Chapter 1

Elements of general topology.

Schedule. Definition of topology and topological space, induced topology, subspace of topological space, discrete and anti-discrete topologies, metric spaces and metric topology, standard topology on Euclidean space, base of topology, cover of set and subset, Zariski topology, Sorgenfrey topology, subbase of topology, disjoint union of topological spaces, Cartesian product of topological spaces, Tychonoff or product topology, quotient topology and quotient space, Vietoris topology, continuous mapping, homeomorphism, embedding, convergence of sequences, closure, interior, boundary, dense subsets, separability, separated or Hausdorff topological space, connected and disconnected topological spaces, connected components, path-connected topological spaces, open cover, subcover, compact and sequentially compact topological spaces, bounded metric spaces, hyperspaces.

In this chapter we present an introduction to general topology.

1.1 Basic facts of general topology

For a set X, let 2^X denote the collection of all subsets of X. If $\mathcal{A} \subset 2^X$ is a family of subsets of X, then $\cup\mathcal{A}$ and $\cap\mathcal{A}$ denote the union and the intersection of the elements from $\mathcal{A}$, respectively. If $\mathcal{A}$ is an indexed family, i.e., $\mathcal{A} = \{A_i\}_{i\in I}$, then we use $\cup_{i\in I}A_i$ and $\cap_{i\in I}A_i$ for the union and the intersection. If different elements of $\mathcal{A}$ do not intersect each other (such family $\mathcal{A}$ is called *disjoint*), then we write $\sqcup\mathcal{A}$ instead of $\cup\mathcal{A}$, to emphasize that $\mathcal{A}$ is disjoint; similarly, we write $\sqcup_{i\in I}A_i$ instead of $\cup_{i\in I}A_i$ for indexed families. We can define $\sqcup_{i\in I}A_i$ also in the case when some differen A_i intersect each other, in particular, when they coincide. In this situation we simply consider A_i for different i as nonintersecting sets. This can be done in a formal way if we change the elements $a_i \in A_i$ to (a_i, i) and identify A_i with the set $\big\{(a_i, i)\big\}_{a_i\in A_i}$.

Definition 1.1. A set $\tau = \{U_\alpha\}_{\alpha\in A} \subset 2^X$ is called *a topology on* X if τ satisfies the following properties:

(1) $\emptyset$, $X \in \tau$;

(2) for any $\mathcal{A} \subset \tau$ we have $\cup\mathcal{A} \in \tau$ (the union of arbitrary collection of elements from τ belongs to τ);

(3) for any **finite** $\mathcal{A} \subset \tau$ it holds $\cap\mathcal{A} \in \tau$ (the intersection of arbitrary finite collection of elements from τ belongs to τ).

Definition 1.2. A set X with a given topology $\tau \subset 2^X$ is called *a topological space*. It is also convenient to denote the topological space X as the pair (X, τ). Also, speaking about the topological space X, we will often denote the topology defined on it by τ_X, without specifically mentioning it every time.

The elements of X are usually called *points*, and the elements of τ are called *open sets*. A set $F \subset X$ is called *closed* if its complement is open.

Problem 1.1. Show that the family of all closed subsets of a topological space X contains $\emptyset$ and X, and that the intersection of any collection of closed subsets, as well as the union of any finite collection of closed subsets are some closed sets.

Take an arbitrary $Y \subset X$ and consider the family $\tau_Y := \{U \cap Y : U \in \tau_X\}$.

Problem 1.2. Prove that τ_Y is a topology on Y.

Definition 1.3. The τ_Y is called *the topology on* Y *induced from* X. The Y with the topology τ_Y is usually called *a subspace* of the topological space X.

Generally speaking, there are many different topologies on each set X, and the inclusion relation generates a partial order on the set of all such topologies. The smallest topology in this order consists of two elements: $\tau_a = \{\emptyset, X\}$. It is called *anti-discrete*. The largest topology consists of all subsets: $\tau_d = 2^X$. It is called *discrete*. If $\mathcal{T}$ is a collection of topologies defined on the same set X, then $\cap\mathcal{T}$ is a topology as well; it is smaller than each topology $\tau \in \mathcal{T}$; for any topology τ' on X that is smaller than all topologies from $\mathcal{T}$ it holds $\tau' \subset \cap\mathcal{T}$.

The most important for us example of topology will be generated by metrics. Namely, suppose that for a set X a function $\rho\colon X \times X \to \mathbb{R}$ is given, which has the following properties:

(1) for any $x, y \in X$ it holds $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$ (positive definiteness);

(2) for any $x, y \in X$ we have $\rho(x, y) = \rho(y, x)$ (symmetry);

(3) for any $x, y, z \in X$ it holds $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (triangle inequality).

Then ρ is called *a metric*, and the set X with the metric ρ is called *a metric space*. It is also convenient to denote the metric space X as the pair (X, ρ). Each $Y \subset X$ endowed with the restriction of ρ to $Y \times Y$ is called *a subspace of* X.

Example 1.4. In calculus, the standard example of a metric space is the arithmetic space $\mathbb{R}^n$ with the Euclidean metric defined on it: for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ it holds $\rho(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. This metric is called *Euclidean*. We will also call *Euclidean* the arithmetic space itself, endowed with the Euclidean metric.

Let X be a metric space with a metric ρ. For every $x \in X$ and $r > 0$ we put

$$U_r(x) := \big\{y \in X : \rho(x, y) < r\big\}$$

and call *an open ball of radius* $r > 0$ *and center* x. Using the metric ρ, we construct the natural topology τ_ρ, called *the metric topology*: we assign the subset $U \subset X$ to open sets of the metric topology τ_ρ if and only if U is either empty, or U is a union of open balls. Equivalent definition: $U \in \tau_\rho$ if and only if for any point $x \in U$ there exists $r > 0$ such that $U_r(x) \subset U$ (together with each point the set U contains some open ball with the center at this point).

Problem 1.3. Prove that the family τ_ρ is a topology.

Remark 1.5. Unless otherwise stated, on the real line $\mathbb{R}$ and, more generally, on the arithmetic space $\mathbb{R}^n$, we consider the topologies generated by the Euclidean metric (see. Example 1.4). This topology is called *standard*.

1.2 Base and subbase

The construction of metric topology described above leads to the following important notion. Similar to linear algebra, where to describe a linear space it is enough to choose a family of vectors that, using linear combinations, generates the whole space, to define a topology, one can also select a subfamily of open sets and generate the topology by means of set-theoretic operations.

Definition 1.6. A family $\beta \subset \tau$ is called *a base of the topology* τ if every nonempty open set $U \in \tau$ is representable as a union of some elements from β.

Thus, by the definition of metric topology, its base is the family of all open balls.

Remark 1.7. Note that a given topology can have many different bases. For example, not all balls can be selected as the base of the metric space, but only, say, of radii not exceeding 1, or of only rational radii, or of only radii of the form $1/n$, etc. On the Euclidean line, for example, only rational numbers can be selected as centers.

We note two important properties of the topology base on the set X:

(1) each point $x \in X$ is contained in some element from the base (otherwise the set X cannot be obtained as the union of some elements from the base);

(2) a nonempty intersection of any two elements of the base is representable as the union of some elements from the base (otherwise this intersection will not belong to the topology).

It turns out that these two properties completely characterize the families that are the bases of some topologies. Before formulating the corresponding criterion, we introduce a definition of cover, which will be useful to us both here and hereinafter.

Definition 1.8. A family $\mathcal{A} \subset 2^X$ is called *a cover of the set* X if $X = \cup\mathcal{A}$. A family $\mathcal{A} \subset 2^X$ is called *a cover of* $Y \subset X$ if $Y \subset \cup\mathcal{A}$.

It is clear that each base of a topological space X is a cover of X.

Problem 1.4. Prove that a family $\beta \subset 2^X$ is a base of some topology τ on X if and only if β is a cover of X, and for any intersecting $B_1, B_2 \in \beta$ their intersection $B_1 \cap B_2$ is the union of some elements from β. Moreover, each family satisfying these properties, generates a unique topology.

Notice that a collection β of open sets in a topological space X which satisfies the conditions of Problem 1.4 may generate a topology τ different from τ_X. What do we need to add for to be sure that $\tau = \tau_X$? The answer can be obtained from the following more general result that is often used in proving the coincidence of topologies.

Problem 1.5. Let some topologies τ_1 and τ_2 with bases β_1 and β_2 be given on a set X. Then $\tau_1 = \tau_2$ if and only if for any $x \in X$ the following condition is fulfilled: for any $B_2 \in \beta_2$, $x \in B_2$, there is $B_1 \in \beta_1$ for which $x \in B_1 \subset B_2$, and vice versa, for any $B_1 \in \beta_1$, $x \in B_1$, there exists $B_2 \in \beta_2$ for which $x \in B_2 \subset B_1$. In particular, for a topological space X, a collection β of open sets satisfying the conditions of Problem 1.4 is a base of the topology τ_X if and only if for each open set $U \in \tau_X$ and any point $x \in U$ there exists some $B \in \beta$ such that $x \in B \subset U$.

Recall that two sets are called *equivalent* if there exists a bijection between them. The equivalence classes of sets are called *cardinalities* or *cardinal numbers*. The cardinality of a set X will be denoted by $\#X$.

Example 1.9. Let X be an infinite set of cardinality n, and m be an infinite cardinal number, with $m \le n$. Consider a family $\mathcal{F}$ of all $F \subset X$ such that $\#F < m$, and let $\beta_m = \{X \setminus F : F \in \mathcal{F}\}$. Then β_m is a base of some topology τ, which we call *the Zariski topology of the weight* m.

Problem 1.6. Prove that the family β_m from Example 1.9 is a base of some topology.

Example 1.10. As we noted above, on the Euclidean line we can take the family of all intervals as the base of the standard topology. Another interesting example of topology is obtained if, instead of intervals, we take all possible half-intervals of the form $[a, b)$. The corresponding topology is called *the arrow topology* or *the Sorgenfrey topology.*

Remark 1.11. The Sorgenfrey topology contains the standard topology of the line, since each interval (a, b) can be represented as a union of half-intervals $[a + 1/n, b)$, $n \in \mathbb{N}$.

Problem 1.7. Show that the collection of all possible half-intervals of the form $[a, b) \subset \mathbb{R}$ form a base of some topology that contains the standard topology.

If, to generate a topology, we allowed also to use finite intersections, then the generating family can, generally speaking, be reduced.

Definition 1.12. A family $\sigma \subset \tau$ is called *a subbase of the topology* τ if the set of all finite intersections of elements from σ forms a base of the topology τ.

It is clear that, like the base, each subbase of a topological space X is a cover of X.

Problem 1.8. Prove that a family $\sigma \subset 2^X$ is a subbase of some topology on X if and only if σ is a cover of X. Moreover, each cover of X generates a unique topology.

Example 1.13. The family of all subsets of the real line $\mathbb{R}$, each of which is an open ray, forms a subbase of the standard topology and is not a base of this topology.

1.3 Standard constructions of topologies

This section provides examples of standard constructions that allow to build new examples of topological spaces from existing ones.

Construction 1.1. Let σ be an arbitrary family of subsets of a set X, and $\mathcal{T}_\sigma$ be the family of all topologies on X containing σ. Then $\tau := \cap\mathcal{T}_\sigma$ is the smallest topology containing σ. If σ is a cover of X, then σ is a subbase of τ.

Construction 1.2 (Disjoint union)**.** Let $\mathcal{A} = \{X_i\}_{i \in I}$ be a family of topological spaces. We define a topology on $\sqcup\mathcal{A} = \sqcup_{i \in I} X_i$, setting its base to be equal to $\sqcup_{i \in I} \tau_{X_i}$. The set $\sqcup_{i \in I} X_i$ with the corresponding topology is called *the disjoint union of the topological spaces* X_i.

Construction 1.3 (Cartesian product). Let $\mathcal{A} = \{X_i\}_{i\in I}$ be an arbitrary family of topological spaces. The set of all mappings $w\colon I \to \sqcup_{i\in I} X_i$ such that $w(i) \in X_i$ for every $i \in I$ is called *the Cartesian product* of the spaces X_i and is denoted by $W := \prod \mathcal{A} = \prod_{i\in I} X_i$. In particular, if all X_i are equal to the same space X, then $W = X^I$, where the latter, recall, denotes the set of all mappings from I to X. For convenience, we will often write w_i instead of $w(i)$, and we will call this value *the i-th coordinate of the point* $w \in \prod_{i\in I} X_i$.

We define a topology on W, choosing as its subbase the family of all products $\prod_{i\in I} U_i$, $U_i \in \tau_{X_i}$, in which only one U_i can differ from X_i. The corresponding base consists of $\prod_{i\in I} U_i$, $U_i \in \tau_{X_i}$, in which only a finite number of U_i can differ from X_i. This topology is called *the product topology* or *the Tychonoff topology*.

In the case when I is a finite set, say, $I = \{1, \dots, n\}$, then the Cartesian product of the spaces X_i is denoted by $X_1 \times \dots \times X_n$. In particular, in this way one can define the standard topology on the n-dimensional arithmetic space $\mathbb{R}^n$.

Problem 1.9. Show that the standard topology of the Euclidean space $\mathbb{R}^n$ coincides with the topology of the Cartesian product $\mathbb{R} \times \dots \times \mathbb{R}$ of real lines endowed with the standard topology.

Example 1.14. *Sorgenfrey space* is the Cartesian product of the Sorgenfrey lines from Example 1.10 and is used in general topology to illustrate numerous exotic possibilities.

Construction 1.4 (Quotient topology). Let X be a topological space, and ν be some equivalence relation on X. Denote by X/ν the set of classes of this equivalence. For each $x \in X$ denote by $[x] \in X/\nu$ the ν-equivalence class containing x, and let $\pi\colon X \to X/\nu$, $\pi\colon x \mapsto [x]$, be the canonical projection. Then the family of all $U \subset X/\nu$ such that $\pi^{-1}(U) \in \tau_X$ forms a topology called *the quotient topology*. The set X/ν endowed with the quotient topology is called *the quotient space*.

Construction 1.5 (Vietoris topology). Let X be an arbitrary topological space. For each finite collection of open sets $U_1, \dots, U_n \in \tau_X$ we put

$$\langle U_1, \dots, U_n\rangle = \{Y \subset X : Y \subset \cup_{i=1}^n U_i \text{ and } Y \cap U_i \neq \emptyset \text{ for all } i = 1, \dots, n\}.$$

Note that if at least one of U_i is empty, then $\langle U_1, \dots, U_n\rangle = \emptyset$.

Problem 1.10. Show that the families

$$\sigma = \big\{\langle U\rangle : U \in \tau_X\big\} \cup \big\{\langle X, U\rangle : U \in \tau_X\big\} \quad\text{and}\quad \beta = \big\{\langle U_1, \dots, U_n\rangle : U_1, \dots, U_n \in \tau_X\big\}$$

form respectively a subbase and the corresponding base of some topology on 2^X.

The topology on 2^X defined in Problem 1.10 is called *the Vietoris topology*.

Remark 1.15. Usually, Vietoris topology is defined on the family of all nonempty closed subsets of a topological space.

Problem 1.11. Prove that each construction of these section provides a topology.

1.4 Continuous mappings

As a rule, all considered mappings between topological spaces are consistent with topologies. Such mappings are called continuous. We give three equivalent definitions of continuity.

Definition 1.16. *A neighborhood of a point* $x \in X$ of a topological space X is an arbitrary open set $U \subset X$ containing x. *A neighborhood of a subset* Z of a topological space X is an arbitrary open set $U \subset X$ containing Z.

Remark 1.17. For convenience, we denote an arbitrary neighborhood of a point $x \in X$ by U^x.

Definition 1.18. A mapping $f\colon X \to Y$ between topological spaces *is continuous at* $x \in X$ if for any neighborhood $U^{f(x)}$ there exists a neighborhood U^x such that $f(U^x) \subset U^{f(x)}$. The mapping f, continuous at all points $x \in X$, is called *continuous*.

Definition 1.19. A mapping $f\colon X \to Y$ between topological spaces *is continuous* if for any open set $U \subset Y$ its preimage $f^{-1}(U) \subset X$ is open (the preimage of each open set is open).

Definition 1.20. A mapping $f\colon X \to Y$ between topological spaces *is continuous* if for any closed set $F \subset Y$ its preimage $f^{-1}(F) \subset X$ is closed (the preimage of each closed set is closed).

Problem 1.12. Prove that the definitions 1.18, 1.19, and 1.20 are equivalent.

Problem 1.13. Let $f\colon X \to Y$ be a mapping of topological spaces and σ a subbase of the topology on the space Y. Prove that f is continuous if and only if f-preimage of each element from the subbase σ is open in X.

Remark 1.21. When we speak of the continuity of a function $f\colon X \to \mathbb{R}$ or, more generally, of a vector-valued mapping $f\colon X \to \mathbb{R}^n$ from a topological space X, then, unless otherwise stated, on $\mathbb{R}$ and $\mathbb{R}^n$ we consider the standard topologies (see. Remark 1.5).

Problem 1.14. Show that the identity mapping and the composition of continuous mappings are continuous.

Problem 1.15. Let X be a topological space, and $Z \subset X$ be its subspace. Show that the inclusion mapping $i\colon Z \to X$, $i(z) = z$ for each point $z \in Z$, is continuous.

Problem 1.16. Let X, Y be topological spaces, $W \subset Y$ be a subspace of Y, and $f\colon X \to W$ be a continuous mapping. Let $g\colon X \to Y$ be a mapping coinciding with f: for each $x \in X$ it holds $f(x) = g(x)$. Prove that the mapping g is continuous.

Let $f\colon X \to Y$ be an arbitrary mapping of sets. Choose arbitrary subsets $Z \subset X$ and $W \subset Y$ such that $f(Z) \subset W$. Then *the restriction* $f|_{Z,W}$ *of the mapping* f *to* Z *and* W is the mapping $g\colon Z \to W$ that coincides on the domain with the mapping f, i.e., for any $x \in Z$ it holds $f(x) = g(x)$.

Problem 1.17. Let $f\colon X \to Y$ be a continuous mapping of topological spaces, $Z \subset X$, $W \subset Y$, $f(Z) \subset W$. Then the restriction $f|_{Z,W}\colon Z \to W$ is also continuous as the mapping of the topological spaces Z and W with the topologies induced on them from X and Y, respectively.

Problem 1.18. Let $\{X_i\}_{i\in I}$ be a cover of a topological space X by open subsets X_i, and $f\colon X \to Y$ a mapping to a topological space Y. Show that f is continuous if and only if all the restrictions $f|_{X_i}$ are continuous. In particular, this holds when $X = \sqcup_{i\in I} X_i$ is the disjoint union of some topological spaces. Will this result remain true if we replace $\{X_i\}$ with a cover of X by arbitrary sets?

Problem 1.19. Let $X = \sqcup_{i\in I} X_i$ be the disjoint union of some topological spaces and $f\colon X \to Y$ be a map into a topological space Y. Show that f is continuous if and only if all its restrictions $f|_{X_i}$ are continuous.

Problem 1.20. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$. We define the canonical projection $\pi_i\colon X \to X_i$ by setting $\pi_i(x) = x_i$. Prove that the product topology on X is the smallest of those topologies in which all the projections π_i are continuous.

Problem 1.21. Let $\{Y_i\}_{i\in I}$ be a family of topological spaces, $Y = \prod_{i\in I} Y_i$, and $f_i\colon X \to Y_i$ be mappings from some topological space X. We construct the mapping $F := \prod_{i\in I} f_i\colon X \to Y$ by associating with each point $x \in X$ the element $y \in Y$ defined as follows: $y_i = f_i(x)$. Prove that the mapping F is continuous if and only if all f_i are continuous.

Problem 1.22. Let $A \subset \mathbb{R}^n$ be an arbitrary subset, $(x^1, \dots, x^n)$ the Cartesian coordinates on $\mathbb{R}^n$, $f\colon A \to \mathbb{R}^m$ a continuous mapping, $(y^1, \dots, y^m)$ the Cartesian coordinates on $\mathbb{R}^m$, and $y^i = y^i(x^1, \dots, x^n)$ the coordinate functions of the mapping f. Prove that the mapping f is continuous if and only if all the coordinate functions $y^i = y^i(x^1, \dots, x^n)$ are continuous.

Problem 1.23. Describe all continuous functions on a topological space with Zariski topology.

1.5 Homeomorphisms, embeddings

An important particular case of continuous mapping is a homeomorphism.

Definition 1.22. A mapping $f\colon X \to Y$ of topological spaces is called *a homeomorphism* if it is bijective, and both the maps f and f^{-1} are continuous. Topological spaces between which there is a homeomorphism are called *homeomorphic*.

Remark 1.23. A homeomorphism, being a bijection, identifies not only points of spaces, but also identifies topologies, establishing a one-to-one correspondence between them. For clarity, we can imagine that the homeomorphism $f : X \to Y$ is a replacement for the names of points in the space X: the point $x \in X$ is "renamed" to $f(x)$, without changing the topology. From these considerations it follows that all topological properties of homeomorphic spaces are the same.

The next exercise follows directly from Problem 1.17.

Problem 1.24. Let $f\colon X \to Y$ be a homeomorphism, and $Z \subset X$, $W = f(Z)$. Prove that the restriction $f|_{Z,W}\colon Z \to W$ is also a homeomorphism.

An injective mapping $f\colon X \to Y$ of topological spaces is called *an embedding of X into Y* if the restriction $f|_{X,f(X)}$ is a homeomorphism.

Problem 1.25. Show that every embedding is continuous. Give an example of a continuous injective mapping of topological spaces that is not an embedding.

1.6 Convergence of sequences

A sequence in a set X is an arbitrary mapping $x\colon \mathbb{N} \to X$ from the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. For convenience, the points $x(n)$ are usually denoted by x_n. Also, for brevity, it is customary to say that *a sequence of points x_n* is given.

Definition 1.24. A sequence of points x_n in a topological space X is called *convergent* if, for some $x \in X$, called *a limit of this sequence*, the following holds: for any neighborhood U^x there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $x_n \in U^x$. If the sequence is not convergent, then it is called *divergent*.

Problem 1.26. Let ω be a character not contained in $\mathbb{N}$. We define a topology on the set $\bar{N} = \mathbb{N} \cup \{\omega\}$, taking as a base all points from $\mathbb{N}$, as well as all sets of the form $\{n \geq N\} \cup \{\omega\}$, $N \in \mathbb{N}$. Show that a sequence $x\colon \mathbb{N} \to X$ converges if and only if the mapping x can be extended to a continuous mapping on $\bar{\mathbb{N}}$.

Problem 1.27. Show that a continuous mapping $f\colon X \to Y$ of topological spaces takes convergent sequences to convergent ones. Show that if X is a metric space, then every mapping $g\colon X \to Y$ that takes convergent sequences into convergent ones is continuous. Give an example of a topological space X and a mapping $h\colon X \to Y$ into a topological space Y, which takes convergent sequences into convergent ones, but is not continuous nonetheless.

Problem 1.28. Let $x_1, x_2, \ldots$ be a sequence of points in a metric space X. Suppose that for some point $x \in X$ each neighborhood of x intersects the set $\{x_i\}_{i=1}^{\infty} \setminus \{x\}$. Prove that the sequence $x_1, x_2, \ldots$ contains a convergent subsequence. Prove that if a sequence of points in a metric space does not contain any convergent subsequence, then for each $x \in X$ there exists $r > 0$ such that the open ball $U_r(x)$ does not contain points of this sequence other than x.

1.7 Closure, interior, boundary, dense subsets, separability

Let Y be a subset of a topological space X. A point $x \in X$ is called *an adherent point*, or *a closure point*, or *a contact point for Y* if every neighborhood of x intersects Y. The set of all adherent points of the set Y is called its *closure* and is denoted by $\overline{Y}$.

Problem 1.29. Prove that the closure of a set $Y \subset X$ is the smallest closed subset of X containing Y, i.e., $\overline{Y}$ is the intersection of all closed sets containing Y.

A subset of Y of a topological space X is called *everywhere dense in X* if $\overline{Y} = X$.

Example 1.25. The set of all rational numbers, like the set of all irrational numbers, are everywhere dense in the real line.

Problem 1.30. Let the topology of Zariski of weight m be given on an infinite set X. Then a subset $Y \subset X$ is everywhere dense in X if and only if $\#Y \geq m$.

A topological space is called *separable* if it contains an everywhere dense sequence.

Example 1.26. Each finite space is separable. The Euclidean space $\mathbb{R}^n$ is also separable: as an everywhere dense sequence we can take arbitrary numbered set of all points with rational coordinates. Each space with a countable Zariski topology is separable. Sorgenfrey space (Example 1.14) is separable.

Problem 1.31. Show that in metric space, separability is equivalent to having a countable base. Extract from this that every subset of a separable metric space is separable. Show that an open subset of an arbitrary separable topological space is separable. Give an example of a separable topological space containing an non-separable subset (use the Sorgenfrey plane).

A point x from a subset Y of a topological space X is called *interior for* Y if some neighborhood of x is contained in Y. The family of all interior points of the set Y is called its *interior* and is denoted by $\operatorname{Int} Y$.

Problem 1.32. Show that the interior $\operatorname{Int} Y$ is the largest open subset of X contained in Y.

Problem 1.33. Prove that a subset Y of the topological space X is closed if and only if $Y = \overline{Y}$, and is open if and only if $Y = \operatorname{Int} Y$.

A point $x \in X$ of a topological space X is called *a boundary point for a subset* $Y \subset X$ if each neighborhood of x intersects both Y and its complement $X \setminus Y$. The set of all boundary points of the set Y is called its *boundary* and is denoted by ∂Y.

Problem 1.34. Prove that the boundary ∂Y is a closed subset of X, and

$$\partial Y = \overline{Y} \setminus \operatorname{Int} Y = \overline{Y} \cap \overline{X \setminus Y}.$$

1.8 Separated spaces

There are a number of separation axioms that generate various classes of topological spaces. We will not dwell on this in detail here, but formulate only one axiom of separation, which will be useful to us in the future.

A topology on a set X, as well as the topological space X itself, is called *Hausdorff* or *separated* if any two points of X have disjoint neighborhoods.

Example 1.27. Each discrete topology is Hausdorff. Each metric space is Hausdorff. If the set X consists of more than one point, then the anti-discrete topology is not Hausdorff. Also, the Zariski topology is not a Hausdorff topology (see Example 1.9).

Problem 1.35. Show that in a Hausdorff topological space every point is closed. Give an example of a non-Hausdorff topological space in which all points are closed.

Problem 1.36. Show that the disjoint union and the Cartesian product of Hausdorff topological spaces are also Hausdorff.

Problem 1.37. Show that in a Hausdorff topological space the limit of a convergent sequence is uniquely defined. Give an example of a topological space in which each sequence converges to each point.

Problem 1.38. Describe what sequences in a space with Zariski topology are convergent, and what limits each convergent sequence has.

1.9 Connected spaces

We say that a set X *is partitioned into subsets* $\{X_i\}_{i\in I}$ if $X = \sqcup_{i\in I} X_i$.

A topological space (its topology) is called *disconnected* if it can be partitioned into two nonempty open (equivalently, closed) sets. If such a partition does not exist, then the topological space is called *connected*. A subset of a topological space is *connected* (*disconnected*), if such is the topology induced on it. In other words, a subset Y of a topological space X is *disconnected* if there exist $U, V \in \tau_X$ such that $Y \subset U \cup V$, and both intersections $Y \cap U$ and $Y \cap V$ are nonempty and do not intersect each other.

Problem 1.39. Prove that each segment $[a, b] \subset \mathbb{R}$ is connected.

Problem 1.40. Prove that the closure of a connected subset of a topological space is connected.

Problem 1.41. Let $\{A_i\}_{i\in I}$ be a family of connected pairwise intersecting subsets of a topological space X, then the set $\cup_{i\in I} A_i$ is connected.

Problem 1.42. Show that the image of a connected topological space under a continuous mapping is also connected.

Problem 1.43. Prove that every continuous function on a connected topological space takes all intermediate values.

The maximum (by inclusion) connected subset of a topological space is called a *connected component* of this space.

Problem 1.44. Show that each connected component is closed, and that each topological space is uniquely partitioned into its connected components. If such a partition is finite, then connected components are also open. Give an example of a topological space in which some connected components are not open.

1.10 Path-connected spaces

A curve in a topological space X is an arbitrary continuous map $\gamma\colon [a,b] \to X$. It is said that the curve γ *joins the points* $\gamma(a)$ *and* $\gamma(b)$.

A topological space X is called *path-connected* if any two of its points can be connected by a curve.

Problem 1.45. Prove that a path-connected topological space is connected. Give an example of a connected space that is not path-connected.

1.11 Compact and sequentially compact spaces

A subcover of a cover is a subfamily of a cover, which itself is a cover. A cover of a topological space composed of open sets is called *open*.

Definition 1.28. A topological space X is called *compact* if a finite subcover can be found in any of its open covers.

Remark 1.29. To define a cover for a subset Y of a topological space X, it is more convenient to modify Definition 1.28, rather than reduce it to the corresponding concept for the induced topology. Namely, an open cover of Y is a family $\{U_a\}_{a\in A}$ of open subsets of X such that $Y \subset \cup_{a\in A} U_a$. Other definitions do not change.

Problem 1.46. Show that a finite union of compact subsets of a topological space is compact.

Problem 1.47. Prove the following statements:

(1) the image under a continuous mapping from a compact topological space is compact;

(2) a closed subset of a compact topological space is compact;

(3) a compact subset of a Hausdorff topological space is closed;

(4) a continuous bijective mapping from a compact topological space to a Hausdorff space is a homeomorphism;

(5) give an example of an infinite topological space in which all subsets are compact. Note that in such a space there are compact subsets that are not closed;

(6) give an example of a continuous bijective mapping of topological spaces that is not a homeomorphism.

Problem 1.48 (Alexander subbase theorem)**.** Let X be a topological space and σ its subbase. Prove that X is compact if and only if each cover of X by elements of the subbase σ has a finite subcover.

Problem 1.49 (Tychonoff's theorem)**.** Prove that the Cartesian product $\prod_{i\in I} X_i$ of topological spaces X_i, endowed with Tychonoff topology, is compact if and only if all X_i are compact.

Problem 1.50. Prove that each segment $[a,b] \subset \mathbb{R}$ is compact.

A subset of a metric space is called *bounded* if it is contained in some ball.

Problem 1.51. Prove that a subset of a Euclidean space is compact if and only if it is closed and bounded.

Problem 1.52. Prove that every compact metric space is bounded. Prove that a continuous function on a compact topological space is bounded and takes its largest and smallest values.

Definition 1.30. A topological space is called *sequentially compact* if every sequence of its points has a convergent subsequence.

Problem 1.53. Prove that every sequentially compact metric space is bounded. Prove that a continuous function on a sequentially compact topological space is bounded and takes its largest and smallest values.

Remark 1.31. Note that compactness and sequential compactness in the case of general topological spaces are not related to each other: there are spaces that have one of these properties and do not have the other. Since the examples are quite complicated, we will omit them here. However, everything is much simpler for metric spaces, see Theorem 2.20 in Chapter 2.

1.12 Hyperspaces

A family of various subsets of a topological or metric space endowed with a certain topology or metric is called *a hyperspace*. In Construction 1.5 above, we defined the Vietoris topology on the set 2^X of all subsets of a topological space X. Thus, 2^X is a special case of hyperspace.

Here are a few more examples of hyperspaces (in all these spaces, the topology is induced from 2^X in the standard way):

- by $\mathcal{P}_0(X) \subset 2^X$ we denote the set of all nonempty subsets of X;
- by $\mathcal{CL}(X) \subset \mathcal{P}_0(X)$ we denote the set of all nonempty closed subsets of X;
- by $\mathcal{C}(X) \subset \mathcal{CL}(X)$ the set of all nonempty closed connected subsets of X;
- by $\mathcal{C}_n(X) \subset \mathcal{CL}(X)$ the set of all nonempty closed subsets of X having at most n components;
- by $\mathcal{C}_\infty(X) \subset \mathcal{CL}(X)$ the set of all nonempty closed subsets of X, each of which has finitely many components;
- by $\mathcal{K}(X) \subset \mathcal{P}_0(X)$ the set of all nonempty compact subsets of X;
- by $\mathcal{F}_n(X) \subset \mathcal{K}(X)$ the set of all nonempty at most n-point subsets of X;
- by $\mathcal{F}_\infty(X) \subset \mathcal{K}(X)$ the set of all nonempty finite subsets of X.

There are numerous, usually obvious, connections between these spaces. For example, if the space X is Hausdorff, then $\mathcal{K}(X) \subset \mathcal{CL}(X)$; if X is Hausdorff and compact, then $\mathcal{K}(X) = \mathcal{CL}(X)$.

A connected nonempty compact Hausdorff topological space is called *a continuum*. If the space X is Hausdorff, then $\mathcal{K}(X) \cap \mathcal{C}(X)$ is the set of all continua. In some literature, for example, in [2], this space is denoted by $\mathcal{C}(X)$.

Problem 1.54. Let $X = \{a, b\}$. We define the following topology on X as follows: $\tau = \big\{\emptyset, X, \{a\}\big\}$. Find out what the space $\mathrm{CL}(X)$ is.

Definition 1.32. A topological space is called *a space of class T_0* if, for any two different points of this space, at least one of them has a neighborhood that does not contain the second point.

Problem 1.55. Prove that the space $\mathrm{CL}(X)$ is always a space of class T_0.

Problem 1.56. A topological space is called *a space of class T_1* if, for any two different points of this space, each of them has a neighborhood that does not contain the remaining point.

Problem 1.57. Prove that if X is a space of class T_1, then $\mathrm{CL}(X)$ is also a space of class T_1. Give an example that demonstrates that the converse is not true.

Problem 1.58. Prove that the space $\mathcal{P}_0(X)$ belongs to the class T_1 if and only if the space X is discrete.

References to Chapter 1

[1] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[2] S. Nadler, *Hyperspaces of Sets*, 1978.

Exercises to Chapter 1

Exercise 1.1. Show that the family of all closed subsets of a topological space X contains $\emptyset$ and X, and that the intersection of any collection of closed subsets, as well as the union of any finite collection of closed subsets are some closed sets.

Exercise 1.2. Let X be a topological space and $Y \subset X$. Consider the family $\tau_Y := \{U \cap Y : U \in \tau_X\}$. Prove that τ_Y is a topology on Y.

Exercise 1.3. For a metric space (X, ρ) define $\tau_\rho \subset 2^X$ as the collection consisting of the empty set and all possible unions of open balls. Prove that the family τ_ρ is a topology.

Exercise 1.4. Prove that a family $\beta \subset 2^X$ is a base of some topology τ on X if and only if β is a cover of X, and for any intersecting $B_1, B_2 \in \beta$ their intersection $B_1 \cap B_2$ is the union of some elements from β. Moreover, each family satisfying these properties, generates a unique topology.

Exercise 1.5. Let some topologies τ_1 and τ_2 with bases β_1 and β_2 be given on a set X. Then $\tau_1 = \tau_2$ if and only iff for any $x \in X$ the following condition is fulfilled: for any $B_2 \in \beta_2$, $x \in B_2$, there is $B_1 \in \beta_1$ for which $x \in B_1 \subset B_2$, and vice versa, for any $B_1 \in \beta_1$, $x \in B_1$, there exists $B_2 \in \beta_2$ for which $x \in B_2 \subset B_1$. In particular, for a topological space X, a collection β of open sets satisfying the condition of Exercise 1.4 is a base of the topology τ_X if and only if for each open set $U \in \tau_X$ and any point $x \in U$ there exists some $B \in \beta$ such that $x \in B \subset U$.

Exercise 1.6. Let X be an infinite set of cardinality n, and m be an infinite cardinal number, with $m \le n$. Consider a family $\mathcal{F}$ of all $F \subset X$ such that $\#F < m$, and let $\beta_m = \{X \setminus F : F \in \mathcal{F}\}$. Prove that the family β_m is a base of some topology.

Exercise 1.7. Show that the collection of all possible half-intervals of the form $[a, b) \subset \mathbb{R}$ form a base of some topology that contains the standard topology.

Exercise 1.8. Prove that a family $\sigma \subset 2^X$ is a subbase of some topology on X if and only if σ is a cover of X. Moreover, each cover of X generates a unique topology.

Exercise 1.9. Show that the standard topology of the Euclidean space $\mathbb{R}^n$ coincides with the topology of the Cartesian product $\mathbb{R} \times \cdots \times \mathbb{R}$ of real lines endowed with the standard topology.

Exercise 1.10. Let X be an arbitrary topological space. For each finite collection $U_1, \ldots, U_n \in \tau_X$ we put

$$\langle U_1, \ldots, U_n \rangle = \{Y \subset X : Y \subset \cup_{i=1}^n U_i, \text{ and } Y \cap U_i \neq \emptyset \text{ for all } i = 1, \ldots, n\}.$$

Show that the families

$$\sigma = \big\{\langle U \rangle : U \in \tau_X\big\} \cup \big\{\langle X, U \rangle : U \in \tau_X\big\} \quad \text{and} \quad \beta = \big\{\langle U_1, \ldots, U_n \rangle : U_1, \ldots, U_n \in \tau_X\big\}$$

form respectively a subbase and the corresponding base of some topology on 2^X.

Exercise 1.11. Prove that each construction from the section "Standard constructions of topologies" provides a topology.

Exercise 1.12. Prove that the definitions 1.18, 1.19, and 1.20 are equivalent.

Exercise 1.13. Let $f \colon X \to Y$ be a mapping of topological spaces and σ be a subbase of the topology on the space Y. Prove that f is continuous if and only if f-preimage of each element from the subbase σ is open in X.

Exercise 1.14. Show that the identity mapping and the composition of continuous mappings are continuous.

Exercise 1.15. Let X be a topological space, and $Z \subset X$ be its subspace. Show that the inclusion mapping $i \colon Z \to X$, $i(z) = z$ for each point $z \in Z$, is continuous.

Exercise 1.16. Let X, Y be topological spaces, $W \subset Y$ be a subspace of Y, and $f \colon X \to W$ be a continuous mapping. Let $g \colon X \to Y$ be a mapping coinciding with f: for each $x \in X$ it holds $f(x) = g(x)$. Prove that the mapping g is continuous.

Exercise 1.17. Let $f\colon X \to Y$ be a continuous mapping of topological spaces, $Z \subset X$, $W \subset Y$, $f(Z) \subset W$. Then the restriction $f|_{Z,W}\colon Z \to W$ is also continuous as the mapping of the topological spaces Z and W with the topologies induced on them from X and Y, respectively.

Exercise 1.18. Let $\{X_i\}_{i\in I}$ be a cover of a topological space X by open subsets X_i, and $f\colon X \to Y$ a mapping to a topological space Y. Show that f is continuous if and only if all the restrictions $f|_{X_i}$ are continuous. In particular, this holds when $X = \sqcup_{i\in I} X_i$ is the disjoint union of some topological spaces. Will this result remain true if we replace $\{X_i\}$ with a cover of X by arbitrary sets?

Exercise 1.19. Let $X = \sqcup_{i\in I} X_i$ be the disjoint union of some topological spaces and $f\colon X \to Y$ be a map into the topological space Y. Show that f is continuous if and only if all its restrictions $f|_{X_i}$ are continuous.

Exercise 1.20. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$. We define the canonical projection $\pi_i\colon X \to X_i$ by setting $\pi_i(x) = x_i$. Prove that the product topology on X is the smallest of those topologies in which all the projections π_i are continuous.

Exercise 1.21. Let $\{Y_i\}_{i\in I}$ be a family of topological spaces, $Y = \prod_{i\in I} Y_i$, and $f_i\colon X \to Y_i$ are mappings from some topological space X. We construct the mapping $F := \prod_{i\in I} f_i\colon X \to Y$ by associating with each point $x \in X$ the element $y \in Y$ defined as follows: $y_i = f_i(x)$. Prove that the mapping F is continuous if and only if all f_i are continuous.

Exercise 1.22. Let $A \subset \mathbb{R}^n$ be an arbitrary subset, $(x^1, \dots, x^n)$ the Cartesian coordinates on $\mathbb{R}^n$, $f\colon A \to \mathbb{R}^m$ a continuous mapping, $(y^1, \dots, y^m)$ the Cartesian coordinates on $\mathbb{R}^m$, and $y^i = y^i(x^1, \dots, x^n)$ the coordinate functions of the mapping f. Prove that the mapping f is continuous if and only if all the coordinate functions $y^i = y^i(x^1, \dots, x^n)$ are continuous.

Exercise 1.23. Describe all continuous functions on a topological space with Zariski topology.

Exercise 1.24. Let $f\colon X \to Y$ be a homeomorphism, and $Z \subset X$, $W = f(Z)$. Prove that the restriction $f|_{Z,W}\colon Z \to W$ is also a homeomorphism. Show that the letters b, c, f, g, i, h, o are pairwise non-homeomorphic.

Exercise 1.25. Show that every embedding is continuous. Give an example of a continuous injective mapping of topological spaces that is not an embedding.

Exercise 1.26. Let ω be a character not contained in $\mathbb{N}$. We define a topology on the set $\bar{N} = \mathbb{N} \cup \{\omega\}$, taking as a base all points from $\mathbb{N}$, as well as all sets of the form $\{n \geq N\} \cup \{\omega\}$, $N \in \mathbb{N}$. Show that a sequence $x\colon \mathbb{N} \to X$ converges if and only if the mapping x can be extended to a continuous mapping on $\bar{\mathbb{N}}$.

Exercise 1.27. Show that a continuous mapping $f\colon X \to Y$ of topological spaces takes convergent sequences to convergent ones. Show that if X is a metric space, then every mapping $g\colon X \to Y$ that takes convergent sequences into convergent ones is continuous. Give an example of a topological space X and a mapping $h\colon X \to Y$ into a topological space Y, which takes convergent sequences into convergent ones, but is not continuous nonetheless.

Exercise 1.28. Let $x_1, x_2, \dots$ be a sequence of points in a metric space X. Suppose that for some point $x \in X$ each neighborhood of x intersects the set $\{x_i\}_{i=1}^{\infty} \setminus \{x\}$. Prove that the sequence $x_1, x_2, \dots$ contains a convergent subsequence. Extract from this that if a sequence of points in a metric space does not contain any convergent subsequence, then for each $x \in X$ there exists $r > 0$ such that the open ball $U_r(x)$ does not contain points of this sequence other than x.

Exercise 1.29. Prove that the closure of a set $Y \subset X$ is the smallest closed subset of X containing Y, i.e., $\overline{Y}$ is the intersection of all closed sets containing Y.

Exercise 1.30. Let the topology of Zariski of weight m be given on an infinite set X. Then a subset $Y \subset X$ is everywhere dense in X if and only if $\#Y \geq m$.

Exercise 1.31. Show that in metric space, separability is equivalent to having a countable base. Extract from this that every subset of a separable metric space is separable. Show that an open subset of an arbitrary separable topological space is separable. Give an example of a separable topological space containing an non-separable subset (use the Sorgenfrey plane).

Exercise 1.32. Show that the interior $\operatorname{Int} Y$ is the largest open subset of X contained in Y.

Exercise 1.33. Prove that a subset Y of the topological space X is closed if and only if $Y = \overline{Y}$, and is open if and only if $Y = \operatorname{Int} Y$.

Exercise 1.34. Prove that the boundary ∂Y is a closed subset of X, and

$$\partial Y = \overline{Y} \setminus \operatorname{Int} Y = \overline{Y} \cap \overline{X \setminus Y}.$$

Exercise 1.35. Show that in a Hausdorff topological space every point is closed. Give an example of a non-Hausdorff topological space in which all points are closed.

Exercise 1.36. Show that the disjoint union and the Cartesian product of Hausdorff topological spaces are also Hausdorff.

Exercise 1.37. Show that in a Hausdorff topological space the limit of a convergent sequence is uniquely defined. Give an example of topological space in which each sequence converges to each point.

Exercise 1.38. Describe what sequences in a space with Zariski topology are convergent, and what limits each convergent sequence has.

Exercise 1.39. Prove that each segment $[a, b] \subset \mathbb{R}$ is connected.

Exercise 1.40. Prove that the closure of a connected subset of a topological space is connected.

Exercise 1.41. Let $\{A_i\}_{i \in I}$ be a family of connected pairwise intersecting subsets of a topological space X, then the set $\cup_{i \in I} A_i$ is connected.

Exercise 1.42. Show that the image of a connected topological space under a continuous mapping is also connected.

Exercise 1.43. Prove that every continuous function on a connected topological space takes all intermediate values.

Exercise 1.44. Show that each connected component is closed, and that each topological space is uniquely partitioned into its connected components. If such a partition is finite, then connected components are also open. Give an example of a topological space in which some connected components are not open.

Exercise 1.45. Prove that a path-connected topological space is connected. Give an example of a connected space that is not path-connected.

Exercise 1.46. Show that a finite union of compact subsets of a topological space is compact.

Exercise 1.47. Prove the following statements:

(1) the image under a continuous mapping from a compact topological space is compact;

(2) a closed subset of a compact topological space is compact;

(3) a compact subset of a Hausdorff topological space is closed;

(4) a continuous bijective mapping from a compact topological space to a Hausdorff space is a homeomorphism;

(5) give an example of an infinite topological space in which all subsets are compact. Note that in such a space there are compact subsets that are not closed;

(6) give an example of a continuous bijective mapping of topological spaces that is not a homeomorphism.

Exercise 1.48 (Alexander subbase theorem)**.** Let X be a topological space and σ its subbase. Prove that X is compact if and only if each cover of X by elements of the subbase σ has a finite subcover.

Exercise 1.49 (Tychonoff's theorem)**.** Prove that the Cartesian product $\prod_{i \in I} X_i$ of topological spaces X_i, endowed with Tychonoff topology, is compact if and only if all X_i are compact.

Exercise 1.50. Prove that each segment $[a, b] \subset \mathbb{R}$ is compact.

Exercise 1.51. Prove that a subset of a Euclidean space is compact if and only if it is closed and bounded.

Exercise 1.52. Prove that every compact metric space is bounded. Prove that a continuous function on a compact topological space is bounded and takes its largest and smallest values.

Exercise 1.53. Prove that every sequentially compact metric space is bounded. Prove that a continuous function on a sequentially compact topological space is bounded and takes its largest and smallest values.

Notation. The following matrix groups consist of real matrices of size $n \times n$ and are considered as subsets of $\mathbb{R}^{n^2}$ with the induced topology (their rows or columns are written out one after another and form vectors): $\mathrm{O}(n)$ consists of all orthogonal matrices (*orthogonal group*); $\mathrm{SO}(n)$ consists of all orthogonal matrices with determinant 1 (*special orthogonal group*); $\mathrm{GL}(n)$ consists of all nondegenerate matrices (*general linear group*); $\mathrm{SL}(n)$ consists of all matrices with determinant 1 (*special linear group*).

Exercise 1.54. Find out which of the following matrix groups are connected, which are compact:

$$\mathrm{O}(n),\ \mathrm{SO}(n),\ \mathrm{GL}(n),\ \mathrm{SL}(n).$$

Exercise 1.55. Let $X = \{a, b\}$. We define the following topology on X: $\tau = \big\{\emptyset, X, \{a\}\big\}$. Find out what the space $\mathrm{CL}(X)$ is.

Definition 1.33. A topological space is called *a space of class* T_0 if, for any two different points of this space, at least one of them has a neighborhood that does not contain the second point.

Exercise 1.56. Prove that the space $\mathrm{CL}(X)$ is always a space of class T_0.

Definition 1.34. A topological space is called *a space of class* T_1 if, for any two different points of this space, each of them has a neighborhood that does not contain the remaining point.

Exercise 1.57. Prove that if X is a space of class T_1, then $\mathrm{CL}(X)$ is also a space of class T_1. Give an example that demonstrates that the converse is not true.

Exercise 1.58. Prove that the space $\mathcal{P}_0(X)$ belongs to the class T_1 if and only if the space X is discrete.