A remark on nonsymmetric compact Riemann surfaces

By

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Let G be a group of automorphisms of a compact Riemann surface of genus $g > 1$. It is well known that G is a factor group of a Fuchsian group Γ of signature

$$(h;\, l_1, l_2, \ldots, l_n),$$

that means a factor group of

$$x_1^{l_1} = x_2^{l_2} = \cdots = x_n^{l_n} = \prod_{i=1}^{n} x_i = \prod_{i=1}^{h} [a_i, b_i] = 1$$

where $[a, b] = aba^{-1}b^{-1}$. The kernel N is a group of signature $(g; -)$.

We use the notation of Singerman [3]. G is called a *K-automorphism group* if Γ is of signature $(0;\, l_1, l_2, 2)$. There is a close connection between K-automorphism groups and regular maps. A Riemann surface is called *nonsymmetric* if it admits no anticonformal involution.

Singerman [3; Theorem 1] has shown that for every automorphism group G of a compact Riemann surface S for which Γ is not a triangle group $(0;\, l_1, l_2, l_3)$ there exists an automorphism group $G_1 \cong G$ of a nonsymmetric Riemann surface S_1 homeomorphic to S. Thus it is of interest to study the nonsymmetric Riemann surfaces where Γ is a triangle group. They are rather exceptional. Singerman gives one of genus $g = 17$ and some others of higher genus.

In this note we classify all K-automorphism groups of compact Riemann surfaces of genus 7. Together with earlier results of Coxeter-Moser, Sherk and the author (see [2]) one then knows all K-automorphism groups of genus $2 \leqq g \leqq 7$. Apart from isomorphic copies there exist exactly two nonsymmetric Riemann surfaces of genus $2 \leqq g \leqq 7$ whose group is a K-automorphism group. They belong to two surface kernels in $(0;\, 9, 6, 2)$ and are of genus 7. As P. Bergau told me, one of these has been known to him before.

Our enumeration is based on the methods described in [2]. Firstly one determines the arithmetical possibilities of $(0;\, l_1, l_2, 2) \to G$ by means of the Riemann-Hurwitz formula

$$(1) \qquad 2g - 2 = |G| \left(\frac{1}{2} - \frac{1}{l_1} - \frac{1}{l_2} \right).$$

By grouptheoretical reasoning one then decides wether there exist torsionfree normal subgroups of index $|G|$ in $(0;\, l_1, l_2, 2)$. We illustrate this procedure by proving:

Proposition. *There exist exactly three classes of conformally equivalent compact Riemann surfaces of genus 7 whose automorphism group is of type $G \cong (0;\, 9,\, 6,\, 2)/N$ with torsionfree N. They are defined by the relations*

(i) $\quad x_1^9 = x_2^6 = x_3^2 = x_1 x_2 x_3 = 1; \quad [x_1, x_2^2] = x_1^3;$

(ii) $\quad x_1^9 = x_2^6 = x_3^2 = x_1 x_2 x_3 = 1; \quad [x_1, x_2^2] = x_1^{-3};$

(iii) $\quad x_1^9 = x_2^6 = x_3^2 = x_1 x_2 x_3 = 1; \quad [x_1, x_2^2] = 1.$

(i) *and* (ii) *belong to nonsymmetric Riemann surfaces. So they give irreflexible regular maps.*

Proof. Using (1) we get $|G| = 54$. The 3-Sylow subgroup S_3 of G is normal. The corresponding normal subgroup in (0; 9, 6, 2) is defined by

$$u = x_1, \quad v = x_2 x_1 x_2^{-1} \mid \; u^9 = v^9 = (uv)^3 = 1.$$

(Use Reidemeister-Schreier.) If S_3 is abelian we have $x_1 \rightleftarrows x_2^2$. Taking N as the normal closure of $[x_1, x_2^2]$ in (0; 9, 6, 2) we get the K-automorphism group (iii).

Now let S_3 be nonabelian. The commutator quotient group S_3/S_3' is of order 9. In fact: S_3' can't be of order 9. Otherwise $S_3' \cong C_9$ or $S_3' \cong C_3 \times C_3$, and so S_3 would have a normal subgroup of index 9.

S_3/S_3' cannot be cyclic of order 9. As the automorphism group of S_3' has order 2 the centralizer of S_3' in S_3 would be S_3. Hence S_3 would be abelian. Thus S_3/S_3' is elementary abelian of type (3, 3).

The commutator quotient group of $\langle u, v\rangle$ is isomorphic to $C_9 \times C_3$. Thus the normal subgroup H in $\langle u, v\rangle$ which corresponds to S_3' is uniquely determined. Hence $u^3 \in H$, $uv^{-1}u^{-1}v \in H$. The cosets $\bar{u}^3$ and $\bar{u}\bar{v}^{-1}\bar{u}^{-1}\bar{v}$ in G necessarily generate the same cyclic group of order 3, so

$$\bar{u}\bar{v}^{-1}\bar{u}^{-1}\bar{v} = \bar{u}^3 \quad \text{or} \quad \bar{u}\bar{v}^{-1}\bar{u}^{-1}\bar{v} = \bar{u}^{-3}. \tag{2}$$

Adding the relations which correspond to (2) to those of (0; 9, 6, 2) one gets cases (i) and (ii). They satisfy the well-known criterion of nonsymmetry resp. irreflexibility (see [2; p. 41] and [3; Theorem 2]) whereas (iii) does not.

We want to remark that (i) and (ii) are 'mirror images' of each other.

The following list gives all K-automorphism groups of genus 7.

	$(l_1, l_2, 2)$ $l_1 \geqq l_2$	$\lvert G \rvert$	Defining relators of N in $(0; l_1, l_2, 2)$	References
1.	(28, 28, 2)	28	$x_1^{13} x_2^{-1}$	[1; p. 105] $\{28, 28\}_{1,0}$
2.	(30, 15, 2)	30	$x_1^{14} x_2^{-1}$	[1; p. 139] $\{30, 15\}_2$
3.	(16, 16, 2)	32	$x_1^2 x_2^2$	[1; p. 139] $\{16, 16\}_2$
4.	(16, 16, 2)	32	$x_1^{10} x_2^2$	[2; Satz 6.2]
5.	(21, 6, 2)	42	$x_1^7 x_2^{-2}$	[2; Satz 6.2]
6.	(12, 6, 2)	48	$x_1^6 x_2^3$; $[x_1^3, x_2]$	[2; Satz 6.4]
7.	(9, 6, 2)	54	$x_1^{-3} [x_1, x_2^2]$	—
8.	(9, 6, 2)	54	$x_1^3 [x_1, x_2^2]$	—
9.	(9, 6, 2)	54	$[x_1, x_2^2]$	—
10.	(7, 7, 2)	56	$x_1^3 x_2^2 (x_1^{-3} x_2)^2$	[1; p. 103] Edmonds' map
11.	(28, 4, 2)	56	$x_1^{14} x_2^2$	[1; p. 115] $\{28, 4\}_{1,1}$
12.	(16, 4, 2)	64	$(x_1^{-1} x_2)^2$	[1; p. 110] $\{16, 4 \mid 2\}$
13.	(16, 4, 2)	64	$x_1^2 x_2 x_1^{-6} x_2^{-1}$	[2; Satz 6.4]
14.	(12, 3, 2)	144	$x_1^3 [x_1^2, x_2] x_1 [x_1^2, x_2]$	—
15.	(7, 3, 2)	504	$\{(x_1^2 x_2^{-1})^5 x_1^{-1}\}^2$	[4; p. 70]

References

[1] H. S. M. COXETER and W. O. J. MOSER, Generators and relations for discrete groups. 2nd edition. Berlin 1965.

[2] D. GARBE, Über die regulären Zerlegungen geschlossener orientierbarer Flächen. J. reine angew. Math. **237**, 39—55 (1969).

[3] D. SINGERMANN, Symmetries of Riemann surfaces with large automorphism group. Math. Ann. **210**, 17—32 (1974).

[4] A. SINKOV, Necessary and sufficient conditions for generating certain simple groups. Amer. J. Math. **59**, 67—76 (1937).

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