opposite oriented edges were replaced by one (non-oriented) edge. For such Cayley graph, Construction 2.6 defines a metric on G, which is used to determine the growth rate of the group G.

Remark 2.12. In what follows, we will always consider only those S that satisfies the both conditions: $S = S^{-1}$ and S does not contain the neutral element of G. However, we will not list in S all inverses of its elements supposing that this holds by default.

Problem 2.9. Describe the Cayley graphs for the following groups G and generating sets S:

(1) $G = \mathbb{Z}$ and $S = \{1\}$;

(2) $G = \mathbb{Z}_m$ and $S = \{1\}$;

(3) $G = \mathbb{Z}^2$ and $S = \big\{(1, 0)\, (0, 1)\big\}$;

(4) $G = \mathbb{Z}^2$ and $S = \big\{(1, 0)\, (0, 1),\, (1, 1)\big\}$;

(5) G is a free group with generators a and b.

Construction 2.9 (Quotient spaces)**.** Let (X, ρ) be a metric (pseudometric) space, and $\sim$ an equivalence relation on X. Define the following *quotient distance function* on X:

$$\rho_\sim(x, y) = \inf\Big\{\sum_{i=0}^n \rho(p_i, q_i) : p_0 = x,\, q_n = y,\, n \in \mathbb{N},\, q_i \sim p_{i+1} \text{ for all } i\Big\}.$$

The terms in the right-hand side can be naturally visualized as $p_0 - q_0 \sim p_1 - q_1 \sim \cdots \sim p_n - q_n$, where $p_i - q_i$ indicates that we calculate the distance between p_i and q_i, and $q_i \sim p_{i+1}$ says that these points are taken from the same equivalency class, and thus the distance between them is zero. We call such sequences $p_0 - q_0 \sim p_1 - q_1 \sim \cdots \sim p_n - q_n$ an *admissible* one joining the classes $[x]$ and $[y]$, and the value $\sum_{i=0}^n \rho(p_i, q_i)$ *the weight of this admissible sequence.*

Problem 2.10. Prove that $\rho_\sim$ is a pseudometric on X.

Now, let $x \sim y$, then we can put $p_0 = x = q_0$ and $p_1 = q_1 = y$, so $\rho_\sim(x, y) \le \rho(x, x) + \rho(y, y) = 0$, thus $\rho_\sim(x, y) = 0$. Therefore, each equivalence class of $\sim$ consists of points on $\rho_\sim$-zero distance from each other, thus $\rho_\sim$ generates correctly a pseudometric on $X/\!\sim$ which we also denote by $\rho_\sim$ and call in *the quotient pseudometric w.r.t.* $\sim$; the space $X/\!\sim$ we will call the *the quotient pseudometric space w.r.t.* $\sim$. The next step is to consider the space $X/\rho_\sim$ instead of the $X/\!\sim$: these two spaces are different when the distance between some distinct classes of $\sim$ vanishes. As above, we denote by the same $\rho_\sim$ the corresponding metric on $X/\rho_\sim$, and the $\rho_\sim$ and $X/\rho_\sim$ we call *the quotient metric space and the quotient metric space w.r.t.* $\sim$, respectively.

To work effectively with the $\rho_\sim$, let us define the following notions. The admissible sequence $p_0 - q_0 \sim p_1 - q_1 \sim \cdots \sim p_n - q_n$ from definition of $\rho_\sim$ is called *reducible* if it is possible to delete a part of it in such a way that its ends remains to belong to the classes $[x], [y] \in X/\!\sim$, and its weight did not increase. Otherwise, its is called *irreducible.* It is evident that to calculate the distance $\rho_\sim$, it is sufficient to consider irreducible admissible sequences only. If a and b not equivalent, then we write it as $a \not\sim b$.

Problem 2.11. Let $\xi = (p_0 - q_0 \sim p_1 - q_1 \sim \cdots \sim p_n - q_n)$ be an irreducible admissible sequence. Prove that

(1) for any $i < j$ we have $p_i \not\sim p_j$, $q_i \not\sim q_j$;

(2) $q_i \sim p_j$ if and only if $j = i + 1$;

(3) for any i we have $q_i \ne p_{i+1}$.

Example 2.13. Let us show how Problem 2.11 can be applied. Let $X = [0, a] \subset \mathbb{R}$ be a segment of real line. Identify its ends. This means, that $x, y \in X$ are equivalent if and only if either $x = y$, or $\{x, y\} = \{0, a\}$. Describe all irreducible admissible sequences ξ. The are two possibilities (verify):

(1) ξ consists of two different points, say x and y, and its weight equals $|xy|$;

(2) ξ consists of 4 points, its ends are distinct from 0 and a, but the middle points are these 0 and a; w.l.o.g., $\xi = (x - 0 \sim a - y)$, thus its weight equals $|0x| + |ya|$.

Thus, on the topological circle $[0, a]/\sim$ we introduce the distance in the same way as we do for the standard circle when we choose the length of the shortest arc between the points.

Problem 2.12. Let $\sim$ be the trivial equivalence on a metric space (X, ρ), i.e., $x \sim y$ if and only if $x = y$. Prove that $\rho_\sim = \rho$.

Problem 2.13. Let $\sim$ be an equivalence on a pseudometric space (X, ρ). Prove that for any $x, y \in X$ it holds $\rho_\sim(x, y) \le \rho(x, y)$. Thus, if we define the function $b\colon X \to X \to \mathbb{R}$ such that $b(x, y) = 0$ for $x \sim y$, and $b(x, y) = \rho(x, y)$ otherwise, then $\rho_\sim \le b$.

Construction 2.10 (Generalization of distance)**.** It is useful to allow infinite distances. A function $\rho\colon X \times X \to [0, \infty]$ that satisfies the axioms of distance (pseudometric, metric) will be called *generalized* one. The corresponding spaces X with such distances we call *generalized* as well.

Remark 2.14. In some monographs the authors work with generalized distances from the very beginning and because of that they call such distances without the word “generalized”. However, the distances with values in $[0, \infty)$ they call *finite* ones.

The generalization of distance gives rise more metric constructions.

Construction 2.11 (Disjoint union of spaces)**.** Let $\big\{(X_i, \rho_i)\big\}_{i\in I}$ be an arbitrary family of generalized spaces. Consider on $\sqcup_{i\in I} X_i$ the distance function that is equal to ρ_i on X_i, and to ∞ for any pair of points from different i. The resulting space is called *the disjoint union of spaces X_i*.

Evidently, if X_i are generalized pseudometric (metric) spaces, then $\sqcup_{i\in I} X_i$ is the space of the same type.

Let X be a generalized pseudometric space. So, we can define two equivalence relations: $x \sim_1 y$ if and only if $|xy| < \infty$; and $x \sim_2 y$ if and only if $|xy| = 0$.

Problem 2.14. Prove that each class of equivalence $\sim_1$ is a pseudometric space (with finite distance), and that the distance between points from different classes equals ∞. Thus, if we denote by X_i the classes of equivalence $\sim_1$, then $X = \sqcup X_i$. Prove that the space $X/{\sim_2}$ equals the disjoint union $\sqcup(X_i/{\sim_2})$ of metric spaces $X_i/{\sim_2}$.

Now we combine disjoint union and quotient operation.

Example 2.15. Let X and Y be metric spaces, $Z \subset X$, and $f\colon Z \to Y$ a mapping. Consider on $X \sqcup Y$ with generalized metric ρ the equivalence relation $\sim$ which identifies each $a \in f(Z)$ with all $b \in f^{-1}(a)$. The quotient space $X \sqcup_f Y := (X \sqcup Y)/\rho_\sim$ is called the result of *gluing the spaces X and Y over the mapping f*.

Problem 2.15. Suppose that f is isometric. Prove that the restrictions of $\rho_\sim$ to X and Y coincides with the initial metrics of X and Y, respectively.

Problem 2.16. Let $y_0 \in Y$ and $f(X) = y_0$. Prove that $X \sqcup_f Y$ is isometric to Y.

Construction 2.12 (Maximal pseudometric)**.** Consider an arbitrary function $b\colon X \times X \to [0, \infty]$, and denote by $\mathcal{D}_b$ the set of all generalized pseudometrics $d\colon X \times X \to [0, \infty]$ such that $d(x, y) \le b(x, y)$ for any $x, y \in X$.

Lemma 2.16. *There exists and unique $d_b \in \mathcal{D}_b$ such that $d_b \ge d$ for all $d \in \mathcal{D}_b$.*

Proof. We put $d_b(x, y) = \sup_{d\in\mathcal{D}_b} d(x, y)$. It is evident that d_b is nonnegative, symmetric, $d_b(x, x) = 0$ for all $x \in X$, and $d_b \le b$. It remains to verify the triangle inequality:

$$d_b(x, z) = \sup_{d\in\mathcal{D}_b} d(x, z) \le \sup_{d\in\mathcal{D}_b} \big(d(x, y) + d(y, z)\big) \le \sup_{d\in\mathcal{D}_b} d(x, y) + \sup_{d\in\mathcal{D}_b} d(y, z) = d_b(x, y) + d_b(y, z).$$

The uniqueness of d_b is evident. □

We denote d_b by $\sup \mathcal{D}_b$ and call it *maximal pseudometric w.r.t. the function b*.

Theorem 2.17. *Let (X,ρ) be a pseudometric space, and $\sim$ an equivalence relation. We put*

$$b(x,y) := b_\sim(x,y) = \begin{cases} 0, & \text{if } x\sim y,\\ \rho(x,y), & \text{otherwise.}\end{cases}$$

Then $d_b = \sup \mathcal{D}_b = \rho_\sim$, thus $X/\rho_\sim = X/d_b$.

Proof. By Problem 2.13, we have $\rho_\sim \le b$, thus $\rho_\sim \in \mathcal{D}_b$. It remains to prove that $\rho_\sim \ge d$ for each $d\in\mathcal{D}_b$. To do that, we write down

$$\rho_\sim(x,y) = \inf\Big\{\sum_{i=0}^n \rho(p_i,q_i) : p_0 = x,\, q_n = y,\, n\in\mathbb{N},\, q_i\sim p_{i+1} \text{ for all } i\Big\} \ge$$
$$\ge \inf\Big\{d(p_0,q_0)+d(q_0,p_1)+\dots+d(q_{n-1},p_n)+d(p_n,q_n) : n\in\mathbb{N},\, q_i\sim p_{i+1} \text{ for all } i\Big\} \ge d(x,y).$$

□

Now we apply the quotation technique to obtain a few more important classes of metric spaces.

Construction 2.13 (Groups actions). Let X be a metric space and $G\subset \mathrm{Iso}(X)$ a subgroup of its isometry group. Consider the action of G on X, i.e., the mapping $\varphi\colon G\times X\to X$ such that $\varphi(g,x) =: g(x)$ satisfies the following conditions:

(1) $e(x) = x$ for the neutral element $e\in G$ and any $x\in X$;

(2) $(hg)(x) = h\big(g(x)\big)$ for any $g,h\in G$ and $x\in X$.

Consider the following equivalence relation on X: $x\sim y$ if and only if $g(x) = y$ for some $g\in G$. We say that the equivalence $\sim$ *is generated by the action of G on X*. The corresponding quotient space $X/\!\sim$ is usually denoted by X/G. The sets $G(x)$ are called *orbits*, they are elements of X/G; the set X/G is called *the orbit-space*.

Problem 2.17. Let X be a metric space and $G\subset\mathrm{Iso}(X)$ a subgroup of its isometry group. For each two elements $G(x), G(y)\in X/G$ we set $d\big(G(x),G(y)\big) = \inf\big\{|x'y'| : x'\in G(x),\, y'\in G(y)\big\}$. Prove that $d = \rho_\sim$, where the equivalence $\sim$ is generated by the action of G on X.

Problem 2.18. Let S^1 be the standard unit circle in the Euclidean plane. As a distance between $x,y\in S^1$ we take the length of the shortest arc of S_1 between x and y. By *the standard torus* we mean the direct product $T^2 = S^1\times S^1$ (with the Euclidean binder). We describe the points on the both S^1 by their polar angles φ_1 and φ_2, defined up to 2π. So, *the shifts* $s_{a,b}\colon(\varphi_1,\varphi_2)\mapsto(\varphi_1+a,\varphi_2+b)$ are isometries of T^2. Fix some $(a,b)\in\mathbb{R}^2$ and consider a subgroup $G_{a,b}\subset\mathrm{Iso}(T^2)$ consisting of all shifts $s_{ta,tb}$, $t\in\mathbb{R}$. For different a and b, find the corresponding pseudometric and metric quotient spaces.

Construction 2.14 (Metrized graphs). Take an arbitrary collection of segments $\big\{[a_k,b_k]\big\}_{k\in K}$, and on the set $\sqcup\{a_k,b_k\}$ consider an equivalence $\sim$. The generalized pseudometric metric space $\big(\sqcup[a_k,b_k]\big)/\sim$ is called a *metrized graph*. To obtained its combinatorial structure, we represent it as we did before, namely, as a triple $G = (V,E,i)$. To do that, we put $V := \big(\sqcup\{a_k,b_k\}\big)/\sim$, $E := \big\{[a_k,b_k]\big\}_{k\in K}$, and $\pi\colon \sqcup\{a_k,b_k\}\to\big(\sqcup\{a_k,b_k\}\big)/\sim$ be the canonical projection, then $i\big([a_i,b_i]\big) := \pi\big(\{a_i,b_i\}\big)$.

Construction 2.15 (Polyhedral spaces). By *a polyhedron of dimension n* we mean an intersection of a finite number of half-spaces in $\mathbb{R}^n$ that has nonempty interior. For each point x of a polyhedron W we define its *dimension* $\dim_W x$ as follows: consider all affine subspaces L containing x such that x is an interior point of $W\cap L$, and put $\dim_W x$ to be the maximal dimension of such L. Denote by W^k the subset in W consisting of all points of dimension k. One can show that W^k consists of connected component each of which belongs of an affine subspace L of dimension k. The closures of these connected components are called *the faces of W of dimension k*.

Now, consider an arbitrary collection $\{W_k\}_{k\in K}$ of polyhedra (probably, of different dimensions), and for some pairs (W_k,W_l), $k\ne l$, choose some faces $F_k^p\subset W_k$ and $F_l^q\subset W_l$ such that there exists an isometry $\varphi\colon F_k^p\to F_l^q$. Consider the generalized metric space $\sqcup_{k\in K}W_k$ and the equivalence relation generated by the isometries φ: we put in one class each point $x\in F_k^p$ and $\varphi(x)\in F_l^q$, for all isometries φ (we glue the faces by the isometry φ). The resulting quotient generalized space is called *a polyhedron space*.

Problem 2.19. Represent the standard torus from Problem 2.18 as a polyhedron space.

Problem 2.20. Verify that the above constructions actually, as declared, define (pseudo-)metrics.

A huge collection of metric spaces can be found in [1].

2.4 Convergence and completeness

Since each metric space is Hausdorff, the following result holds.

Proposition 2.18. *If a sequence converges in a metric space, then its limit is uniquely determined.*

A sequence $x_1, x_2, \ldots$ of points of a metric space X is called *fundamental* if for any $\varepsilon > 0$ there exists N such that for all $m, n \geq N$ the inequality $|x_m x_n| < \varepsilon$ holds. If every fundamental sequence in a metric space X is convergent, then such X is called *complete*.

Problem 2.21. Show that a subspace of a complete metric space is complete if and only if it is closed.

Theorem 2.19. *Each metric space X is an everywhere dense subspace of some complete space called a* ***completion of*** *X. The completion of the space X is uniquely defined: for any two completions $X' \supset X$ and $X'' \supset X$ there is an isometry $f\colon X' \to X''$ that is identical on X.*

Construction 2.16 (Completion)**.** The standard construction of completion of a metric space X is as follows:

- the set W of all fundamental sequences in the space X is considered;
- on W, a pseudometric is set equal to the limit of the distances between the points of two fundamental sequences (the existence of the limit follows from the fundamentality);
- X can be realized as a subset of W by associating each point $x \in X$ with the constant sequence $x, x, \ldots$.
- the desired completion is obtained by factorization as in Problem 2.1.

The uniqueness of completion implies the following result.

Problem 2.22. Let X be an arbitrary subspace of a complete metric space. Then the closure $\overline{X}$ of the set X is a completion of the space X.

Problem 2.23. Let $f\colon X \to Y$ be a bi-Lipschitz mapping of metric spaces. Prove that X is complete if and only of Y is complete. Construct a homeomorphism of metric spaces that does not preserves completeness.

Problem 2.24. Show that a metric space is complete if and only if the following condition holds: for any sequence of closed subsets $X_1 \supset X_2 \supset X_3 \supset \cdots$ such that $\operatorname{diam} X_n \to 0$ as $n \to 0$, the intersection $\cap_{i=1}^{\infty} X_i$ is not empty (in fact, it consists of unique element). Show that the condition $\operatorname{diam} X_n \to 0$ is essential.

Problem 2.25 (Fixed-point theorem)**.** Let $f\colon X \to X$ be a C-Lipschitz mapping of a complete metric spaces X. Prove that for $C < 1$ there exists and unique a point x_0 such that $f(x_0) = x_0$ (it is called *the fixed point of the mapping f*).

2.5 Compactness and sequential compactness

Recall that a topological space is called *compact* if a finite subcover can be extracted from any of its open cover, and *sequentially compact* if any sequence of its points has a convergent subsequence. As noted in Section 1, in the case of general topological spaces, compactness and sequential compactness are different concepts. However, for metric spaces this is not so.

Theorem 2.20. *A metric space is compact if and only if it is sequentially compact.*

Proof. Let X be a compact metric space and $x_1, x_2, \ldots$ be an arbitrary sequence of points from X. We must show that there is a convergent subsequence in this sequence.

Suppose this is not so, then

- the sequence x_i contains an infinite number of different points;
- for each point $x \in U := X \setminus \{x_1, x_2, \ldots\}$ there is an (open) neighborhood U^x that does not contain points of the sequence x_i, therefore $U = \cup_{x \in U} U^x$ and, thus, U is an open set;
- for each x_i there is a neighborhood U^{x_i} for which $U^{x_i} \cap \{x_1, x_2, \ldots\} = \{x_i\}$.

But then the family $\{U, U^{x_1}, U^{x_2}, \ldots\}$ is an open cover of X from which it is impossible to choose a finite subcover (each finite subcover contains only finitely many points of the sequence x_i). The obtained contradiction shows that X is sequentially compact.

Now let the space X be sequentially compact. Suppose that there exists an open covering $\mathcal{U} = \{U_\alpha\}$ of the space X that does not contain finite subcovers. On X we define a function $\rho\colon X \to \mathbb{R}$ as follows:

$$\rho(x) = \sup\{r \in \mathbb{R} \mid \exists U_\alpha \in \mathcal{U} : \mathrm{U}_r(x) \subset U_a\}.$$

This function is everywhere finite, since the sequentially compact space is bounded, and everywhere positive by virtue of the definition of metric topology.

Lemma 2.21. *The function ρ defined above is 1-Lipschitz and, therefore, continuous.*

Proof. Assume the contrary, i.e., that for some $x, y \in X$ the inequality $|\rho(y) - \rho(x)| > |xy|$ holds. Without loss of generality, we assume that $\rho(y) > \rho(x)$, then $\rho(y) > \rho(x) + |xy|$. Increase the number $\rho(x)$ a little to ρ' and slightly decrease the number $\rho(y)$ to ρ'' so that $\rho'' > \rho' + |xy|$ is still true, then

- $U_{\rho'}(x) \subset U_{\rho''}(y)$, because for an arbitrary point $z \in U_{\rho'}(x)$ we have

$$|zy| \le |zx| + |xy| < \rho' + |xy| < \rho'';$$

- there is $U_\alpha \in \mathcal{U}$ for which $U_{\rho''}(y) \subset U_\alpha$.

But then $U_{\rho'}(x) \subset U_\alpha$, therefore $\rho(x) \ge \rho'$ by the definition of the function ρ, a contradiction. □

By virtue of Problem 1.53, the function ρ achieves the smallest value ρ_0, which is, therefore, strictly positive. Put $r = \rho_0/2$. Then for each point $x \in X$ there exists $U_\alpha \in \mathcal{U}$ such that $U_r(x) \subset U_\alpha$.

Choose an arbitrary point $x_1 \in X$, and let $U_1 \in \mathcal{U}$ be such that $U_r(x_1) \subset U_1$. There is $x_2 \in X \setminus U_1$. Choose $U_2 \in \mathcal{U}$ such that $U_r(x_2) \subset U_2$. In general, if $x_1, \ldots, x_n$ and $U_1, \ldots, U_n$ are selected, then there is $x_{n+1} \in X \setminus \cup_{i=1}^n U_i$ and $U_{n+1} \in \mathcal{U}$ such that $U_r(x_{n+1}) \subset U_{n+1}$. Since $\mathcal{U}$ does not have a finite subcover, we construct an infinite sequence $x_1, x_2, \ldots$, and it is clear that every point x_{n+1} lies outside $\cup_{i=1}^n U_r(x_i)$, so for any x_i and x_j, $i \ne j$, we have $|x_i x_j| \ge r$. But such a sequence does not contain a convergent subsequence, which contradicts the sequential compactness of X. □

Problem 2.26. Give an example of a topological space that is

(1) compact, but not sequentially compact;

(2) sequentially compact, but not compact.

Problem 2.27 (Lebesgue's lemma)**.** Let X be a compact metric space. Prove the following statement: for any open cover $\{U_i\}_{i\in I}$ of X there exists $\rho > 0$ such that for any $x \in X$ one can find U_i with $B_\rho(x) \subset U_i$.

Problem 2.28. Show that each continuous mapping $f\colon X \to Y$ from a compact metric space to an arbitrary metric space is uniformly continuous.

Problem 2.29. Prove that the diameter $\operatorname{diam} X$ of a compact metric space X if finite, and that there exist $x, y \in X$ such that $\operatorname{diam} X = |xy|$.

2.5.1 Completeness and compactness

Theorem 2.20 easily yields the following result.

Corollary 2.22. *A compact metric space is complete.*

Proof. Let X be a compact metric space. In the space X we choose an arbitrary fundamental sequence $x_1, x_2, \ldots$. By Theorem 2.20, there exists a subsequence $x_{i_1}, x_{i_2}, \ldots$, converging to some point $x \in X$. Due to fundamentality of $x_1, x_2, \ldots$, the whole this sequence converges to x, therefore the space X is complete. □

Definition 2.23. For $\varepsilon > 0$, a subset S of a metric space X is called *an ε-net* if for any point $x \in X$ there exists $s \in S$ such that $|xs| < \varepsilon$. A metric space is called *totally bounded* if for any $\varepsilon > 0$ there exists a finite ε-net in it.

Theorem 2.24. *A metric space is compact if and only if it is complete and totally bounded.*

Proof. Let X be a compact metric space. Then, by virtue of Corollary 2.22, it is complete. We choose an arbitrary $\varepsilon > 0$, then $\{U_\varepsilon(x)\}_{x\in X}$ is an open cover of X. Since X is compact, there exists a finite subcover $\{U_\varepsilon(x_i)\}_{i=1}^n$. But then $\{x_1,\dots,x_n\}$ is a finite ε-net. Thus, the space X is totally bounded.

Now let X be a complete and totally bounded metric space. We prove that X is sequentially compact and apply Theorem 2.20. Consider an arbitrary sequence x_i in X. For each $n\in\mathbb{N}$, consider a finite $1/n$-net S_n. The balls $\{U_1(s)\}_{s\in S_1}$ cover X, so there is a ball U_1 among them that contains infinitely many elements of the sequence x_i. The balls $\{U_{1/2}(s)\}_{s\in S_2}$ cover U_1, so there is a ball U_2 among them for which $C_2 = U_1\cap U_2$ contains infinitely many x_i. If $U_1, U_2, \dots, U_k$ are already selected so that $C_k = \cap_{j=1}^k U_j$ contains infinitely many x_i, then the family $\{U_{1/(k+1)}(s)\}_{s\in S_{k+1}}$, since it covers C_k, contains a ball U_{k+1} for which there are infinitely many x_i in $C_{k+1} = \cap_{j=1}^{k+1} U_j$.

Now, we choose an arbitrary $x_{i_1}\in C_1$. Since there are infinitely many points of our sequence in C_2, there exists $i_2 > i_1$ such that $x_{i_2}\in C_2$. Continuing this process, we construct a subsequence $x_{i_1}, x_{i_2},\dots$ such that $x_{i_k}, x_{i_{k+1}},\dots\in C_k$ for each k, and since $\operatorname{diam} C_k \to \emptyset$ as $k\to\infty$, this subsequence is fundamental and, therefore, converges because X is complete. Thus, X is sequentially compact. □

Problem 2.30. Prove that every compact metric space is separable.

A subset of a topological space is called *nowhere dense* if its closure has empty interior.

Problem 2.31 (Baire's theorem)**.** Prove that a complete metric space cannot be covered by at most countably many nowhere dense subsets. Moreover, the complement of the union of at most countably many nowhere dense subsets is everywhere dense.

Problem 2.32. Prove that a compact metric space X cannot be isometrically mapped to a subspace $Y\subset X$ such that $Y\neq X$. In other words, each isometric mapping $f\colon X\to X$ for a compact metric space X is surjective.

Problem 2.33. Let X be a compact metric space and $f\colon X\to X$ be a mapping. Prove that

(1) if f is surjective and nonexpanding, then f is an isometry;

(2) if $\bigl|f(x)f(y)\bigr| \ge |xy|$ for all $x,y\in X$, then f is an isometry.

2.6 Canonical isometric embeddings of metric spaces

Let X be an arbitrary metric space. We denote by $C(X)$ the vector space of functions continuous on X, and by $C_b(X)$ the subspace of $C(X)$ composed of all bounded functions, and consider on $C_b(X)$ the norm $\|f\|_\infty = \sup_{x\in X}\bigl|f(x)\bigr|$ and the corresponding metric $|fg|_\infty = \|f-g\|_\infty$.

We define a mapping $\nu\colon X\to C_b(X)$ as follows. For each point $x\in X$, by d_x we denote the function $d_x\colon X\to[0,\infty)$ defined by the rule $d_x(y) = |xy|$. This function is 1-Lipschitz and, therefore, $d_x\in C(X)$. Now we fix some point $p\in X$ and consider the function $d_x - d_p$. Then for each $y\in X$ we have $\bigl|d_x(y) - d_p(y)\bigr| \le |xp|$, so the function $d_x - d_p$ is bounded and, thus, belongs to $C_b(X)$.

Theorem 2.25 (Frechet, Kuratowski)**.** *The mapping $\nu\colon X\to C_b(X)$ defined by the formula $\nu\colon x\mapsto d_x - d_p$ is an isometric embedding.*

Proof. By virtue of Problem 2.5, it suffices to verify that the mapping ν is isometric.

Choose arbitrary $x,y\in X$, then

$$\bigl|(d_x-d_p)(d_y-d_p)\bigr|_\infty = \sup_{z\in X}\bigl|d_x(z)-d_y(z)\bigr| \le |xy|$$

by triangle inequality. On the other hand, if $z=y$, then $\bigl|d_x(z)-d_y(z)\bigr| = |xy|$, which implies the required result. □

Let us prove a finer result. Recall that by ℓ_∞ we denote the space of all bounded sequences with the metric defined by the norm $\|\cdot\|_\infty$.

Theorem 2.26 (Frechet)**.** *Let X be a separable metric space, then X can be isometrically embedded into ℓ_∞.*

Proof. Consider an everywhere dense sequence $x_1, x_2, \ldots$ in X, which exists due to separability. Choose an arbitrary point $p \in X$ and associate with each point $x \in X$ a sequence $\nu(x)$ obtained by restricting the function $d_x - d_p$ to the sequence $x_1, x_2, \ldots$, namely, $\nu(x)(i) = d_x(x_i) - d_p(x_i)$. Since the functions $d_x - d_p$ are bounded, then $\nu(x) \in \ell_\infty$. The inequality $\bigl|\nu(x)\nu(y)\bigr|_\infty \le |xy|$ is verified in the same way as in the proof of Theorem 2.25. The converse inequality follows from the fact that the subset $\{x_i\}$ is everywhere dense in X: consider $x_{i_k} \to x$, then

$$\bigl|\nu(x)\nu(y)\bigr|_\infty = \sup_{z\in X}\bigl|d_x(z) - d_y(z)\bigr| \ge \bigl|d_x(x_{i_k}) - d_y(x_{i_k})\bigr| \to d_y(x) = |xy| \text{ as } k \to \infty.$$

The theorem is proved. □

References to Chapter 2

[1] M.M.Deza, E.Deza, *Encyclopedia of Distances*. Springer, 2009.

Exercises to Chapter 2

Exercise 2.1. Let X be a pseudometric space and $\sim$ is the natural equivalence relation: $x \sim y$ if and only if $|xy| = 0$. For each $x \in X$ denote by $[x]$ the equivalence class containing x. Prove that for any $x, y \in X$, $x' \in [x]$, and $y' \in [y]$ it is true that $|x'y'| = |xy|$. Thus, on the set $X/\sim$ the corresponding distance function is correctly defined: $\big|[x][y]\big| = |xy|$. Show that this distance function is a metric.

Exercise 2.2. Let X be an arbitrary metric space, $x, y \in X$, $r \geq 0$, $s, t > 0$, and $A \subset X$ be nonempty. Verify that

(1) $U_s\big(\{x\}\big) = U_s\big(x\big)$ and $B_r\big(\{x\}\big) = B_r\big(x\big)$;

(2) the functions $y \mapsto |xy|$, $y \mapsto |yA|$ are continuous;

(3) an open neighborhood $U_s(A)$ is an open subset of X, and a closed neighborhood $B_r(A)$ is a closed subset of X;

(4) $U_t\big(U_s(A)\big) \subset U_{s+t}(A)$ and construct an example demonstrating that the left-hand side can be different from the right-hand side;

(5) $B_t\big(B_s(A)\big) \subset B_{s+t}(A)$ and construct an example demonstrating that the left-hand side can be different from the right-hand side;

(6) $\partial U_s(x)$, $\partial B_s(x)$ are not related by any inclusion; $\partial U_s(x) \subset S_s(x)$ and $\partial B_r(x) \subset S_r(x)$; the both previous inclusions can be strict;

(7) $\operatorname{diam} U_s(x) \leq \operatorname{diam} B_s(x) \leq 2s$;

(8) $\operatorname{diam} U_s(A) \leq \operatorname{diam} B_s(A) \leq \operatorname{diam} A + 2s$.

Exercise 2.3. Let $\mathcal{L}(f) \subset \mathbb{R}$ be the set of all Lipschitz constants for a Lipschitz mapping f. Prove that $\inf \mathcal{L}(f)$ is also a Lipschitz constant.

Exercise 2.4. Show that each Lipschitz map is uniformly continuous, and each uniformly continuous map is continuous.

Exercise 2.5. Show that each isometry is a homeomorphism, in particular, each isometric mapping of one metric space into another one is an embedding, i.e., we recall, it is a homeomorphism with an image.

Exercise 2.6. Verify that the identity map, the composition of isometries, and the inverse mapping to an isometry are also isometries, i.e., the set of all isometries of an arbitrary metric space forms a group.

Exercise 2.7. Let X be an arbitrary metric space, $x, y \in X$ and $A \subset X$ be nonempty. Prove that $|Ax| + |xy| \geq |Ay|$, so that the function $\rho_A(x) = |Ax|$ is 1-Lipschitz and, therefore, uniformly continuous.

Exercise 2.8. Describe the Cayley graphs for the following groups G and generating sets S:

(1) $G = \mathbb{Z}$ and $S = \{1\}$;

(2) $G = \mathbb{Z}_m$ and $S = \{1\}$;

(3) $G = \mathbb{Z}^2$ and $S = \big\{(1,0)\,(0,1)\big\}$;

(4) $G = \mathbb{Z}^2$ and $S = \big\{(1,0)\,(0,1),\,(1,1)\big\}$;

(5) G is a free group with generators a and b.

Exercise 2.9. Let (X, ρ) be a metric (pseudometric) space, and $\sim$ an equivalence relation on X. Define the following *quotient distance function* on X:

$$\rho_\sim(x, y) = \inf\Big\{\sum_{i=0}^{n} \rho(p_i, q_i) : p_0 = x,\ q_n = y,\ n \in \mathbb{N},\ q_i \sim p_{i+1} \text{ for all } i\Big\}.$$

Prove that $\rho_\sim$ is a pseudometric on X.

Exercise 2.10. Let $\xi = (p_0 - q_0 \sim p_1 - q_1 \sim \cdots \sim p_n - q_n)$ be an irreducible admissible sequence. Prove that

(1) for any $i < j$ we have $p_i \not\sim p_j$, $q_i \not\sim q_j$;

(2) $q_i \sim p_j$ if and only if $j = i + 1$;

(3) for any i we have $q_i \neq p_{i+1}$.

Exercise 2.11. Let $\sim$ be the trivial equivalence on a metric space (X, ρ), i.e., $x \sim y$ if and only if $x = y$. Prove that $\rho_\sim = \rho$.

Exercise 2.12. Let $\sim$ be an equivalence on a pseudometric space (X, ρ). Prove that for any $x, y \in X$ it holds $\rho_\sim(x, y) \le \rho(x, y)$. Thus, if we define the function $b\colon X \times X \to \mathbb{R}$ such that $b(x, y) = 0$ for $x \sim y$, and $b(x, y) = \rho(x, y)$ otherwise, then $\rho_\sim \le b$.

Exercise 2.13. Let X be a generalized pseudometric space. We can define two equivalence relations: $x \sim_1 y$ if and only if $|xy| = \infty$, and $x \sim_2 y$ if and only if $|xy| = 0$. Prove that each class of equivalence $\sim_1$ is a pseudometric space (with finite distance), and that the distance between points from different classes equals ∞. Thus, if we denote by X_i the classes of equivalence $\sim_1$, then $X = \sqcup X_i$. Prove that the space $X/\!\sim_2$ equals the disjoint union $\sqcup(X_i/\!\sim_2)$ of metric spaces $X_i/\!\sim_2$.

Exercise 2.14. Let X and Y be metric spaces, $Z \subset X$, and $f\colon Z \to Y$ is an isometric mapping. Let ρ be the metric on the $X \sqcup_f Y$. Prove that the restrictions of ρ to X and Y coincides with the initial metrics of X and Y, respectively.

Exercise 2.15. Let $y_0 \in Y$ and $f(X) = y_0$. Prove that $X \sqcup_f Y$ is isometric to Y.

Exercise 2.16. Let X be a metric space and $G \subset \mathrm{Iso}(X)$ a subgroup of its isometry group. For each two elements $G(x), G(y) \in X/G$ we set $d\big(G(x), G(y)\big) = \inf\big\{|x'y'| : x' \in G(x),\, y' \in G(y)\big\}$. Prove that $d = \rho_\sim$, where the equivalence $\sim$ is generated by the action of G on X.

Exercise 2.17. Let S^1 be the standard unit circle in the Euclidean plane. As a distance between $x, y \in S^1$ we take the length of the shortest arc of S_1 between x and y. By *the standard torus* we mean the direct product $T^2 = S^1 \times S^1$ (with the Euclidean binder). We describe the points on the both S^1 by their polar angles φ_1 and φ_2, defined up to 2π. So, *the shifts* $s_{a,b}\colon (\varphi_1, \varphi_2) \mapsto (\varphi_1 + a, \varphi_2 + b)$ are isometries of T^2. Fix some $(a, b) \in \mathbb{R}^2$ and consider a subgroup $G_{a,b} \subset \mathrm{Iso}(T^2)$ consisting of all shifts $s_{ta,tb}$, $t \in \mathbb{R}$. For different a and b, find the corresponding pseudometric and metric quotient spaces.

Exercise 2.18. Represent the standard torus from Exercise 2.17 as a polyhedron space.

Exercise 2.19. Verify that the constructions given in Section 2.3 do define (pseudo-)metrics, as declared.

Exercise 2.20. Let d be a metric. Find the least possible c such that $d + c$ is a pseudometric. Verify that for such c and any $c' > c$ the function $d + c'$ is a metric.

Exercise 2.21. Show that a subspace of a complete metric space is complete if and only if it is closed.

Exercise 2.22. Let X be an arbitrary subspace of a complete metric space. Then the closure $\overline{X}$ of the set X is a completion of the space X.

Exercise 2.23. Let $f\colon X \to Y$ be a bi-Lipschitz mapping of metric spaces. Prove that X is complete if and only of Y is complete. Construct a homeomorphism of metric spaces that does not preserves completeness.

Exercise 2.24. Show that a metric space is complete if and only if the following condition holds: for any sequence of closed subsets $X_1 \supset X_2 \supset X_3 \supset \cdots$ such that $\operatorname{diam} X_n \to 0$ as $n \to 0$, the intersection $\cap_{i=1}^\infty X_i$ is not empty (in fact, it consists of unique element). Show that the condition $\operatorname{diam} X_n \to 0$ is essential.

Exercise 2.25 (Fixed-point theorem)**.** Let $f\colon X \to X$ be a C-Lipschitz mapping of a complete metric spaces X. Prove that for $C < 1$ there exists and unique a point x_0 such that $f(x_0) = x_0$ (it is called *the fixed point of the mapping* f).

Exercise 2.26. Give an example of a topological space that is

(1) compact, but not sequentially compact;

(2) sequentially compact, but not compact.

Exercise 2.27 (Lebesgue's lemma)**.** Let X be a compact metric space. Prove the following statement: for any open cover $\{U_i\}_{i\in I}$ of X there exists $\rho > 0$ such that for any $x \in X$ one can find U_i with $B_\rho(x) \subset U_i$.

Exercise 2.28. Show that each continuous mapping $f\colon X \to Y$ from a compact metric space to an arbitrary metric space is uniformly continuous.

Exercise 2.29. Prove that the diameter $\operatorname{diam} X$ of a compact metric space X if finite, and that there exist $x, y \in X$ such that $\operatorname{diam} X = |xy|$.

Exercise 2.30. Prove that every compact metric space is separable.

Exercise 2.31 (Baire's theorem)**.** A subset of a topological space is called *nowhere dense* if its closure has empty interior. Prove that a complete metric space cannot be covered by at most countably many nowhere dense subsets. Moreover, the complement of the union of at most countably many nowhere dense subsets is everywhere dense.

Exercise 2.32. Prove that a compact metric space X cannot be isometrically mapped to a subspace $Y \subset X$ such that $Y \neq X$. In other words, each isometric mapping $f\colon X \to X$ for a compact metric space X is surjective.

Exercise 2.33. Let X be a compact metric space and $f\colon X \to X$ be a mapping. Prove that

(1) if f is surjective and nonexpanding, then f is an isometry*;*

(2) if $\big|f(x)f(y)\big| \geq |xy|$ for all $x, y \in X$, then f is an isometry.