**Mathematical olympiad 15.05.2010 for students**

**Faculty of Mechanics and Mathematics, Moscow State University**

**1.** Let $f\colon [0,1] \to [0,1]$ and $g\colon [0,1] \to [0,1]$ be continuous functions such that $f(g(x)) = g(f(x))$ for all $x \in [0,1]$. Suppose also that $g$ is continuously differentiable in $(0,1)$ and $g'(x) \le 1$ if $x \in (0,1)$. Prove that there exists a point $t \in [0,1]$ such that $f(t) = g(t) = t$.

**2.** Fix vectors $v_1, \dots, v_n \in \mathbb{R}^k$. Denote by $(v_1, \dots, v_n)$ the $k \times n$-matrix formed by the coordinates of $v_1, \dots, v_n$. Prove that there exist $m$ and $w_1, \dots, w_n \in \mathbb{R}^m$ with the following two properties:
(i) $(w_1, \dots, w_n)(v_1, \dots, v_n)^T = 0 \in \mathbb{R}^{mk}$;
(ii) for every $l$ and any elements $u_1, \dots, u_n \in \mathbb{R}^l$ such that $(u_1, \dots, u_n)(v_1, \dots, v_n)^T = 0 \in \mathbb{R}^{lk}$, there exists a linear map $T\colon \mathbb{R}^m \to \mathbb{R}^l$ such that $T(w_i) = u_i$ for all $i = 1, \dots, n$.

**3.** Let $A = (a_{ij})$ be a square $n \times n$-matrix with real entries. Suppose that for each subset $I \subseteq \{1, \dots, n\}$ and the matrix $A_I := (a_{ks})_{k,s \in I}$ we have $A_I^n = 0$. Prove that one can make the matrix $A$ upper-triangular with zeroes at the principal diagonal by finitely many transformations of the following type: one transposes any two lines in the matrix and then transposes the columns with the same numbers.

**4.** Let $SABC$ be a pyramid. Consider spheres $X$ inscribed into the trihedral corner with the vertex $S$. Describe the set of the intersection points of the three planes tangent to $X$ passing through lines $AB$, $BC$, $CA$ and not containing the pyramid faces. ($S, A, B, C$ are fixed while $X$ varies.)

**5.** For which (a) odd $n$ (b) even $n$

can one color the vertices of the $n$-dimensional cube into $n$ colors so that for each vertex all $n$ neighboring vertices have different colors?

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